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Abstract

We prove that computing a minimum-dilation (Euclidean) Hamilton circuit or path on a given set of points in the plane is NP-hard.

1 Introduction

Let P be a set of n points in $\mathbb{R}^2$ and G be a geometric network on P, i.e., an undirected graph $G(P, E)$ drawn with straight line edges on the plane, where the weight of an edge $pq \in E$ equals the Euclidean distance $|pq|$. The dilation $\delta_G(p, q)$ of a pair of points p, q in G is defined as $\delta_G(p, q) = d_G(p, q)/|pq|$, where $d_G(p, q)$ is the weight or length of a shortest path from p to q in G. The *vertex-to-vertex dilation* or *stretch factor* $\delta(G)$ of G is defined as

$$\delta(G) = \max_{p,q \in P,\ p \neq q} \delta_G(p, q).$$

For a real number $t \geq 1$, we say that G is a t-spanner for P if $\delta(G) \leq t$.

The cost of a network can be measured by the number of edges, the weight, the diameter, or the maximum degree. Constructing low-cost geometric networks of small dilation, as alternatives to the 'expensive' complete Euclidean graphs, is a problem that has been studied extensively. For example, for any given n-point set in the plane and any $\epsilon > 0$, a $(1 + \epsilon)$-spanner with $O(n/\epsilon)$ edges can be constructed in $O((n \log n + n/\epsilon^2)) \log(1/\epsilon)$ time [4]; see also the surveys by Eppstein [6] and Smid [11], as well as the forthcoming book by Narasimhan and Smid [10]. To the other end, one can ask what is the minimum dilation that can be achieved by a network with a given number of edges and other additional properties, and how do we compute such a network. We are interested in the complexity of the following problem:

Minimum-dilation tour (path): Given a set P of points in the plane, compute a minimum-dilation Euclidean Hamilton circuit (path) on P.

Related work. Klein and Kutz [9] have recently proved that computing a minimum-dilation geometric network on a point set in the plane, using not more than a given number of edges, is NP-hard, no matter whether edge crossings are allowed or not. Moreover, Cheong et al. [5]

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showed that the problem remains NP-hard even for the minimum-dilation spanning tree. On the other hand, Eppstein and Wortman [7] gave polynomial time algorithms for the minimum-dilation star problem. Several hardness results exist also for the related problem of finding spanners of given small dilation and weight in general weighted graphs [2, 3, 1].

Results. We prove that the minimum-dilation tour (and path) problem is (strongly) NP-hard. The problem requires the use of exactly n ($n-1$, for a path) edges and that every vertex have degree 2 (except for the start and end-point in the case of a path). Note that the proofs of the results by Klein and Kutz and Cheong et al., mentioned above, cannot handle our problem since the former creates graphs with more than one cycle, while the latter works only for trees with no restriction on the maximum degree. Also, both of these results use reductions from Set Partition, while we use a different approach and reduce from the Hamilton Circuit problem on grid graphs [8]. A collorary of our reduction is that the minimum-dilation tour (and path) problem does not admit an FPTAS.

2 Reduction

For a point $p \in \mathbb{R}^2$, we denote by $p(1), p(2)$ its x and y-coordinate respectively.

Let G^∞ be the infinite graph whose vertex set contains all points of the plane with integer coordinates and in which two vertices are connected if and only if the Euclidean distance between them is equal to 1. A *grid graph* is a finite, node-induced subgraph of G^∞. Note that a grid graph is completely specified by its vertex set. It is well-known that deciding whether a given grid graph has a Hamilton circuit is an NP-hard problem [8]. We reduce the Hamilton circuit problem in grid graphs to the decision version of the minimum-dilation tour problem. Our main result is the following:

Theorem 1 *Given a set P of points in the plane and a parameter $\delta > 1$, the problem of deciding whether there exists a Euclidean Hamilton circuit on P with dilation at most δ is NP-hard.*

Proof: Let G be a grid graph with vertex set V and $|V| = n$. Using V, we construct a point set P such that, for some δ, a Hamilton circuit on P with dilation at most δ exists if and only if G has a Hamilton circuit.

We assume that G has no degree-0 or 1 vertices, since, otherwise, there is no Hamilton circuit in G; this can be checked in polynomial time. Consider the smallest enclosing rectangle R of G, see Fig. 1. Since G is finite and $|V| = n$, R has finite dimensions and its height is at most n. Let $v \in V$ be the vertex that is closest to the lower-left corner of R and lies on the left vertical edge of R. Then, v must be a degree-2 vertex and have a neighbor on the same edge of R; let u be this vertex. We append two point-sets, called 'handles', S and T to G as shown in Fig. 1. Each handle has one horizontal and one vertical part consisting of 2 and $n+1$ points respectively, and the two parts have one point in common. We have

$$S = \{s_1 = (u(1)-1, u(2)), s_2 = (u(1)-2, u(2))\} \cup \{s_i = (u(1)-2, u(2)+i-3) | i = 3, \ldots, n+3\}$$

and

$$T = \{t_1 = (v(1)-1, v(2)), t_2 = (v(1)-2, v(2))\} \cup \{t_i = (v(1)-2, v(2)-i+3) | i = 3, \ldots, n+3\}.$$

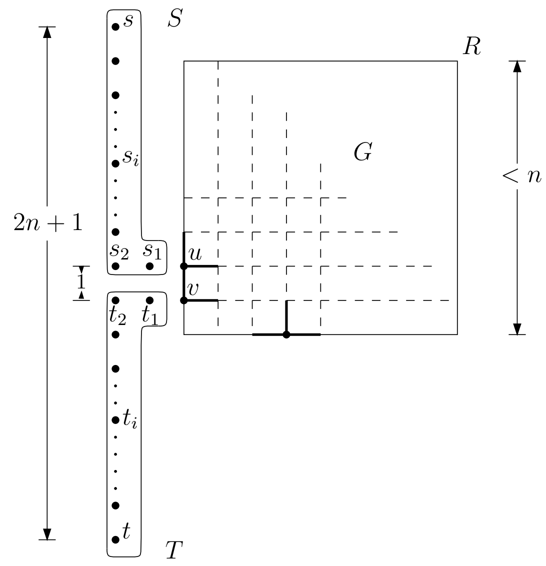

Figure 1: A grid graph G, its smallest enclosing rectangle R, and the point-sets ('handles') S and T.

Let $W = V \cup S \cup T$. We have that $|W| = 3n + 4$. Copies of W will be included later in P. Consider the points $s, t \in W$ with $s = s_{n+3}$ and $t = t_{n+3}$. We have that $|st| = 2n + 1$. We start with a simple lemma.

Lemma 2 *There exists a Hamilton s-t path on W with length $3n+3$ if an only if there exists a Hamilton circuit in G.*

Proof: Assume that there exists a Hamilton s-t path on W with length $3n + 3$. Since W contains $3n + 4$ points, any such path must contain only edges with length 1. Every point s_i with $i = 3, \ldots, n + 2$ is at distance one only from two points, namely, s_{i+1} and s_{i-1}. Hence, the s-t path must contain the edges $s_{i+1}s_i$ and s_is_{i-1}. Similarly, the path must contain the edges $t_{i+1}t_i$ and t_it_{i-1} for $i = 3, \ldots, n + 2$. The edge s_2t_2 cannot be in the path, since, otherwise, the path cannot visit all points in W. Thus, s_2 and t_2 have to connect to s_1 and t_1 respectively. Similarly, s_1t_1 cannot be in the path, and so, the edges s_1u and t_1v must be in the path. The remaining of the s-t path must have a length of $3n + 3 - 2(n + 2) = n - 1$ and visit the remaining $n - 2$ vertices of V starting from u and ending at v. This implies that there is a u-v Hamilton path H_G in G. Since u and v are neighbors in G, edge uv and the u-v path H_G form a Hamilton circuit in G.

Conversely, assume that there is a Hamilton circuit in G. Since v has degree two, any such circuit contains uv. Thus, there is a u-v Hamilton path on V with length $n - 1$. We append to the latter path the edges s_1u, s_2s_1, t_1v, t_2t_1, and $s_{i+1}s_i$, s_is_{i-1}, $t_{i+1}t_i$, t_it_{i-1} for $i = 3, \ldots, n + 2$. This forms a Hamilton s-t path on W with length $3n + 3$. □

We continue with the construction of point set P. First, we choose points on a rectangle R' of width α and height β, with

$$\alpha = (2n^2 + 1)n^6 + 2n^2n^3 \quad \text{and} \quad \beta = 2n^6 + 3n^3.$$

Let a, b, c, and d be the upper-right, upper-left, bottom-left, and bottom-right corner points of R' respectively. Consider a straight-line segment of length n^6. We choose a set B of points on

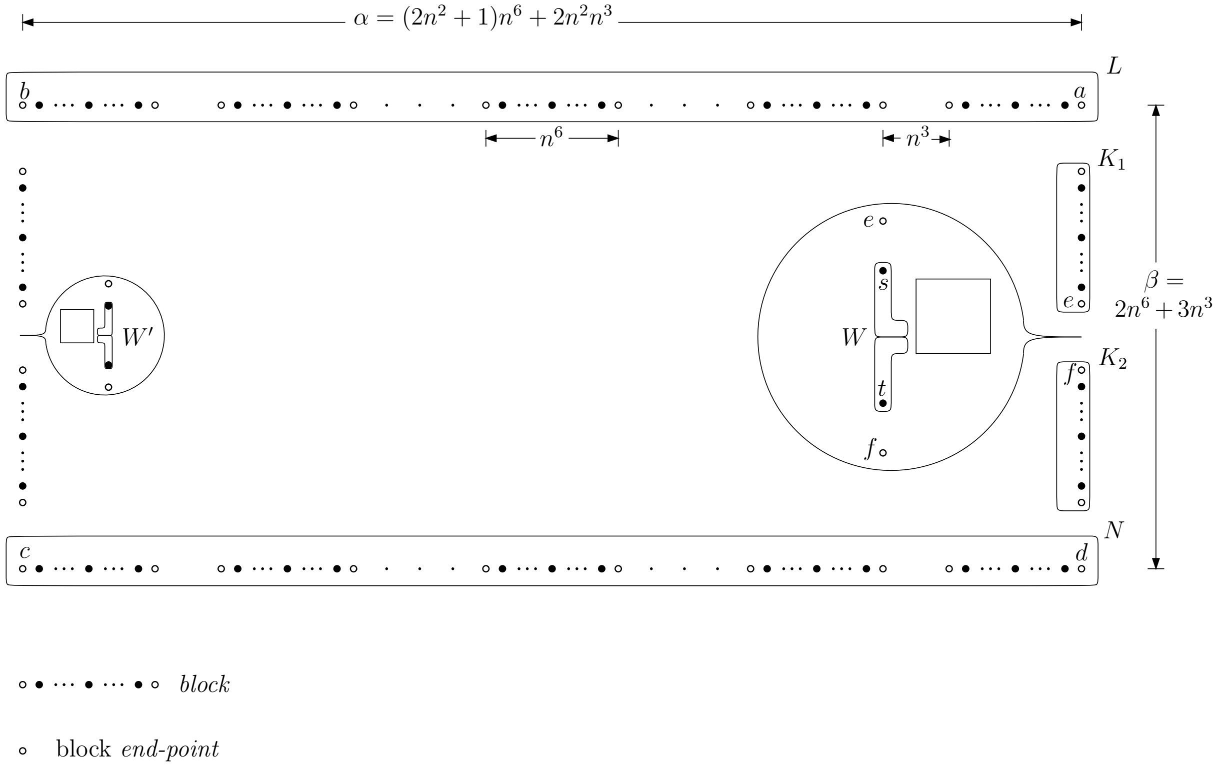

Figure 2: Constructing point set P.

the segment at regular intervals such that the distance between any two consecutive points is $n/2$. We have that $|B| = 2n^5+1$. We use B as a 'building' block: starting on the upper side of the rectangle, from a, we choose copies of B, simply referred to as *blocks*, at regular intervals of length n^3; see Fig. 2 (to avoid cluttering, the edges of the rectangle are not shown). Let K, L, M, and N be the sets of points on the right, upper, left, and lower side of the rectangle respectively. Sets K and M are unions of two vertical blocks each, while L and N are unions of $2n^2+1$ horizontal ones. The right and left-most point of an horizontal block are called the right and left *end-points* of the block. Similarly, the lower and upper-most point of a vertical block are called the lower and upper end-points of the block. Let $K = K_1 \cup K_2$, where K_1, K_2 is the upper and lower block respectively, as shown in Fig. 2. Also, let e be the lower end-point of K_1 and f be the upper end-point of K_2. In the empty interval, i.e., the gap, between K_1 and K_2, we place point set W such that its handles S and T lie on the right side of R'. Additionally, we require that

$$|es| = |ft| = (n^3 - |st|)/2 = (n^3 - 2n - 1)/2.$$

Since the height of the minimum enclosing rectangle R of V is at most n, the distance between any point of a block and any point of W is at least $(n^3 - 2n - 1)/2$ as well. A reflected copy of W, denoted by W', is placed between the two blocks (subsets) of M in a similar way. Let

$$P = K \cup L \cup M \cup N \cup W \cup W'$$

and

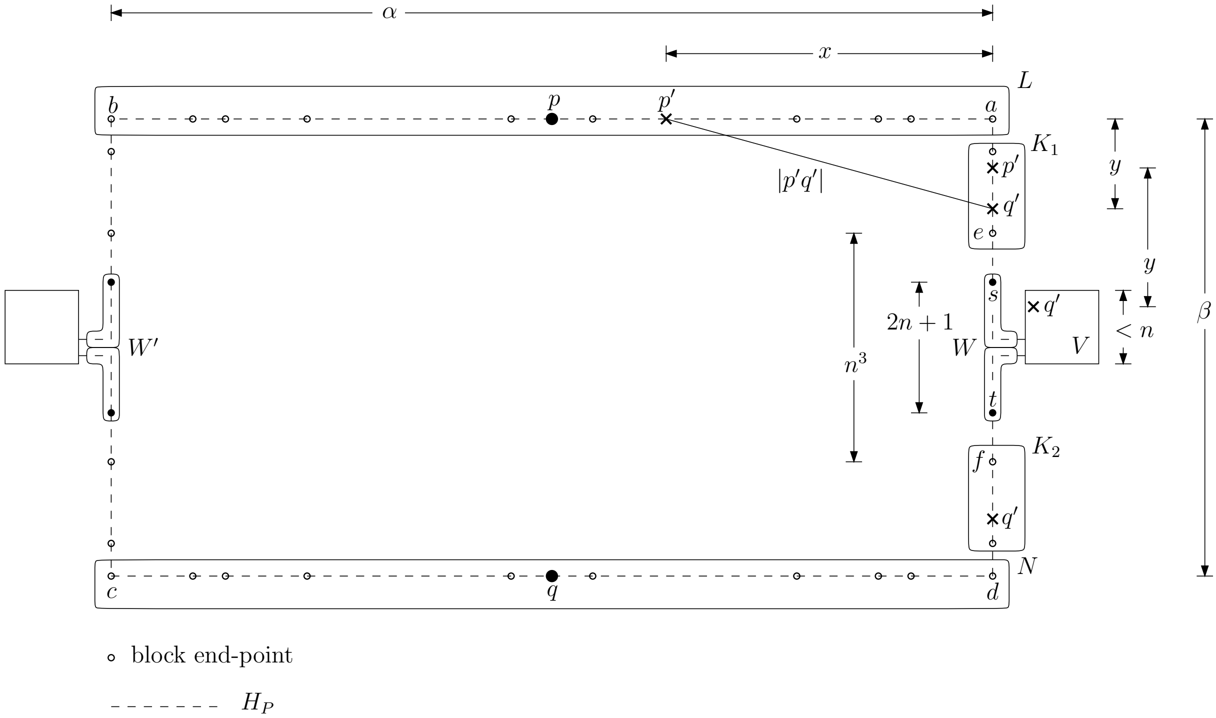

Figure 3: The Hamilton circuit H_P and example positions of points p' and q'.

$$\delta = \frac{\alpha + \beta - (2n+1) + 3n + 3}{\beta} = 1 + n^2 + g(n) + h(n),$$

with

$$g(n) = \frac{n^3 - n^2}{2n^3 + 3} \text{ and } h(n) = \frac{n+2}{(2n^3+3)n^3}.$$

Note that $g(n), h(n) < 1$ for every $n \geq 1$. We have that $|P| = (2n^2+1)(2n^5+1)+2(2n^5+1) = O(n^7)$.

Lemma 3 *If there is a Hamilton s-t path on W with length $3n+3$, then there is a Hamilton circuit on P with dilation at most δ.*

Proof: Let H_W be a Hamilton s-t path on W with length $3n+3$. We construct a Hamilton circuit H_P on P by simply connecting the points in K, L, M, N in the 'canonical' way along the sides of rectangle R', as shown in Fig. 3. First, every two consecutive points in each block are connected by an edge. Second, in L, N, the left end-point of each block is connected to the right end-point of its immediate neighbor block. Finally, the upper end-point of K_1 and the lower end-point of K_2 are connected to points a and d respectively, while e connects to s and f connects to t; the blocks of M are connected to b, c, and the point set W' in a similar way. We prove that $\delta(H_p) \leq \delta$. Let p and q be the 'middle' points of L and N respectively. That is,

$$p = (a+b)/2 \text{ and } q = (c+d)/2.$$

Note that any path from p to q in H_P must go through either W or W'. By the symmetry of the construction of H_P, we have that

$$\begin{aligned} d_{H_P}(p,q) &= |pa| + |ad| + |dq| - |ef| + |es| + |ft| + d_{H_W}(s,t) \\ &= \alpha + \beta - |st| + d_{H_W}(s,t) \\ &= \alpha + \beta - (2n+1) + 3n + 3 = \alpha + \beta + n + 2. \end{aligned}$$

We also have that

$$\begin{aligned} \delta_{H_P}(p,q) &= \frac{d_{H_P}(p,q)}{|pq|} = \frac{\alpha+\beta+n+2}{\beta} = 1 + \frac{\alpha}{\beta} + \frac{n+2}{(2n^3+3)n^3} \\ &= 1 + \frac{(2n^2+1)n^6 + 2n^2n^3}{2n^6+3n^3} + h(n) = 1 + n^2 + \frac{n^3-n^2}{2n^3+3} + h(n) \\ &= 1 + n^2 + g(n) + h(n) = \delta. \end{aligned}$$

We now prove that for any other pair of points $p', q' \in P$, $\delta_{H_P}(p', q') \leq \delta$. We distinguish the following cases, see Fig. 3:

(i) p', q' lie on opposite sides of R', or $p' \in W'$ and $q' \in K$ (symmetrically, $p' \in M$ and $q' \in W$), or $p' \in W'$ and $q' \in W$. In this case we have that $|p'q'| \geq |pq|$. Note that the total length of H_P is $2d_{H_P}(p,q)$, hence $d_{H_P}(p', q') \leq d_{H_P}(p,q)$. Thus, $\delta_{H_P}(p', q') = d_{H_P}(p', q')/|p'q'| \leq d_{H_P}(p,q)/|pq| = \delta$.

(ii) p', q' lie on non-opposite sides of R' and there is a path in H_P connecting them that visits no point in W (symmetrically, W'). If p', q' lie on the same side of R', then $d_{H_P}(p', q') = |p'q'|$, hence $\delta_{H_P}(p', q') = 1$. If p', q' lie on different, i.e., vertical to each other, sides, then $d_{H_P}(p', q') = |p'a| + |aq'| < 2|p'q'|$, hence $\delta_{H_P}(p', q') < 2$.

(iii) p', q' lie on non-opposite sides of R' and any path in H_P connecting them must visit a point in W (symmetrically, W'). First, $|p'q'| \geq |es| = (n^3 - |st|)/2 = (n^3 - 2n - 1)/2$. Let $x = |p'(1) - a(1)|$ and $y = |p'(2) - q'(2)|$. Then,

$$d_{H_P}(p', q') < x + y + d_{H_W}(s,t) < 2|p'q'| + 3n + 3.$$

Thus, $\delta_{H_P}(p', q') < 2 + (3n+3)/(n^3 - 2n - 1) < 3$, for any $n \geq 3$.

(iv) Finally, when $p', q' \in W$ or W', we have that $d_{H_P}(p', q') \leq d_{H_W}(s,t) = 3n+3$ and $|p'q'| \geq 1$, hence $\delta_{H_P}(p', q') \leq 3n + 3 \leq \delta$, for any $n \geq 4$. ◻

Conversely, we now prove the following.

Lemma 4 *If there is a Hamilton circuit on P with dilation at most δ, then there is a Hamilton s-t path on W with length $3n+3$.*

Proof: Let H_P be a Hamilton circuit on P with $\delta(H_P) \leq \delta$. Also, let $p = (a+b)/2$ and $q = (c+d)/2$. We prove that H_P must contain a path from p to q that is 'locally optimal' in the sense that firstly, it connects p to s and t to q in the 'canonical' way on the sides of R' (as is was described the proof of Lemma 3), and secondly, it connects s to t via a Hamilton path on W with length $3n+3$. In particular, we show that δ is small enough to ensure that the following requirements be met:

(i) Once inside a block, H_P visits all the points of the block before leaving it. To see this, consider a block B and a point $p_i \in B$ for which there is an edge op_i with $o \in P \setminus B$ (such a

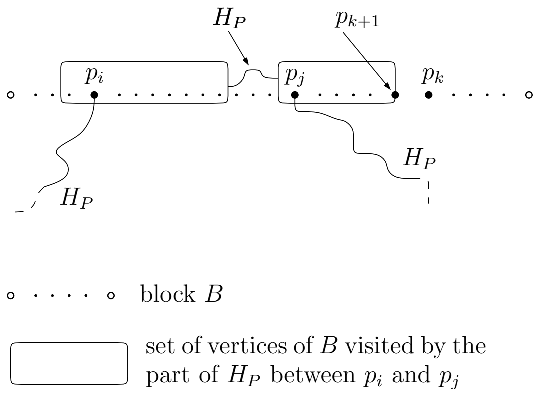

Figure 4: Case (i) in the proof of Lemma 4.

point must exist since H_P must visit all the points of P); see Fig. 4. We trace H_P starting from o and entering B via op_i. Assume that H_P leaves B before having visited all its points and let p_j be the last point visited and p_k be a point that is left out. Then, at least one neighbor point, say p_{k+1} of p_k in B is not connected to p_k via this part of H_P inside B. Also, H_P must visit p_k but only after it has visited at least one point outside B. Let o' be such a point connected to p_j with an edge p_jo'. Note that o' must be in some block other than B or in W or in W'; the same holds for o. Recall that the distance between any two blocks is at least n^3 and that the distance between any block and W or W' is at least $|es| = (n^3 - 2n - 1)/2$. Hence, $|p_jo'| \geq (n^3 - 2n - 1)/2$ and $|op_i| \geq (n^3 - 2n - 1)/2$ as well. We have that $d_{H_P}(p_{k+1}, p_k) > 2\min\{|p_jo'|, |op_i|\} \geq 2(n^3 - 2n - 1)$ and

$$\begin{aligned}\delta_{H_P}(p_{k+1}, p_k) &= \frac{d_{H_P}(p_{k+1}, p_k)}{|p_{k+1}p_k|} > \frac{2(n^3 - 2n - 1)}{n} = 2n^2 - 4 - \frac{2}{n} \\ &> 2n^2 - 6 > n^2 + 2 > \delta,\end{aligned}$$

for any $n \geq 3$.

(ii) Once inside W (or W'), H_P visits all the points of W (or W') before leaving it. This can be seen by using arguments similar to the ones in case (i). Note, first, that every point in W is at distance 1 from at least one other point in W; for each of s, t, there is exactly one such point, while for any other point there are exactly two. If H_P visits W and leaves without having visited some point $p' \in W$, then there is a point $q' \in W$ with $|p'q'| = 1$ that is not connected to p via this part of H_P inside W. Then $d_{H_P}(p', q') > 2|es| = 2(n^3 - 2n - 1)$ and

$$\delta_{H_P}(p', q') = \frac{d_{H_P}(p', q')}{|p'q'|} > 2(n^3 - 2n - 1) > n^2 + 2 > \delta,$$

for any $n \geq 3$.

(iii) Any two blocks that are consecutive along the sides of R' must be 'connected' by an edge in H_P, as long as W or W' does not lie between the two blocks. To see this, consider a block B and a neighbor of it, B', and let p' and q' be the endpoints of B and B' respectively, with $|p'q'| = n^3$. Assume that H_P contains no edge connecting a point of B to a point of B'; see Fig 5. Then, any path from p' to q' in H_P must visit some other block, different from B and B', and, hence, $d_{H_P}(p', q') > n^6$, where n^6 is the diameter of any block. Thus, $\delta_{H_P}(p', q') > n^6/n^3 = n^3 > n^2 + 2 > \delta$, for any $n \geq 2$.

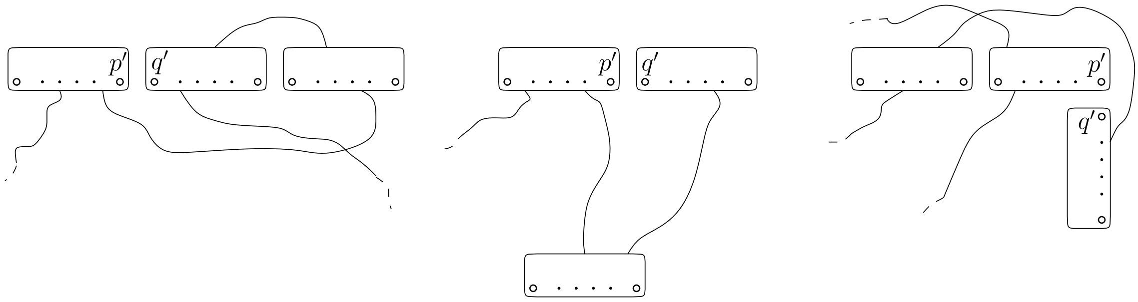

Figure 5: Examples of the case (iii) in the proof of Lemma 4.

(iv) Blocks K_1, K_2 must be connected to W by an edge in H_P; this holds also for the blocks in M and W'. Similarly to the case (iii), assume, for example, that H_P contains no edge 'connecting' K_1 and and W. Then, any path from e to s in H_P must visit some other block as well. By case (i), once in the latter block, H_P has to visit all its points before exiting. Hence, $d_{H_P}(e,s) > n^6$ and

$$\delta_{H_P}(e,s) = \frac{d_{H_P}(e,s)}{|es|} > \frac{n^6}{(n^3 - 2n - 1)} > n^3 > \delta,$$

for any $n \geq 2$.

All the above requirements assert that H_P does not contain 'long' edges (jumps) between any two points of P that belong to different blocks or between a point of a block and a point of W or W'. Note that, since all points of a block lie on the same straight-line segment, the minimum-length Hamilton path on a block has length n^6; any detour increases this length by at least n. Also, as already noted before, s and t are the points of W that are closest to e and f respectively. Case (ii) also asserts that the part of H_P inside W (W') forms a Hamilton s-t path on W (W'); let l be its length. Consider now the pair p, q. By combining all the above, we have that

(1) $d_{H_P}(p,q) \geq |pa| + |ae| + |es| + l + |tf| + |fd| + |dq| = \alpha + \beta - (2n+1) + l$.

Since $\delta(H_P) \leq \delta$, we have that $\delta_{H_P}(p,q) \leq \delta$ as well. From the proof of Lemma 3, this implies that

(2) $d_{H_P}(p,q) \leq \alpha + \beta + n + 2$.

From (1), (2), we have that $l \leq 3n+3$. However, any Hamilton path on W has length at least $3n+3$, and the lemma follows.

$\square$

Note that all points in P have rational coordinates with numerators and denominators bounded by a polynomial in n. Also, the construction of P takes $O(|P|) = O(n^7)$ time. Combining Lemmata 2, 3, and 4, concludes the proof of the theorem. $\square$

The proof of Theorem 1 is based on a reduction from the Hamilton Circuit problem in a grid graph G. As already mentioned above, any Hamilton path on W has length at least $3n+3$ and, from Lemma 2, this value is achieved if and only if there is a Hamilton circuit in

G. On the other hand, if there is no Hamilton circuit in G, then any Hamilton path on W has length at least $(3n+3) - 1 + \sqrt{2}$: in this case, at least one diagonal of the grid must be used by the path. This observation implies that the minimum-dilation tour problem admits no FPTAS.

Corollary 5 *The minimum-dilation tour problem does not admit an FPTAS.*

Proof: Assume that that there is an FPTAS. When run on P, the algorithm computes, in $O((1/\epsilon)^c n^k)$ time, for constants c, k, a Hamilton circuit H_P and its dilation δ_{apx} with $\delta_{\text{apx}} \leq (1+\epsilon)\delta_{\text{opt}}$, for any $\epsilon > 0$, where δ_{opt} is the dilation of an optimal tour. Consider any $\epsilon < (\sqrt{2}-1)/(\beta\delta)$, where β, δ are as in the proof of Theorem 1.

Note that $\delta_{\text{apx}} \leq (1+\epsilon)\delta$ implies that

$$\delta_{\text{apx}} < \delta + \frac{\sqrt{2}-1}{\beta} = \frac{\alpha + \beta - (2n+1) + 3n + 3 + (\sqrt{2}-1)}{\beta}.$$

Consider the 'middle' points p, q. Since $|pq| = \beta$, we have that $d_{H_P}(p,q) < \alpha + \beta - (2n+1) + 3n + 3 + (\sqrt{2}-1)$. It is easy to check that term $\sqrt{2}-1$ is small enough to leave no other alternative to H_P but the form imposed by the cases (i), (ii), (iii), and (iv) in the proof of Theorem 1. Moreover H_P must visit W through its the points s and t: if any other point in W is visited first instead, $d_{H_P}(p,q)$ will increase by at least 2. Hence, H_P contains a Hamilton path on W of length $3n+3$.

On the other hand, if $\delta_{\text{apx}} > (1+\epsilon)\delta$, then $\delta_{\text{opt}} > \delta$. But, for any Hamilton circuit H on P we have that $\delta(H) \geq \delta_{\text{opt}} > \delta$, which, by Lemma 3, implies that there is no Hamilton path of length $3n+3$ on W. ◽

As it is easy to see, the proofs of Theorem 1 and Corollary 5 hold also for the decision version of the minimum-dilation path problem by considering a point set that contains p and q, the points of P that lie on R' to the right side of p and q, and the points in W.

Corollary 6 *Given a set P of points in the plane and a parameter $\delta > 1$, the problem of deciding whether there exists a Euclidean Hamilton path on P with dilation at most δ is NP-hard. The minimum-dilation path problem does not admit an FPTAS.*

3 Concluding remarks

We have proved that computing a minimum-dilation (Euclidean) Hamilton circuit or path on a given set of points in the plane is NP-hard, and that the problem does not admit an FPTAS. Does it have a PTAS? Note that no constant-factor, polynomial-time approximation algorithm is even known for this problem as well as the (general) minimum-dilation graph and spanning tree problems. Finally, can we devise fast exact algorithms?

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