On enumeration of chord diagrams and asymptotics of Vassiliev invariants

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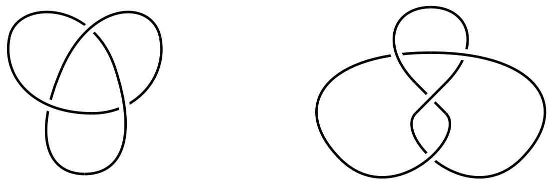
1 Vassiliev Invariants for knots

For many years the nature of knots has fascinated many people, especially mathematicians. Trying to rigorously understand and describe this phenomenon led to the development of knot theory as a branch of mathematics, more precisely of mathematical topology.

Initiated by VASSILIEV's considerations of the homology of knot spaces [Va], in the last years the theory of VASSILIEV invariants has become an interesting and intensively studied field of knot theory. It aims to approach the problem of topological classification of knots by using simpler and easier understandable algebraic and combinatorical methods. In this way this many-sided new field opens interesting relations to classical fields of mathematics, such as the homological algebra, the theory of Lie groups, graph theory and topological quantum field theories (TQFT's).

1.1 The classification problem of knots

Consider a knot, i. e. an oriented embedding of S^1 in $\mathbb{R}^3$. Let us assume, that this embedding is C^1, which particularly means tame, i. e. the knot is isotopic to a piecewise linear embedding. This does not constrain too much the complexity of knots, it only excludes some ugly pathological cases. The following picture shows the two simplest, but non-trivial knots.



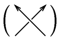
trefoil figure 8 knot

Thinking of a knot as of a real object, say a closed piece of rope, one gets an idea of transforming one knot into another by a sequence of pulling and twisting its strands, but not cutting the knot somewhere. One might consider knots as equal, if they are in this sense transformable into each other. This is mathematically described by the notion of (ambient) isotopy. So, in mathematical terms, knots are called topologically equal, if they are ambient isotopic, and to topologically classify a given knot means to determine uniquely its isotopy class.

Trying to classify knots led to the search for knot invariants, which, at least in some particular cases, can distinguish knots. Some approaches have been made to construct such invariants topologically. The most famous invariants of this kind are the polynomial invariants like the ALEXANDER/CONWAY [Al, Co, Ka3] and HOMFLY [H, LM] polynomials, and also CASSON's invariant [AM]. Unfortunately, these topological approaches often meet major difficulties, resulting from the analytical methods they involve.

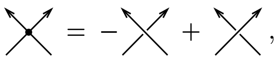
On this background the main advantage of the theory of VASSILIEV invariants appears to be, that it offers a new view of the topological structure of knots in a combinatorial and therefore more discrete and directly accessible way and makes it possible to apply simpler algebraic methods for exploring them.

1.2 The filtration of the knot space

Consider the linear space $\mathcal{V}$, (freely) generated by all the (isotopy classes of) knot embeddings. There is a self-suggesting way how to denote them. Every embedding class of a knot is determined by its projection on a 2-plane in $\mathbb{R}^3$, where all the crossings are transversal and equipped with the additional information, whether they are over- () or under- () crossings. Let $\mathcal{V}^p$ be the space of singular knots with exactly p double points  (up to isotopy).

Except for the last section, it is convenient to assume it to be a vector space over a field of characteristic zero, else it can be a module over any commutative ring with unit.

$\mathcal{V}^p$ can be identified with a linear subspace of $\mathcal{V}$ by resolving the singularities into the difference of an over- and an undercrossing via the rule

 (1.1)

where all the rest of the knot projections are assumed to be equal (one can show that the result does not depend on the order in which all the double points are resolved). This yields a filtration of $\mathcal{V}$

$$\mathcal{V} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \mathcal{V}^2 \supset \mathcal{V}^3 \supset \ldots \tag{1.2}$$

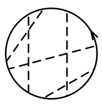
There exists a combinatorial description of the graded vector space,

$$\bigoplus_{i=0}^{\infty} \left(\mathcal{V}^i \big/ \mathcal{V}^{i+1} \right) \tag{1.3}$$

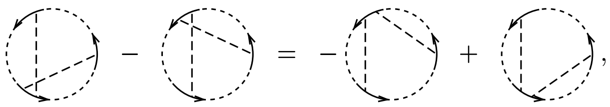
associated to this filtration, namely

$$\mathcal{V}^i \big/ \mathcal{V}^{i+1} \;\simeq\; \mathcal{L}in \,\{ \text{ chord diagrams of degree } i \,\} \Big/ \begin{smallmatrix} 4T \text{ relation} \\ FI \text{ relation} \end{smallmatrix} , \tag{1.4}$$

where the chord diagrams (CD's) are objects like this (an oriented circle with finitely many dashed chords in it)

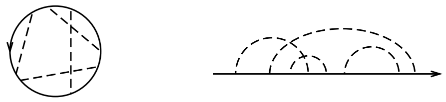


(up to isotopy) and are graded by the number of chords. The $4T$ (4 term) relations have the form

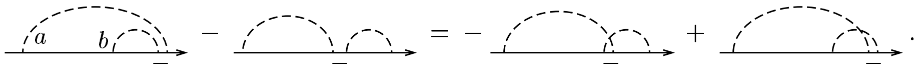


and the FI (framing independence) relation requires, that each chord diagram with an *isolated* chord (i. e., a chord not crossed by any of the others) is zero.

Chord diagrams appear in two forms – circular and linearized. In the latter case they are also called *linearized chord diagrams* or LCD's.



In the linearized form, which we will use henceforth, the 4T relation looks this way:



term 1 term 4 term 3 term 2

$(4T(a, b))$

(Note that the fixed end of chord a can also be to the right or within the chord b.) The meaning of the underlines '$-$' in the previous picture is the following: if in a picture an interval of the baseline is marked by a '$-$', we will not allow basepoints of other chords to end within it, whereas in the other case this can be.

The map which yields this isomorphism is given by the following simple way of how to assign a chord diagram D_K to a singular knot K. Connect in the parameter space of K (which is an oriented S^1) pairs of points with the same image by a chord. Actually, the idea to describe singular knots in this way led to the representation (1.4), and, more generally, many of the further algebraic statements are based on this idea. Once the chord diagrams have been introduced, the $4T$ and FI relations become self-suggesting. Look e. g. at the FI relation. If one takes a singular knot corresponding to a diagram with an isolated chord and resolves the singularity corresponding to this chord, one gets exactly an ambient isotopy relation of (singular) knots.

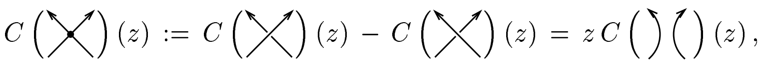
Note that (1.4) implies the finite dimensionality of the filtration (1.2), which considerably simplifies further algebraic considerations.

Definition 1.1 *A* VASSILIEV *invariant of degree m is an element $V \in \mathcal{V}^*$ (where $\mathcal{V}^*$ denotes henceforth the dual space of $\mathcal{V}$), s. t.*

$$V\Big|_{\mathcal{V}^{p+1}} \equiv 0 \quad \textit{and} \quad V\Big|_{\mathcal{V}^{p}} \not\equiv 0\,.$$

It is possible in some sense to consider (1.1) as a way to "differentiate" a knot, and in this language VASSILIEV invariant corresponds to a polynomial invariant, i. e. a function with a vanishing derivative.

This notion is interesting in connection with many other known knot invariants, such as the CONWAY and the HOMFLY polynomials, which have originally come about by topological considerations, but which fulfill certain relations closely connected to (1.1). E. g., the CONWAY polynomial $C_K(z)$ of a knot K satisfies the (skein) relation



(1.5)

from which one concludes, that

Lemma 1.2 *The coefficient $K \mapsto [C_K(z)]_{z^k}$ of z^k in the Conway polynomial is a* VASSILIEV *invariant of degree at most k.*

By similar arguments the same is true for the HOMFLY polynomial in the reparametrization introduced by Jones [J] $F_K(N, e^z)$ (and therefore, setting $N = 2$, for the Jones polynomial [J2] as well) and the Kauffman polynomial [Ka2] in his version called by Kauffman the Dubrovnik polynomial [Ka]. See [BL].

In this way Vassiliev invariants generalize many known (topological) invariants, and that is another reason why Vassiliev invariants are so interesting.

Note, that the grading of the chord diagrams is preserved by $4T$, so the linear space $\mathcal{A}^r$ obtained by taking the direct sum over all i of the right hand side of (1.4) is a finite dimensionally graded space. Let G_m denote the degree-m-piece of $\mathcal{A}^r$.

By definition all VASSILIEV invariants of degree $\leq m$ are sensitive with respect to knot classification only *maximally* up to $\mathcal{V}^{m+1}$. However, one does not know yet, whether *all* Vassiliev invariants are capable of a *complete* topological classification of knots.

Conjecture 1.3 VASSILIEV *invariants separate knots.*

By now conjecture 1.3 has been proved at least for pure braids [BN3, BN4] (see section 1.5), but for knots we know little about it. As it stands it sounds very appealing, but unfortunately, we cannot even yet affirm the following weaker

Question 1.1 *(see [BN2, sect. 7.2]) Do Vassiliev invariants distinguish knot orientation?*

This is one of the hardest problems in knot theory. Yet, there are no easily definable invariants, as quantum and skein invariants, which distinguish knot orientation. As pointed out by BIRMAN [Bi], the fact itself that non-invertible knots exist has been proved only in the 60's by TROTTER [Tr]. In [St4] I tried to enlighten the problems with detecting orientation with Vassiliev invariants in an indepentent way from Bar-Natan's computational arguments (see section 1.6).

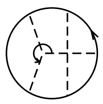
1.3 The Algebra $\mathcal{A}$

Let us for a moment forget about the FI relation and consider $\mathcal{A}$ as the space, obtained from $\mathcal{A}^r$ by not factoring out the FI relation.

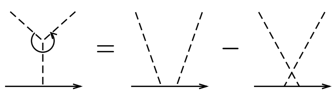
For this space $\mathcal{A}$ at least 3 other descriptions are known (for the proof, that they are all isomorphic, see [BN2]).

1) $$\mathcal{A} = \mathcal{L}in \left\{ \begin{matrix} \text{CCD's (Chinese character or} \\ \text{FEYNMAN diagrams)} \end{matrix} \right\} \Big/ STU \text{ relation},$$

where a Chinese character diagram is something like



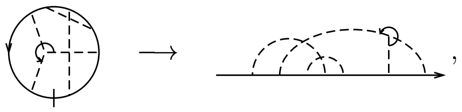
(like a chord diagram, but with oriented internal trivalent vertices allowed, which represent singular triple points) and the STU relation is



,

2) $$\mathcal{A} = \mathcal{L}in \{ \text{ linearized CCD's } \} \Big/ STU \text{ relation},$$

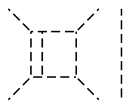
where a linearized Chinese character diagram is a Chinese character diagram, with the solid line cut somewhere



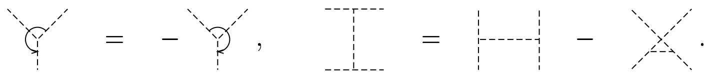
and both spaces are graded by half the number of trivalent vertices (internal or on the solid line) and

3) $\mathcal{A} = \mathcal{L}in \, \{ \text{ CC's (Chinese characters) } \} \Big/ AS \text{ and } IHX \text{ relation} ,$

where the Chinese characters are objects like this



(Chinese character diagrams with the solid line removed) and graded by half the number of vertices (univalent or trivalent), and the AS (antisymmetry) and the IHX relations are



Using the different representations of $\mathcal{A}$ one can define multi- and comultiplication, which make $\mathcal{A}$ into a graded commutative and cocommutative HOPF algebra [MM], including also a sort of ADAMS operations ψ^q, $q \in \mathbb{Z}$.

1.4 Weight systems

We have defined the Vassiliev invariants in terms of spaces of knot embeddings, but we found an easier description of these spaces by diagrams. Then is it possible to describe Vassiliev invariants by combinatorical objects defined entirely with the help of our diagram spaces?

Definition 1.4 *A weight system of degree m is an element $W \in (G_m\mathcal{A})^*$. Let $\mathcal{W}^m$ be the linear space of all weight systems of degree m.*

It is easy to assign to a Vassiliev invariant V of deg $\leq m$ a weight system W of degree m. Consider the (graded) map W_* given by

$$\mathcal{V}_m := (\mathcal{V}^m)^* \xrightarrow{W_m} (G_m\mathcal{A})^* =: \mathcal{W}^m \tag{1.6}$$

$$W_m(V)\,(\text{diagram of deg } m) := V\left(\begin{array}{c}\text{one singular knot with } m \text{ singula-}\\ \text{rities, that represents this diagram}\end{array} \right) \tag{1.7}$$

The kernel of this map is by definition $\mathcal{V}_{m-1} = (\mathcal{V}^{m-1})^*$, i. e. the space of Vassiliev invariants of deg $\leq m-1$, so we have the short exact sequence

$$0 \longrightarrow \mathcal{V}_{m-1} \hookrightarrow \mathcal{V}_m \xrightarrow{W_m} \mathcal{W}_m \longrightarrow 0\,.$$

Theorem 1.5 (Fundamental Theorem of Vassiliev invariants) *This exact sequence splits, i. e. there is a (graded) map V*

$$V_m \,:\, \mathcal{W}_m \longrightarrow \mathcal{V}_m \,,$$

such that

$$Im(\,Id_{\mathcal{V}_m} - V \circ W\,) \subset \mathcal{V}_{m-1} \quad \textit{and} \quad W_m \circ V_m = Id_{\mathcal{W}_m} \,.$$

In some sense the map V can be considered as "integrating" a weight system. If we want to do this, we need to consider another crucial object, closely related to this theorem. Namely, it turns out, that proving that theorem is equivalent to constructing a *universal* VASSILIEV *invariant*, i. e. a map

$$\mathcal{V}^* \xrightarrow{V'} \mathcal{A}^{**}_* \,,$$

where $\mathcal{A}^{**}$ is the graded completion of $\mathcal{A}$, such that for every knot $K \in \mathcal{V}^m$ and any Vassiliev invariant $V \in \mathcal{V}_m$ there holds

$$\big(\,W(V)\big)\,(\,V'(K)\,) \quad = \quad V(K)\,.$$

Therefore, universal Vassiliev invariants play a basic role in the theory of Vassiliev invariants. Universal Vassiliev invariants have been constructed by many people an in different ways, e. g. [BN, Pi]. Though it is known, that the choice of a universal Vassiliev invariant is not unique, surprisingly many of the ansatzes seem to generate the same special one.

Summarizing, Vassiliev invariants form a commutative and co-commutative Hopf algebra $\mathcal{A}^*$ which is the dual algebra to the Hopf algebra $\mathcal{A}$ of *chord diagrams* (CD's) modulo the 4T (4 term) relation.

The primitive (as Hopf algebra elements) Vassiliev invariants $\mathcal{P}(\mathcal{A}^*)$ are the ones behaving additively under connected knot sum. Since co-commutative co-associative Hopf algebras over a field of characteristic zero are primitively generated [MM, corr. 4.18], the projection from $\mathcal{A}^*$ onto $\big(\mathcal{P}(\mathcal{A})\big)^*$ gives an isomorphism between $\big(\mathcal{P}(\mathcal{A})\big)^*$ and $\mathcal{P}(\mathcal{A}^*)$. $\mathcal{P} = \mathcal{P}(\mathcal{A})$ is the linear subspace generated by *connected* chord diagrams. $\mathcal{A}$ and $\mathcal{A}^*$ are finite-dimensionally graded by the number of chords (which we will denote by D).

The degree-1-piece $I := G_1\mathcal{A}$ of $\mathcal{A}$ is one-dimensional and the algebra $\mathcal{A}^r := \mathcal{A}/I\mathcal{A}$ has as dual the *framing-independent* Vassiliev invariants (i. e., exactly the ideal generated by FI relations).

Numerically little is known about $\mathcal{A}$ and $\mathcal{A}^r$. The dimension of their graded pieces $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ is known up to degree $D \leq 9$ [BN2][1]. In general one knows some asymptotical bounds for $\dim G_D\mathcal{A}^r$. It was recently found [CD2] that $\dim G_D\mathcal{P}$ grows faster than any polynomial in D. The best upper bound known up to now is the one obtained by NG [Ng, corollary 4.3][2] : $(D-2)!/2$ for $D > 5$ (see also Ng and STANFORD [NS]), who improved the bound $(D-1)!$ by CHMUTOV and DUZHIN [CD].

In section 5 we're going to prove the upper bound $D!$/any given polynomial in D for both $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ and later improve it to $D!/1.1^D$.

1.5 VASSILIEV invariants for braids and string links

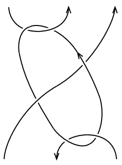
In the same way as for knots, the idea of constructing VASSILIEV invariants can be generalized to other "knot-like objects", i. e. certain classes of 1-dimensional embeddings into $\mathbb{R}^3$,

[1] Some more computational information on these algebras and their generalizations to string links [BN4] and braids [BN3] can be found in [BN7].

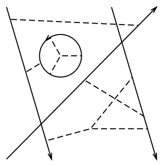
[2] Ng proved the bound only for the non-framed case, however, with a minor modification the proof for a somewhat worse but asymptotically equal bound also works for the framed case.

factorized by an appropriate notion of isotopy, such as tangles, (pure) braids and string links. The resulting diagram spaces $\mathbf{AT}$ (for tangles), $\mathbf{AB}$ (for braids), $\mathbf{AP} = \mathcal{A}^{sl}$ (for string links) and $\mathcal{A}^{pb} = \mathbf{AP}^{hor}$ (for pure braids) are constructed in the same way as for knots and have a similar algebraic structure (here the multiplication is given by stacking up) and properties as the knot diagram algebra $\mathcal{A}$. Here are some typical examples for the several embedding classes and for diagrams, corresponding to these classes.

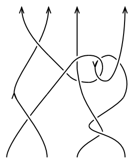
A tangle is an oriented embedding $T : [0,1] \times \mathbb{Z}_n \cup S^1 \times \mathbb{Z}_m \hookrightarrow [0,1] \times \mathbb{R}^2$, such that $T(\{0,1\} \times \mathbb{Z}_n) \subset \{0,1\} \times \mathbb{R}^2$.



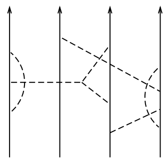
Here is a singular tangle diagram.



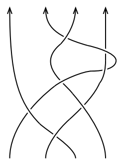
A string link is an oriented embedding $S : [0,1] \times \mathbb{Z}_n \hookrightarrow [0,1] \times \mathbb{R}^2$, such that $S(\{i\} \times \mathbb{Z}_n) \subset \{i\} \times \mathbb{R}^2$ for $i = 0, 1$.



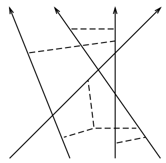
The diagram of a string link contains no cyclic full lines, and all strands are vertical and point to the top.



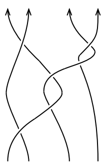
Here is a braid. It is a string link with monotonous 1st component.



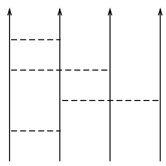
The diagram of a braid contains no chords ending on one strand only.



A pure braid is a braid, which preserves the order of the strands.



The diagram of a pure braid can be simplified to a diagram only with horizontal chords.



$\mathcal{A}^{sl}$ is known to have also a formulation as a space of coloured Chinese characters, i. e. Chinese characters with coloured univalent vertices [BN4].

1.6 Constructing a universal VASSILIEV invariant

There are at least 4 different approaches to this task: a naive topological approach (see [BS]), which fails but comes close, KONTSEVICH's integral formula [Ko], a physical and an algebraic approach. Originally due to DRINFEL'D [Dr] and elucidated further by BAR-NATAN [BN], latter is the probably most elegant solution. (See also [K]).

1.7 Braiding sequences

The combinatorial structure of chord diagrams and Chinese character diagrams considerably simplified our understanding of Vassiliev invariants and was the main tool in the proof of a series of results [BG, Vo]. Despite being therefore much celebrated, this approach has some serious defects. Although many ways exist to prove the Fundamental theorem [BS], they are all rather complicated and at some point unnatural, and their connections are not yet completely understood. So the integration of the (series of) weight system(s) to a Vassiliev invariant is far from being routine work.

But even for itself, although simpler and much friendlier to work with, the combinatorial structure of chord diagrams is far from being easily understandable [BN7].

In an attempt to create an alternative to the (defects of the) classical approach and generalizing some ideas of Dean [De], Trapp [Tr] and Stanford [Sa], in [St4] I introduced the notion of a braiding sequence. It offered a simple direct understanding of the behaviour of Vassiliev invariants on special knot classes, something, which was never worked out using the classical approach. Beside some other facts, it gave relatively simple proofs that the dimension of the space of Vassiliev invariants of degree $\leq n$ on certain knot classes is finite (arborescent knots), and in some cases even exponential upper bounds in n for this dimension (e. g., rational knots, closed 3 braids), something, which was not yet achieved by chord diagrams.

Moreover, while the Kontsevich-Drinfel'd approach (used in [CD]) works only over zero (field) characteristic, our arguments with braiding sequences hold for any zero divisor free ring, in particular the fields $\mathbb{Z}_p$, p prime.

Definition 1.6 *For some odd $k \in \mathbb{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 1). A braiding sequence (associated to a numbered set P of crossings in a diagram D; all crossings by default) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1, \dots, x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done.*

Any Vassiliev invariant v of degree at most k behaves on a braiding sequence as a polynomial of degree at most k in $x_1, \dots, x_{|P|}$ (see [St4] and [Tr]), and this polynomial is called the braiding polynomial of v on this braiding sequence.

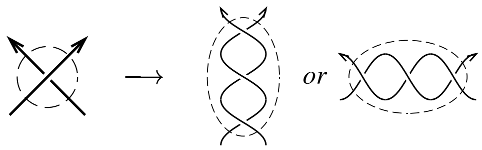


Figure 1. Two ways to do a -3-braiding at a crossing.

2 The results of this thesis

The subject of the present thesis are combinatorics of chord diagrams and asymptotics of Vassiliev invariants.

In sections 3 and 4 we will derive some (purely) enumerative results on special kinds of chord diagrams. Although not directly related to Vassiliev invariants, these results provide a glimpse of the combinatorial complexity of chord diagrams – already for chord diagrams with properties, which are easy to define, the enumeration is rather hard and requires additional ideas.

We show consecutively how to count in a non-brute force way all chord diagrams of given degree, all chord diagrams up to mirroring, all chord diagrams with an isolated chord (the ones sent to zero by the FI relation), all chord diagrams with an (isolated) chord of length one, chord diagrams, whose intersection graph is connected and those for which it is a tree.

In section 5 we will use combinatorial techniques to relate the enumeration of special chord diagrams to the enumeration of Vassiliev invariants and will prove the asymptotical upper bound $D!/1.1^D$ for the number of Vassiliev invariants in degree D.

The basic idea for this improvement is to work with linearized chord diagrams (LCD's) and the order of chord basepoints from left to right.

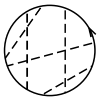
In section 6 we will use the techniques of section 5 and the result of Chmutov and Duzhin [CD2] to deduce a lower bound for the number of all Vassiliev invariants and discuss the relation between the asymptotics of primitive and all Vassiliev invariants. At the same time, we give a summary on what we know about the asymptotics of Vassiliev invariants.

Finally, in section 7 we use the rather different approach of braiding sequences to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots.

Parts of this work can be found in several papers of mine [St2, St6, St8, St9, St10].

3 On the number of chord diagrams

In this section we treat some enumeration problems of certain kinds of chord diagrams. Recall, that a chord diagram (a CD) is an object like this,



i. e. an oriented circle with finitely many dashed chords in it and considered up to isotopy.

The essential difficulty of this enumeration is determining their linearized relatives, called LCD's, fixed by a certain cyclic permutation of the basepoints. This we achieve by introducing some new objects called generalized linearized chord diagrams or short GLCD's.

It should be mentioned, that similar enumeration problems have been treated in another way in several other papers, e. g. [Be, S, NW, Bo, DP, HS].

3.1 Notations

For two numbers $m, n \in \mathbf{N}$ their g.c.d. is denoted (m, n) and $m\%n$ is $m \bmod n$.

If P is a finite set, by the symbol $\#P$ we will denote its cardinality and by $\mathcal{P}(P)$ its power set (set of all subsets).

In the following we will need some number-theoretic functions. $\varphi(n)$ will denote the EULER function, which can be defined by

$$\varphi(n) := \#\{\, 0 < n' \leq n \,;\ (n, n') = 1 \,\} = n \cdot \prod_{\substack{p \text{ prime} \\ p \mid n}} \left(1 - \frac{1}{p}\right) .$$

A well-known property of these values is that for all $n \in \mathbf{N}_+$

$$\sum_{d \mid n} \varphi(d) = n . \tag{3.1}$$

Let

$$(n)_d := \frac{n!}{(n-d)!}$$

denote the number of d-fold ordered choices out of n elements.

The bifactorial $n!!$ of an integral number n is defined by

$$n!! \quad := \quad \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} n - 2i$$

for $n > 0$ and by convention we set $0!! := 1$, $(-1)!! := 1$ and $n!! = 0$ for $n \leq -2$.

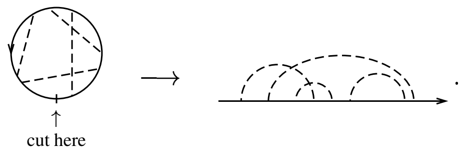
$[P(x)]_d$ will denote the coefficient of x^d in the polynomial (or power series) P in the formal variable x.

By $\lfloor n \rfloor$ we will mean the greatest integer not greater than n.

For two sequences of numbers $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$, the expression $a_i \asymp b_i$ denotes $\lim_{n\to\infty} a_i/b_i = 1$, or, in words, that (a_i) and (b_i) are asymptotically equivalent.

3.2 Linearized chord diagrams

One can obtain a linearized chord diagram (an LCD) from a usual chord diagram by "cutting" the solid line somewhere. Then one has something like this



Both chord diagrams and LCD's are graded by the number of their chords[3], so the picture above is of degree 4.

Let us use the following notations.

$$\begin{array}{rcl} C_D & := & \{ \text{CD's of deg } D \} \\ L_D & := & \{ \text{LCD's of deg } D \} \end{array} \qquad \begin{array}{rcl} \sigma_D & := & \#C_D \\ \lambda_D & := & \#L_D \end{array}$$

A generalization of the LCD's with more than one solid line are the so called *string link* diagrams (for pictures look e. g. in [BN4]). Let

$$L_{D,k} \quad := \quad \{ \text{string link diagrams with } k \text{ strands of deg } D \} \qquad \lambda_{D,k} \quad := \quad \#L_{D,k}$$

The motivation to start these considerations was for me the fact, that the number λ_D of LCD's of deg D can be computed very easily. In fact, it is a simple exercise to show the following

Lemma 3.1

$$\lambda_D \; = \; (2D-1)!!$$

As a generalization of this fact, one can prove the following statement about $\lambda_{D,k}$.

Lemma 3.2

$$\lambda_{D,k} \; = \; \binom{2D+k-1}{2D} (2D-1)!!$$

[3]From the point of view of the following considerations it might appear more natural to take the number of basepoints (i. e. the double number of chords) as graduation but I preferred to keep the original definition of this notion from knot theory.

Hint: Glue all strands into one and place a mark on the point of each gluing.

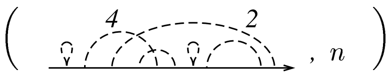
The symmertric group S_{2D} acts on L_D by permuting the order of the base points of the D chords, and in this sense C_D is isomorphic to the orbit space of the cyclic subgroup $\mathbf{Z}_{2D} \subset S_{2D}$ generated by the cycle $z_D := (1\,2\,3\,\ldots 2D)$ on L_D. So, we shall consider the behaviour of LCD's under this action.

Let for $\sigma \in S_{2D}$

$$R_\sigma \quad := \quad \left\{ \text{ LCD's } Y \text{ of deg } D \text{ with } \sigma(Y) = Y \right\} \quad r_\sigma \;:=\; \# R_\sigma$$

3.3 Cyclic CD's and GLCD's

Definition 3.3 *A generalized linearized chord diagram (GLCD) is a pair of the following form*



where $n \in \mathbf{N}_+$ and the first component is something like an LCD, but has the following 2 additional features

- *If n is even, it may contain self-loops*  *, i. e. chords starting and ending onto the same basepoint*
- *Each real chord (a chord which is not a self-loop) is equipped with a number between 0 (in this case we drop the number for convenience) and $n-1$. We will say that it's* coloured *or* labeled *by this number.*

Let the GLCD's be graded by the number of the basepoints (not chords !) and the cyclicity of a GLCD be its second component. So the LCD's are exactly GLCD's with cyclicity 1. Then the above picture has degree 10 and cyclicity n.

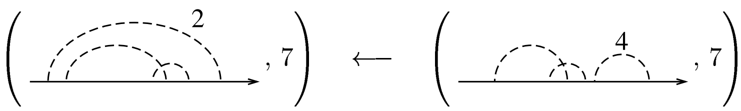
It will be sometimes convenient to drop the cyclicity and take only the first part (what is meant will be clear from the context).

Let

$$\Gamma_{d,c} \quad := \quad \left\{ \text{ GLCD's of deg } d \text{ and cyclicity } c \right\} \quad \text{and} \quad \gamma_{d,c} \quad := \quad \#\Gamma_{d,c}\,.$$

Furthermore, we introduce an action of $\mathbf{Z}_d$ on $\Gamma_{d,c}$ by letting $1 \in \mathbf{Z}_d$ act on a GLCD in the following manner.

- It flips self-loops and real chord ends from the right-most position to the left-most
- Each time it flips one of the ends of a real chord, its number changes from k to $n-1-k$, e. g.



It will turn out as useful to know the cardinality $\gamma_{d,c}$ of $\Gamma_{d,c}$. This is an easy combinatorial task.

If c is odd, then self-loops don't exist and the only non-vanishing case is if d is even. Then we are left with counting LCD's with numbered chords.

If c is even then sum over all possible numbers of  and over the choices to put them between the real chords. Then

$$\gamma_{d,c} = \begin{cases} 0 & 2 \not| c, d \\ c^{\frac{d}{2}}(d-1)!! & 2 \not| c, 2|d \\ \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d}{2i} \cdot c^i \cdot (2i-1)!! & 2|c \end{cases} . \tag{3.2}$$

Let

$$\begin{aligned} L_{D,c} &:= \left\{ \text{LCD's } Y \text{ of deg } D \text{ with } \mathbf{Z}_c \subset \mathrm{stab}_{\mathbf{Z}_{2D}}(Y) \subset \mathbf{Z}_{2D} \right\} \\ C_{D,c} &:= \left\{ \text{CD's } Y \text{ of deg } D \text{ with } \mathbf{Z}_c \subset \mathrm{stab}_{\mathbf{Z}_{2D}}(Y) \subset \mathbf{Z}_{2D} \right\}, \end{aligned}$$

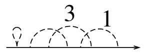
where $C_{D,c}$ counts the chord diagrams obtained by closing up the LCD's counted by $L_{D,c}$. In other words

$$L_{D,c} = R_{z_D^{2D/c}} \quad \text{and} \quad C_{D,c} = R_{z_D^{2D/c}} \Big/ \mathbf{Z}_c .$$

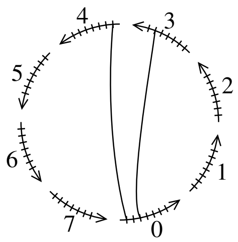
A relation between GLCD's and CD's is given by the following

Theorem 3.4 *There is a bijection*

$$C_{D,c} \quad \overset{\Phi_{D,c}}{\longleftrightarrow} \quad \Gamma_{\frac{2D}{c},c} \Big/ \mathbf{Z}_{\frac{2D}{c}} \quad .$$

Proof. We should best use an example to demonstrate what we are going to do. Given a GLCD, say $\Bigg($  $, n \Bigg)$ construct the corresponding chord diagram in $C_{\frac{7}{2}n,n}$ followingly :

1. Separate an oriented circle into n arcs and mark on each arc 7 basepoints. Number the arcs from 0 to $n-1$ and the basepoints on each arc from 1 to 7.
2. For each real chord in your GLCD and $0 \le m \le n-1$ connect the left end basepoint on the arc m with the right end basepoint on the arc $(m+\mu) \bmod n$, where μ is the number of the chord, e. g. for $n = 8$ in the case of arc 0 and the chord numbered by 3 in the example we use we get the right chord in the following picture



3. For a self-loop, connect the basepoint in arc m with the same basepoint in arc $(m + \frac{n}{2}) \bmod n$, e. g. for arc 0 we get the left chord in the picture above.
4. Glue all the arcs together and remove all markings on them.

Now it is easy to see how to construct the inverse of Φ – separate for a cyclic chord diagram in $C_{D,c}$ the circle into c pieces with $\frac{2D}{c}$ basepoints and assign the unique numbers to the chords in your GLCD, counting the difference between the arc numbers. If chords in your chord diagram start and end on the same position in different arcs (i. e. you obtain a self-loop), then c must be even and the arcs opposite in order the chord diagram to be cyclic. Now check that the action of $\mathbf{Z}_{\frac{2D}{c}}$ factors out exactly the arbitrarity how to choose the splitting of the baseline into arcs. $\square$

3.4 Counting all chord diagrams

Using theorem 3.4 we see that LCD's invariant under $d \in \mathbf{Z}_{2D}$ bijectively correspond to GLCD's with cyclicity $c = 2D/d$. Noticing that an LCD is invariant under $d \in \mathbf{Z}_{2D}$ exactly if it is under $d \cdot l$ where $(l, 2D/d) = 1$, we have

$$r_{z_D^c} = \gamma_{(2D,c),\frac{2D}{(2D,c)}}$$

and by BURNSIDE's lemma on counting orbits of a finite group action we get the following combinatorial expression for σ_D.

Theorem 3.5 *With (3.2) one has*

$$\sigma_D = \frac{1}{2D} \sum_{d \cdot c = 2D} \varphi(c)\, \gamma_{d,c}\,. \tag{3.3}$$

This formula is probably originally due to JEAN BÉTRÉMA [Sl].

3.5 Symmetric chord diagrams

A variation of the enumeration problem is to count chord diagrams up to mirror images (or equivalently, up to change of orientation of the solid line). Let

$$\hat{\sigma}_D := \#\{\text{ CD's of degree } D\ \}\Big/\text{symmetry}$$

and

$$\sigma_D^{\text{sym}} := \#\{\text{ symm. CD's s of degree } D\ \}\,.$$

Then clearly,

$$\hat{\sigma}_D = \frac{\sigma_D + \sigma_D^{\text{sym}}}{2}\,. \tag{3.4}$$

$\hat{\sigma}_D$ can also be computed using Burnside's lemma. In view of (3.4) it is more convenient to give $\hat{\sigma}_D$ in terms of σ_D^{sym}, since the resulting formula for σ_D^{sym} turns out to be surprisingly simple.

Theorem 3.6 *For $D > 0$ we have*

$$\sigma_D^{\text{sym}} = \sum_{i=0}^{\lfloor \frac{D}{2} \rfloor} \frac{(D-1)_{2i}}{i!}\,(D-2i)\,.$$

The resulting formula for $\hat{\sigma}_D$ is originally due to V. Liskovets [Li]. See [S, §4] for discussion of symmetric LCD's.

Proof. We are looking for the orbits of the dihedral group[4]

$$D_{2D} := \langle \omega_D, z_D \rangle \subset S_{2D}\,,$$

where $\omega_D(i) := 2D + 1 - i$, $i \le i \le 2D$. We have

$$\#D_{2D} = \begin{cases} 4D & D > 1 \\ 2 & D = 1 \end{cases}$$

[4] This definition of the dihedral group is in fact orrect only for $D > 1$.

For $D > 1$ we have by Burnside's lemma

$$\hat{\sigma}_D = \frac{1}{4D} \sum_{i=0}^{2D-1} r_{z_D^i} + r_{\omega_D \cdot z_D^i} \, .$$

This is however also true for $D = 1$ (since we count both elements twice and divide by twice the group order).

Then

$$\sigma_D^{\mathrm{sym}} = 2\hat{\sigma}_D - \sigma_D = \frac{1}{2D} \sum_{i=0}^{2D-1} r_{\omega_D \cdot z_D^i}$$

Lemma 3.7 *There is a bijection from $R_{\omega_D \cdot z_D^i}$ to $\Gamma_{D-i\%2,2}$.*

Using this lemma we get

$$\sigma_D^{\mathrm{sym}} = \frac{1}{2}\{\, \gamma_{D,2} + \gamma_{D-1,2} \,\}$$

from which the formula follows by an easy transformation. □

Proof of lemma. The bijection from $\Gamma_{D-i\%2,2}$ to $R_{\omega_D \cdot z_D^i}$ can be described as follows. Note that $\omega_D \cdot z_D^i$ acts as a transposition on $\{1,\dots,i\}$ and $\{i+1,\dots,2D\}$.

Define a map $m_i \,:\, \{1,\dots,D-i\%2\} \,\to\, \mathcal{P}(\{1,\dots,2D\})$ by

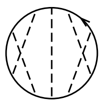
$$m_i(j) := \left\{ \begin{array}{ll} \{j, 1+i-j\} & j \le \lfloor \frac{i}{2} \rfloor \\ \{\lfloor \frac{i+1}{2} \rfloor + j, 1 + 2D + \lfloor \frac{i}{2} \rfloor - j\} & D - i\%2 \ge j > \lfloor \frac{i}{2} \rfloor \end{array} \right\} .$$

For a self-loop at basepoint j make a chord between the two points in $m_i(j)$. For a chord between basepoints j_1 and j_2 connect the four basepoints of $m_i(\{j_1, j_2\})$ in two pairs by connecting $\min(m_i(j_1))$ with $\min(m_i(j_2))$ for a chord labeled by 1 and to $\max(m_i(j_2))$ if the label is 2. Finally, if i is odd, connect basepoints $\lfloor \frac{i+1}{2} \rfloor$ and $\lfloor \frac{i+1}{2} \rfloor + D$.

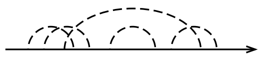
One can check that this procedure indeed describes a bijection. □

3.6 Degenerate CD's and LCD's

Definition 3.8 *Let a chord diagram (or linearized chord diagram) be* degenerate*, if it has an* isolated *chord, i. e. one not crossed by any other, e. g. like in the following diagrams*



or



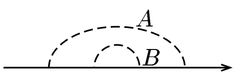
The count of FI relations is the same as the count of degenerate CD's or LCD's. Let

$$\omega_D := \#\{\text{ degenerate CD's of deg } D \,\}\,, \qquad \psi_D := \#\{\text{ degenerate LCD's of deg } D \,\}$$

and $\bar{\omega}_D$ and $\bar{\psi}_D$ the corresponding counts of non-degenerate CD's and LCD's.

For the explanation of our approach it will be helpful to introduce the following appealing notions.

Definition 3.9 *We will say that a chord A of an LCD (or GLCD)* encloses *another chord (or a self-loop) B (or B is* enclosed *by A, or B is* inside *of A), if the endpoints of B are between the endpoints of A.*



Conversely, we will say that B is appearing outside *of A if A does not enclose B (which does* not *mean that B encloses A !). A chord (with a certain property) will be called* minimal *(with this property) if it does not enclose another chord (with the same property). In the same way it will be called* maximal *if there is no other chord enclosing it.*

Another definition we will need later is the following one.

Definition 3.10 *The* length *of a chord A in an LCD is the by 1 augumented number of basepoints of other chords between the two basepoints of A. The length of a chord in a chord diagram will be the minimum of its lengths counted on both circle segments between its endpoints. E.g., the chord diagram in definition 3.8 has 4 chords of length 2 and one of length 5.*

We will start by counting degenerate LCD's of degree D. Applying the inclusion-exclusion principle [Ai, §2.4] and grouping by the number of minimal isolated chords on LCD's we get a recursive formula for ψ_D.

$$\psi_D = \sum_{i=1}^{D}(-1)^{i-1} \sum_{\substack{(j_1,\ldots,j_i,k) \\ j_l,k\geq 0 \\ \sum j_l + k = D-i}} \lambda_{k,i+1} \prod_{l=1}^{i}(\lambda_{j_l}-\psi_{j_l}) \tag{3.5}$$

and $\psi_0 = 0$. Here $\lambda_{k,i}$'s are the numbers of string link diagrams, introduced in lemma 3.2, i is the number of choices of minimal isolated chords, $j_1,\ldots,j_i$ are the degrees of the LCD's enclosed by the i chords, and k is the degree of the remaining string link diagram.

Using the characteristic series $P_{\bar\psi}$ and P_λ defined by

$$P_{\bar\psi}(x) := \sum_{i=0}^{\infty} \bar\psi_i\, x^i$$

and

$$P_\lambda(x,y) := \sum_{i,l=0}^{\infty} x^i y^l\, \lambda_{i,l+1}\,,$$

(3.5) can be rewritten more nicely as

$$P_{\bar\psi}(x) = P_\lambda(x, -x\, P_{\bar\psi}(x))\,.$$

For determining ω_D we have to work a little harder. We will calculate the number $\tilde\gamma_{d,c}$ of GLCD's of degree d and cyclicity c, which produce[5] degenerate chord diagrams.

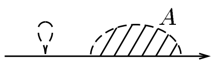
In the case $c = 1$ we have

$$\tilde\gamma_{d,1} := \left\{ \begin{array}{cc} \psi_{d/2} & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\}.$$

So, from now on let $c \geq 2$. We will distinguish 2 cases.

Case 1. All isolated chords in the CD come from a self-loop in the GLCD.

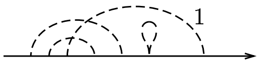
In this case we must have $c = 2$ and exactly one . If we cut the chord diagram just before the chord coming from the self-loop, we get (applying $\Phi_{D,2}^{-1}$ of theorem 3.4) a GLCD which looks like



,

[5]From now on we will always mean this in the sense described in the proof of theorem 3.4.

where A is a non-degenerate LCD[6]. We see that such a GLCD is of odd degree. All other GLCD's producing the same GLCD are generated by the action of $\mathbf{Z}_d$ (described in §3.3) from our special one above. They can be described as follows: Put between the basepoints of a non-degenerate LCD a  and colour the chords by 1 if they enclose the self-loop and by 0 otherwise, e. g.

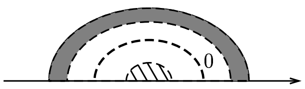


.

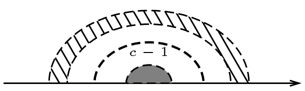
So for odd d there are $d \cdot \bar{\psi}_{(d-1)/2}$ such GLCD's.

Case 2. There is an isolated chord in the chord diagram coming from a chord in the GLCD.

Let $\eta_{c,d}$ be the number of such GLCD's. Take a minimal chord C in the GLCD producing an isolated chord. Then it must be coloured by 0 or $c-1$ (else its c copies would mutually intersect in the chord diagram). If it has label 0 then it only encloses 0-labeled chords, which have to build up a non-degenerate LCD.



If it is labeled $c-1$, we can use the $\mathbf{Z}_d$ action to transform it into a 0-labeled chord. This way we see that outside of a $c-1$-labeled chord there are no self-loops and the chords have a unique labelling: $c-1$ if they enclose C, and 0 otherwise. Furthermore, by forgetting the labels they build up a non-degenerate LCD.



We will count GLCD's of both types by the inclusion-exclusion principle over minimal chords producing isolated chords, that is, we have to count a GLCD with $\geq k$ such chords, so that each GLCD with exactly n chords is counted $\binom{n}{k}$ times.

There are two cases.

Case 2.1. All k chosen minimal chords are labeled by 0. Let $\xi^k_{c,d}$ be the resulting number. We can calculate it by contracting the chords and taking into account the LCD's they enclose. So we have

$$
\begin{aligned}
\xi^k_{c,d} &= \sum_{(e_1,\dots,e_k)\geq 0} \prod_{j=1}^{k} \bar{\psi}_{e_j} \cdot \lambda^c_{d-2k-2\sum e_j,k+1} \\
&= \Big[\big(P_{\bar{\psi}}(x^2)\big)^k \, P_{\lambda^c_{*,k+1}}(x) \Big]_{d-2k} \quad \text{for } 0 \leq 2k \leq d \,,
\end{aligned}
$$

where

$$
\lambda^c_{e,d} \,:=\, \binom{e+d-1}{e} \cdot \gamma_{e,c}
$$

is the number of generalized string link diagrams of d strands, cyclicity c and degree e (with the obvious definition and counted by the same idea as in lemma 3.2) and $P_{\lambda^c_{*,k}}(x)$

[6]From now on a gray filled part in a diagram stands for an *arbitrary* LCD and a shaded part for a *non-degenerate* LCD.

is the characteristic series in x of $\lambda^c_{d,k}$ over the degree[7] d

$$P_{\lambda^c_{*,k}}(x) := \sum_{d=0}^{\infty} x^d \lambda^c_{d,k} .$$

Case 2.2. There are $k-1$ chords with colour 0 and one chord with colour $c-1$. Let $\bar{\xi}^k_{c,d}$ be this number. Such a GLCD we can describe by the GLCD outside of the $c-1$-coloured chord (whose degree we will call e) with a position marked between its basepoints (where the $c-1$-coloured chord and what it encloses is attached) and by the GLCD enclosed by the chord, where $k-1$ chords of colour 0 remain, and which has to be counted as in case 2.1. So

$$\begin{aligned} \bar{\xi}^k_{c,d} &= \sum_{e\geq 0} (2e+1)\,\bar{\psi}_e \cdot \xi^{k-1}_{c,d-(2e+2)} \\ &= \left[\frac{\partial}{\partial x}\left(x \cdot P_{\bar{\psi}}(x^2)\right) \cdot P_{\xi^{k-1}_{c,*}}(x)\right]_{d-2} . \end{aligned}$$

Let

$$\zeta^k_{c,d} := \xi^k_{c,d} + \bar{\xi}^k_{c,d} . \tag{3.6}$$

Now by the inclusion-exclusion principle we get

$$\eta_{c,d} = \sum_{k=1}^{\lfloor d/2 \rfloor} (-1)^{k-1} \zeta^k_{c,d} \tag{3.7}$$

Putting it all together, we find that

$$\begin{aligned} \tilde{\gamma}_{d,2} &= \eta_{2,d} + \left\{ \begin{array}{cc} d\,\bar{\psi}_{(d-1)/2} & \text{if } 2 \not| d \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}_{d,c} &= \eta_{c,d} \text{ for } c > 2 . \end{aligned}$$

Having obtained $\tilde{\gamma}_{d,c}$, we can now apply Burnside's lemma [HC, lemma 14.3 on p. 1058] and get

$$\omega_D = \frac{1}{2D} \sum_{d\cdot c = 2D} \varphi(c)\,\tilde{\gamma}_{d,c} . \tag{3.8}$$

3.7 Chord diagrams with chords of length 1

Let ω^1_D be this number[8]. Determining it is nothing but a slight modification of what we did above.

Following the same strategy, first we compute $\bar{\psi}^1_D$.

We look at an LCD whose closure produces a chord diagram with an isolated minimal chord of length 1. Such an LCD either has such a chord or it has a maximal chord, which is isolated[9]. Once again we apply the inclusion-exclusion principle. For a fixed number k of chords we have again as in subsection 3.6 two cases.

Case 1. All k chosen chords are minimal. By removing them we are left with a string link count.

[7] By P with a subscript containing a '$*$' we will always denote the characteristic series of the expression in the subscript over the variable replacing the '$*$'.

[8] We will henceforth denote the equivalents of the symbols in subsection 3.6 by an additional superscript.

[9] i.e., not enclosed by *any* chord, not only by non-isolated chords!

Case 2. There are $k-1$ minimal chords and one maximal chord. By removing the maximal chord we get back to case 1.

So we get

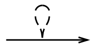
$$\bar{\psi}_D^1 = \sum_{k=0}^{D} (-1)^k (\lambda_{D-k,k+1} + \lambda_{D-k,k}) \,. \tag{3.9}$$

For $D = 1$ the formula gives $\psi_0^1 = -1$, since in

the chord is both of length 1 and maximal and is counted twice. So set $\psi_1^1 := 0$.

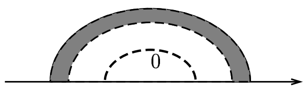
For the computation of $\tilde{\gamma}_{d,c}^1$ for $c \geq 2$ we make the same case distinction as above.

Case 1. There is an isolated chord on the chord diagram coming from a self-loop in the GLCD. This is possible in only one case – $c = 2$, $d = 1$ and the GLCD is

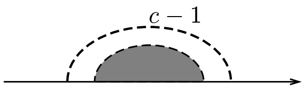
( , 2) .

Case 2. All isolated chords in the chord diagram come from chords in the GLCD.

This enumeration process is as in subsection 3.6, but here we have no enclosed or enclosing LCD's of minimal or maximal chords. The GLCD's look like



and



.

We obtain

$$\begin{aligned} \xi_{c,d}^{k,1} &= \lambda_{d-2k,k+1}^c \text{ for } 0 \leq 2k \leq d \\ \bar{\xi}_{c,d}^{k,1} &= \xi_{c,d-2}^{k-1,1} = \lambda_{d-2k,k}^c \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}_{d,1}^1 &= \left\{ \begin{array}{cc} \bar{\psi}_{d/2}^1 & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}_{d,2}^1 &= \eta_{2,d}^1 + \left\{ \begin{array}{cc} 1 & \text{if } d = 1 \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}_{d,c}^1 &= \eta_{c,d}^1 \text{ for } c > 2 \end{aligned}$$

with the analogous formulas as (3.6) and (3.7) for $\zeta_{c,d}^{k,1}$ and $\eta_{c,d}^1$. The formula for ω_D^1 is then the same as (3.8).

Remark 3.1 The sequence ω_D^1 (and probably also σ_D) was first calculated for $D \leq 9$ without a formula by direct enumeration by D. BAR-NATAN [BN2]. It appeared in the algorithm he uses to compute the dimension of the space of weight systems (see therein the table in section 6.1, 2nd last row).

3.8 Chord diagrams with isolated chords only

We will call such chord diagrams also *fully-degenerate* and will denote their number by ω_D^2.

This enumeration problem and the formula for it are classical. However, I include it here, because we will just see how easily it can be reproduced using our approach.

Let's start once again with the linear case.

It is a classical combinatorial fact, that the number of LCD's of degree D with isolated chords only is the Catalan number

$$\psi_D^2 \,=\, C_D \,:=\, \frac{(2D)!}{D!\,(D+1)!}\,.$$

This number is the number of ways to parenthesize $D+1$ factors in a non-associative algebra or the number of binary planar trees with a basepoint and $D+1$ leaves, as considered in [Lo2, §2.1][10].

To see this, group such LCD's by maximal chords and prove for the generating (or characteristic) series P_C

$$P_C(x) \,=\, \frac{1}{1-xP_C(x)}\,.$$

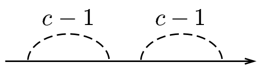
For calculating $\tilde\gamma^2_{d,c}$ for $c \geq 2$ make the following case distinction.

Case 1. There is a self-loop in the GLCD. The argument is analogous to the one in subsection 3.6. We have $c=2$ and d odd, and the count is $d \cdot \psi^2_{(d-1)/2}$.

Case 2. There is no self-loop in the GLCD.

This means that d is even. By the same argument as in subsection 3.6 all chords must be labeled either by 0 or by $c-1$. Furthermore, the GLCD has the following two properties

- a chord of colour 0 encloses only chords of colour 0.
- for each 2 chords of colour $c-1$ one encloses the other, i. e. we never have something like this



.

Then distinguish once again 2 cases.

Case 2.1. All chords are coloured by 0. There are $\psi^2_{d/2}$ such GLCD's.

Case 2.2. There is a chord coloured by $c-1$.

Then by the above properties there exists a *unique* minimal chord coloured by $c-1$ and once given this the colouring of the others is uniquely determined. So such GLCD's correspond to LCD's with a distinguished chord, and their number is $\frac{d}{2}\cdot\psi^2_{d/2}$.

So

$$\eta^2_{c,d} \,=\, \left\{ \begin{array}{cc} \left(\frac{d}{2}+1\right)\psi^2_{d/2} & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\}.$$

[10] These are planar trees with a root vertex of valence 2, internal vertices of valence 3 and leaves of valence 1 modulo isotopies in the plane, which preserve a distinguished order of the 2 edges adjacent to the root vertex.

Hence we get

$$\begin{aligned}
\tilde{\gamma}^2_{d,1} &= \left\{ \begin{array}{cc} \psi^2_{d/2} & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\} \\
\tilde{\gamma}^2_{d,2} &= \eta^2_{2,d} + \left\{ \begin{array}{cc} d\,\psi^2_{(d-1)/2} & \text{if } 2 \not| d \\ 0 & \text{else} \end{array} \right\} \\
\tilde{\gamma}^2_{d,c} &= \eta^2_{c,d} \text{ for } c > 2
\end{aligned}$$

and

$$\begin{aligned}
\omega^2_D &= \frac{1}{2D} \sum_{d \cdot c = 2D} \varphi(c)\, \tilde{\gamma}^2_{d,c} \qquad (3.10) \\
&= \frac{1}{2D} \Big\{ \sum_{\substack{c \cdot d = D \\ c \geq 2}} \varphi(c)\,(d+1)\,C_d \,+\, C_D \Big\} + \left\{ \begin{array}{cc} \frac{1}{2}\,C_{(D-1)/2} & \text{if } 2 \not| D \\ 0 & \text{else} \end{array} \right\},
\end{aligned}$$

which is the classical formula for the number of planar trees with $D + 1$ nodes [Sl]. (If you like, find a direct bijection between the latter set and the set of fully degenerate chord diagrams.)

Remark 3.2 Using similar arguments it should also be possible to count the various kinds of Gauß diagrams (chord diagrams with oriented chords) [Po]. There we have to orient each chord in the GLCD and we have no self-loops.

3.9 Some computations

With the previous formulas it is not hard to compute the beginning of the various integer sequences above[11]. The first 10 values are given in the following table.

D	1	2	3	4	5	6	7	8	9	10
σ_D	1	2	5	18	105	902	9,749	127,072	1,915,951	32,743,182
σ^{sym}_D	1	2	5	16	53	206	817	3,620	16,361	80,218
$\hat{\sigma}_D$	1	2	5	17	79	554	5,283	65,346	966,156	16,411,700
ω_D	1	1	3	11	70	607	6,577	85,198	1,276,563	21,695,178
ω^1_D	1	1	3	11	69	602	6,531	84,737	1,271,143	21,623,667
ω^2_D	1	1	2	3	6	14	34	95	280	854

(If you like, find the two mutually (but not self-) symmetric chord diagrams in degree 4. Which is the only degenerate chord diagram of degree 5 with no chord of length 1?)

3.10 Asymptotics

A first fact to mention is the (not very surprising) observation

Lemma 3.11

$$\sigma_D \asymp \frac{(2D-1)!!}{2D}.$$

[11] A MATHEMATICA™ package doing this is available on my WWW page.

This is, the contribution to σ_D in (3.3) coming from $\gamma_{2D,1} = \lambda_D$ is the dominating one.

Proof. Using the bound

$$\gamma_{d,c} \le \left(1+\sqrt{c}\right)^d (d-1)!!\,,$$

following directly from (3.2), and that the function $\sqrt[n]{1+\sqrt{n}}$ is monotonously decseasing for $n > 0$ we obtain

$$\begin{aligned}\sum_{\substack{c|2D\\ c\ge 2}} \varphi(c)\,\gamma_{2D/c,c} &\le \left(1+\sqrt{2}\right)^D (2D-1)\left(2\left\lfloor\frac{D}{2}\right\rfloor-1\right)!!\\ &= \left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)^D\left(\frac{\sqrt{2}}{D}\right)^{D\%2}(2D-1)\,\frac{D!}{\lfloor\frac{D}{2}\rfloor!}\,.\end{aligned}$$

Therefore

$$\frac{\sum_{c|2D,\,c\ge 2}\varphi(c)\,\gamma_{2D/c,c}}{\gamma_{2D,1}} \le \frac{\left(2+\sqrt{2}\right)^D(2D-1)}{\binom{2D}{D}\left\lfloor\frac{D}{2}\right\rfloor!} \xrightarrow[D\to\infty]{} 0 \qquad \square$$

Something more interesting happens in the case ω_D^1, the number of chord diagrams with chords of length 1. Looking at (3.9) we see that we can write the ratio between λ_D and the k^{th} term in the sum on the r.h.s.

$$\frac{\lambda_{D-k,k+1}+\lambda_{D-k,k}}{\lambda_D} = \frac{1}{k!}\,P(D)\,,$$

where $P(D)$ is a polynomial fraction of degree 0 in D bounded above by 1 and converging to 1 for $D\to\infty$. This means that

$$\frac{\bar{\psi}_D^1}{\lambda_D} \xrightarrow[D\to\infty]{} \frac{1}{e}\,,$$

where e is the Euler number $2.71828\ldots$, and together with lemma 3.11 we get the same result for chord diagrams:

$$\frac{\bar{\omega}_D^1}{\lambda_D} \xrightarrow[D\to\infty]{} \frac{1}{e}\,.$$

Lemma 3.12 *Asymptotically* $\frac{1}{e}$ *of all chord diagrams and LCD's have no isolated chord (or isolated chord of length 1).*

In fact, it is an easy exercise to convince oneself that there are "very few" degenerate chord diagrams with no chord of length 1, that is

$$\frac{\omega_D-\omega_D^1}{\sigma_D} \xrightarrow[D\to\infty]{} 0\,.$$

Unfortunately, computing more carefully the difference

$$\frac{1}{e} - \frac{\bar{\psi}_D^1}{\lambda_D}$$

we see that the dominating term is

$$\frac{1}{2(2D-1)}\,,$$

so we cannot hope for a fast convergence.

Problem. At present I don't know the asymptotics of $\sigma_D^{\rm sym}$.

However, although unimportant for our context, one can obtain the following alternative expression for it.

$$\sigma_D^{\rm sym} \;=\; \frac{d^{D-1}}{dx^{D-1}}\Big((1+x)\,e^{x+x^2}\Big)(0)\,.$$

To see this, look at the nomalized generating series of $\gamma_{*,2}$

$$\tilde P_{\gamma_{*,2}}(x) \;:=\; \sum_{k=0}^{\infty} \frac{\gamma_{k,2}}{k!}\,x^k$$

and prove that $\tilde P_{\gamma_{*,2}}$ is a solution of the differential equation

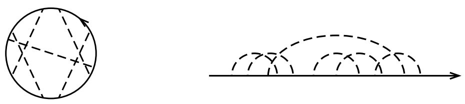
$$xf'(x) \;=\; xf(x) + 2x^2 f(x), \qquad f(0)=1\,.$$

4 Connected and tree-connected chord diagrams

Using the methods of section 3, in this section we discuss the enumeration of two more classes of chord diagrams.

4.1 Connected CD's and LCD's

Definition 4.1 *A CD (or LCD) is called* connected*, if the graph remaining after removing the solid circle line is connected, or equivalently, the intersection graph of the dashed chords is connected.*



Recall the enumeration of degenerate chord diagrams in §3. The present enumeration problem of

$$\omega_D^{\rm nc} := \#\,\{\text{ non-connected CD's of deg } D\ \}\ ,\quad \bar\omega_D^{\rm nc} := \#\,\{\text{ connected CD's of deg } D\ \}$$

is very similar to this[12].

Start with the linear case and look for a formula for

$$\psi_D^{\rm nc} := \#\,\{\text{ non-connected LCD's of deg } D\ \}\ ,\quad \bar\psi_D^{\rm nc} := \#\,\{\text{ connected LCD's of deg } D\ \}\ .$$

Non-connected LCD's have a *virtual* separating arc (i. e., they can be extended by such an arc), which means that the arc is isolated and the enclosed and outside pieces of the LCD (with respect to this arc) are both non-empty.

Apply the inclusion-exclusion principle over the *minimal* virtual separating arcs (i. e., such ones which enclose *connected* LCD's). Note, that such arcs lie beside each other.

The recursive formula you obtain for $\bar\psi_D^{\rm nc}$ is

$$\bar\psi_0^{\rm nc} \;=\; 1$$

[12] All further references to §3 in this section should be understood as references to this enumeration.

and

$$
\begin{aligned}
\bar\psi_D^{\mathrm{nc}} &= \sum_{i=0}^{D} (-1)^i \sum_{\substack{j_1,\dots,j_i,k \\ j_i>0 \\ k\ge 0,\, k>0 \text{ for } i=1 \\ \sum j_i + k = D}} \lambda_{k,i+1} \prod_{l=1}^{i} \bar\psi_{j_l}^{\mathrm{nc}} \\
&= \sum_{i=0}^{D} (-1)^i \left[\left(P_{\bar\psi^{\mathrm{nc}}}(x)-1\right)^i \cdot \left(P_{\lambda_{*,i+1}}(x) - \left\{ \begin{array}{ll} 1 & \text{if } i=1 \\ 0 & \text{else} \end{array} \right\} \right) \right]_{x^D} \quad \text{for } D>0\,,
\end{aligned}
$$

using the characteristic series

$$P_{\bar\psi^{\mathrm{nc}}}(x) := \sum_{i=0}^{\infty} \bar\psi_i^{\mathrm{nc}} x^i\,.$$

Here i is the number of choice of minimal virtual separating chords, $j_1,\dots,j_i$ are the degrees of the LCD's enclosed by the i chords, and k is the degree of the remaining string link diagram. (Recall §3, that string link diagrams arise from LCD's by cutting the solid line.)

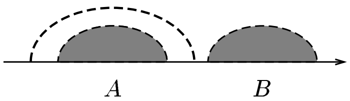
Note, that the condition of the non-emptyness of the enclosing string link diagram is necessary only in the case of 1 virtual separating chord.

The number of non-connected LCD's is then

$$\psi_D^{\mathrm{nc}} = \lambda_D - \bar\psi_D^{\mathrm{nc}}\,.$$

Now look at the number $\tilde\gamma_{d,c}^{\mathrm{nc}}$ of GLCD's of degree d and cycl. $c\ge 2$, producing (in the sense described in §3) non-connected chord diagrams.

Look at the GLCD producing such a chord diagram obtained by cutting the solid line just before one of the endpoints of such an arc[13]. Such a GLCD can be incorporated into the following pattern (where the arc drawn is the virtual one):



,

where B is arbitrary and A has only 0-colored chords[14].

So we need to enumerate such GLCD's as above and the ones resulting from them by the cyclic group action described in §3.

Let us apply as in the case of degenerate chord diagrams the inclusion-exclusion principle on minimal virtual separating chords in the GLCD. As in the case in §3, minimal virtual separating chords can have two *virtual* colorings – 0 (as in the above picture) and $c-1$ (if the action of the cyclic group has reversed the endpoint order; in this case such a chord contains outside of itself a connected GLCD with chords colored by 0 if both endpoints are on the same side of the minimal virtual separating chord and by $c-1$ otherwise).

[13] "before" we mean with respect to the orientation of the solid line.

[14] This is clear in the case where the length of the separating arc is shorter than the degree of the GLCD. In the other case note that the separating arc intersects the one from the next cycle, and smoothing out this intersection we obtain a new separating arc whose length is exactly the degree of the GLCD, so we get back to the picture above with B equal to the empty GLCD.

Let us recompute the numbers $\xi^k_{c,d}$ and $\bar{\xi}^k_{c,d}$ of §3 for our new case. The formulas now look this way:

$$\begin{aligned} \xi^k_{c,d} &= \sum_{(e_1,\dots,e_k)>0} \prod_{j=1}^{k} \bar{\psi}^{\mathrm{nc}}_{e_j} \cdot \lambda^c_{d-2\sum e_j, k+1} \\ &= \left[\left(P_{\bar{\psi}^{\mathrm{nc}}}(x^2) - 1\right)^k P_{\lambda^c_{*,k+1}}(x) \right]_{x^d} \text{ for } 0 \le k \le d \\ \xi^k_{c,d} &= 0 \text{ for } k > d \text{ or } k < 0 \end{aligned} \tag{4.1}$$

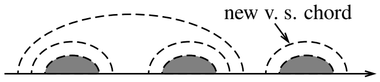
and with

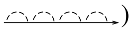
$$P_{\xi^k_{c,*}}(x) := \sum_{d=0}^{\infty} \xi^k_{c,d} x^d$$

$$\begin{aligned} \bar{\xi}^k_{c,d} &= \sum_{e>0} (2e-1)\, \bar{\psi}^{\mathrm{nc}}_e \cdot \xi^{k-1}_{c,d-2e} \\ &= \left[\left(\frac{\partial}{\partial x} \left(x \cdot P_{\bar{\psi}^{\mathrm{nc}}}(x^2) \right) - 2P_{\bar{\psi}^{\mathrm{nc}}}(x^2) + 1 \right) \cdot P_{\xi^{k-1}_{c,*}}(x) \right]_{x^d} . \end{aligned} \tag{4.2}$$

The main differences to the previous case are

1. LCD's enclosed by virtual separating chords are *non-empty*, so $e_j > 0$ in (4.1) and analogously $e > 0$ in (4.2).
2. You get an index translation for the string link diagram in (4.1), since the chords are *virtual*, i. e. they do not really belong to the GLCD.
3. In the case of $\bar{\xi}^k_{c,d}$ you must replace the factor "$(2e+1)$" in §3 by "$(2e-1)$". The reason is the following: If you allow a minimal virtual separating chord of color $c-1$ to contain all basepoints of the LCD outside of it on *one* side only, then you can also draw a minimal virtual separating chord of color 0 around this component.



   This way you obtain diagrams with positions of minimal virtual separating chords which are not incorporated in either counts $\xi^k_{c,d}$ (which counts pictures like ) and $\bar{\xi}^k_{c,d}$ (which counts pictures like ).

   You can avoid this by declaring that you draw a virtual separating chord of color $c-1$ *only* if the component outside of this chord would *enclose* it, i. e., has basepoints on *both* sides of the virtual separating chord (else you draw a virtual separating chord of color 0 around the component).

4. Note, that the non-emptyness of the enclosing string link diagram is not required here.

The rest of the formulas then remain the same as for degenerate chord diagrams.

$$\zeta^k_{c,d} = \xi^k_{c,d} + \bar{\xi}^k_{c,d}$$

$$\eta_{c,d} = \sum_{k=1}^{d} (-1)^{k-1} \zeta^k_{c,d}$$

Then we have

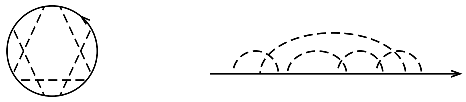
$$\begin{aligned} \tilde{\gamma}^{\mathrm{nc}}_{d,1} &= \psi^{\mathrm{nc}}_{d/2} \text{ for } 2|d \\ \tilde{\gamma}^{\mathrm{nc}}_{d,1} &= 0 \text{ for } 2 \not| d \\ \tilde{\gamma}^{\mathrm{nc}}_{d,c} &= \eta_{c,d} \text{ for } c \geq 2\,, \end{aligned}$$

and finally

$$\omega^{\mathrm{nc}}_D = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\mathrm{nc}}_{2D/c,c}\,.$$

4.2 Tree–connected CD's and LCD's

Definition 4.2 *A chord diagram (or LCD) of degree D is called* tree–connected*, if it is connected and there are exactly $D-1$ chord intersections, i. e., the intersection graph of the dashed chords is a tree.*



This definition appears in a somewhat different form in [NW, §3] and [Be, §2]. See [BG] for a different application of the labelled intersection graph of a LCD.

Remark 4.1 Don't confuse tree–connected chord diagrams with tree–*like* chord diagrams (which are the ones with no chord intersections, i. e. only with isolated chords, and are called in §3 "fully degenerate").

Let

$$\omega^{\mathrm{tc}}_D \quad := \quad \#\,\{\text{ tree–connected CD's of deg } D \text{ }\}\ .$$

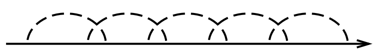
Here the difference to the counting in section 3 is more substantial than in the previous subsection.

Start with the linear case. To enumerate tree–connected LCD's

$$\bar{\psi}^{\mathrm{tc}}_D \quad := \quad \#\,\{\text{ tree–connected LCD's of deg } D \text{ }\}\ ,$$

use the following idea.

Start with the left-most chord of a tree–connected LCD and walk along the chord (in the direction given by the orientation of the solid line) until you intersect a second chord, then walk (in the same direction) along this chord until you get to a third chord, and repeat this way until you walk to the end of the last chord and land back on the solid line. The chords (segments of which) you have passed, form a sub–LCD C which looks like this

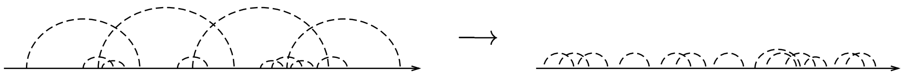


(4.3)

Let us denote for a moment its degree by d. All the remaining chords are associateded at the $2d-2$ inner basepoints of this sub–LCD in such a way that you obtain a (non-empty) tree-connected LCD of lower degree for the fixed basepoint P by applying the following procedure:

1. Call a chord associated to P if it is intersecting the chord ending on P and is enclosed by the chord in (4.3), enclosing P, or if it is intersecting another chord associated to P.
2. For each of the $2d-2$ inner basepoints P build the diagram made up of all chords associated to P and the chord ending on P, putting latter's rightmost basepoint to the left, if the basepoint P was a left basepoint of a chord in (4.3).

Here is an example:



Conversely, from any collection of $2d-2$ non-empty tree-connected LCD's you can reconstruct a tree-conn. LCD by inverting this procedure. The recursive formula for $\bar{\psi}^{\text{tc}}_D$ is then immediate:

$$\begin{aligned} \bar{\psi}^{\text{tc}}_0 &= 1 \\ \bar{\psi}^{\text{tc}}_D &= \sum_{i=1}^{D} \left[(P_{\bar{\psi}^{\text{tc}}}(x) - 1)^{2i-2} \right]_{x^{D+i-2}} \quad \text{for } D > 0\,, \end{aligned} \tag{4.4}$$

with

$$P_{\bar{\psi}^{\text{tc}}}(x) := \sum_{i=0}^{\infty} \bar{\psi}^{\text{tc}}_i x^i\,.$$

Remark 4.2 LEROUX and MILOUDI [LMi, (5.16)] and also DULUCQ and PENAUD [DP, theorem 2.2] found the explicite formula

$$\bar{\psi}^{\text{tc}}_D = \frac{1}{D-1}\binom{3D-3}{D-2}, \qquad D \geq 2\,.$$

In fact, one can derive from (4.4) that

$$g(x) := \frac{P_{\bar{\psi}^{\text{tc}}}(x) - 1}{x}$$

is the solution of the equation (5.15) of [LMi]

$$g(x) = 1 + x g^3(x)\,.$$

Let us now consider cyclicity $c \geq 2$ and count GLCD's producing tree–connected CD's. Let us adopt the convention that we separate the solid line of a chord diagram into $2D$ *equally* long arcs and draw chords always as *straight* lines.

First look at the tree–connected chord diagram (drawn this way) coming from the GLCD. Temporarily remove all chords connecting opposite basepoints (i. e., *self-loop chords*). Then the center of the circle belongs to a component of the complement of the chord diagram.

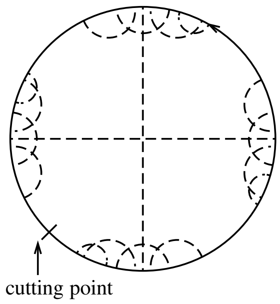
If this component is not bounded by any piece of the solid line then we have an intersection loop of chords and our chord diagram is not tree-connected.

If there is such a piece, then there are at least c of them and the chord diagram has $\geq c$ components. In order for the chord diagram to be connected we must connect all components by reinstalling the self-loop chords. So we need at least 1 self-loop chord. On the other hand there are at most two self-loop chords (since three self-loop chords produce an intersection cycle among themselves).

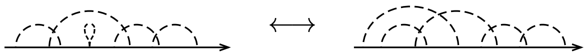
We have two cases.

Case 1. One self-loop . Cyclicity must be 2 (1 self-loop chord) or 4 (2 self-loop chords). Degree must be *odd*.

In this case look at the GLCD produced by cutting the chord diagram at a position which you choose to lie on a piece of the solid line bounding the component of the circle center created after removing the self-loop chords (see example).



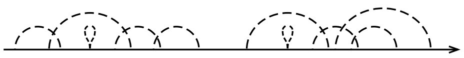
There is a bijection between such GLCD's and tree–connected LCD's by replacing the self-loop by a chord whose left end becomes the leftmost basepoint and whose right end replaces the previous position of the .



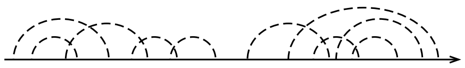
The orbits of the cyclic group action on these GLCD's are free (e. g., since the appears always at different positions if you enumerate all basepoints from left to right), i. e., we get $d\bar{\psi}^{\mathrm{tc}}_{(d+1)/2}$ GLCD's.

Case 2. Two . Cyclicity must be 2. Degree is *even*.

By the procedure of case 1 you obtain a GLCD looking like this:


(4.5)

If you transform the left self-loop into a chord ending on the left–most position and the right self-loop into a chord ending on the right–most position, you get a bijection to diagrams obtained by multiplication of two tree–connected LCD's. E. g., the diagram corresponding to (4.5) is


.

The number of such diagrams is

$$\sum_{\substack{e_1+e_2=d,\\ 2\nmid e_1,e_2}} \bar{\psi}^{\mathrm{tc}}_{(e_1+1)/2}\bar{\psi}^{\mathrm{tc}}_{(e_2+1)/2}\,.$$

Now let the cyclic group $\mathbf{Z}_d$ act on such diagrams and note that each GLCD obtained in this way is obtained always in exactly two ways.

- If first and second component in (4.5) are equal, the action of $\mathbf{Z}_d$ has period $d/2$.
- If first and second component in (4.5) are distinct, $\mathbf{Z}_d$ acts freely, but the orbit is equal to the one of the diagram obtained from (4.5) by swapping the components.

In full completeness, we obtain the following values for $\tilde{\gamma}^{\mathrm{tc}}_{d,c}$ of in our case (compare with §3.7).

$$
\begin{aligned}
\tilde{\gamma}^{\mathrm{tc}}_{d,1} &= \bar{\psi}^{\mathrm{tc}}_{d/2} && \text{for } 2|d \\
\tilde{\gamma}^{\mathrm{tc}}_{d,1} &= 0 && \text{for } 2 \not| d \\
\tilde{\gamma}^{\mathrm{tc}}_{d,2} &= \frac{d}{2} \cdot \sum_{e=1}^{d/2} \bar{\psi}^{\mathrm{tc}}_{e} \bar{\psi}^{\mathrm{tc}}_{d/2+1-e} \\
= \frac{1}{2} \cdot & \left[x \cdot \frac{\partial}{\partial x} \left(\frac{1}{x^2} \cdot \left(P_{\bar{\psi}^{\mathrm{tc}}}(x^2) - 1\right)^2 \right) \right]_{x^d} && \text{for } 2|d \\
\tilde{\gamma}^{\mathrm{tc}}_{d,2} &= d \cdot \bar{\psi}^{\mathrm{tc}}_{(d+1)/2} && \text{for } 2 \not| d \\
\tilde{\gamma}^{\mathrm{tc}}_{d,4} &= d \cdot \bar{\psi}^{\mathrm{tc}}_{(d+1)/2} && \text{for } 2 \not| d \\
\tilde{\gamma}^{\mathrm{tc}}_{d,4} &= 0 && \text{for } 2|d \\
\tilde{\gamma}^{\mathrm{tc}}_{d,c} &= 0 && \text{else}
\end{aligned}
$$

Finally, as always by BURNSIDE's lemma,

$$\omega^{\mathrm{tc}}_D = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\mathrm{tc}}_{2D/c,c} \,.$$

4.3 Some computations

With the previous formulas one can compute the beginning of the various integer sequences above[15]. Here are the first 10 values of each one.

D	1	2	3	4	5	6	7	8	9	10
ω^{tc}_D	1	1	1	2	7	25	108	492	2,431	12,371
ω^{nc}_D	0	1	3	12	74	647	6,961	89,739	1,337,152	22,609,111
$\bar{\omega}^{\mathrm{nc}}_D$	1	1	2	6	31	255	2,788	37,333	578,799	10,134,071

Remark 4.3 By similar arguments as in §3 one can show that

$$\frac{\bar{\omega}^{\mathrm{nc}}_D}{\sigma_D} \xrightarrow[D \to \infty]{} \frac{1}{e} \,.$$

This was first proven for the linear case by STEIN and EVERETT [SE], see also [S, p. 362].

Remark 4.4 The sequence of tree–connected chord diagrams was previously known. According to SLOANE [Sl] it appeared in an enumeration of planar alcohol molecules [LMi] (which correspond to tree–connected chord diagrams by putting on each chord crossing a 'C' and on each chord basepoint an 'H' atom and forgetting the solid line). Leroux and Miloudi used some more general and more elegant PÓLYA theory arguments. Our approach is however geometrically more understandable.

[15] A MATHEMATICA™ package with all formulas is available on my WWW page.

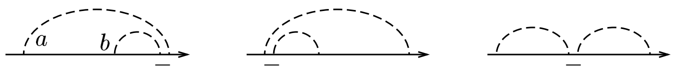
5 Enumeration of chord diagrams and an upper bound for Vassiliev invariants

In this section, we treat an enumeration problem of chord diagrams, which is shown to yield an upper bound for the dimension of the space of Vassiliev invariants for knots. We give an asymptotic estimate for this bound. More precisely, we prove the upper bound $D!$/(any given polynomial in D) for both $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ (where $\mathcal{A}$ and $\mathcal{A}^r$ are the graded algebras of chord diagrams introduced in §1) and later improve it to $D!/1.1^D$. As an aside, we present a trivial proof for the bound $D!$.

In the next section we will discuss further results on the dimension of $\mathcal{A}$.

5.1 Factoring out 4T relations

There are three types of mutual positions of non-intersecting chords in the LCD's with neighboring basepoints.



type I type II type III

It is clear that one can resolve one *fixed* type I and II intervals using the 4T-relation. But this also works with *all* such intervals. This means, we have

Theorem 5.1 *$\mathcal{A}$ is generated by LCD's without type I and II intervals.*

We will call such LCD's *regular*.

Proof. We have to show that we can prform the removing of type I and II intervaals using the 4T relation so that this does not lead to an infinite loop. For this it suffices to define a partial ordering relation on LCD's such that it is always possible to transform an LCD modulo 4T into a linear combination of smaller LCD's.

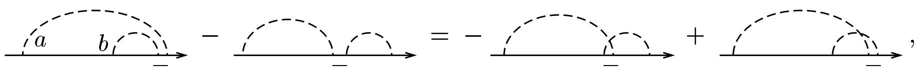
We will use the following relation.

Definition 5.2 *The* length *of a chord in an LCD is the number of enclosed intervals.*

Definition 5.3 *Define a half-ordering relation on LCD's by $a \prec b$ exactly if $(\chi_1(a), -\chi_2(a))$ is lexicographically less than $(\chi_1(b), -\chi_2(b))$, where*

$$\begin{aligned}\chi_1(L) &:= \sum_{i=1}^{D} \textit{len(chord i)}\\ \chi_2(L) &:= \#\ \{\ \textit{chord intersections in } L\ \}\ .\end{aligned}$$

Look at the 4 LCD's of the 4T relation $(4T(a, b))$.



term 1 term 4 term 3 term 2

$(4T(a, b))$

where term 1 has a type I interval. (For the type II interval apply an analogous argument). We have

$$\begin{aligned} \chi_1(\text{term } 4) &< \chi_1(\text{term } 1)\,, \\ \chi_1(\text{term } 2) &= \chi_1(\text{term } 1) \quad \text{ and } \quad \chi_2(\text{term } 2) = \chi_2(\text{term } 1) + 1 \end{aligned}$$

and

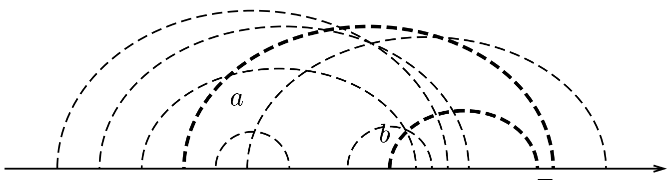
$$\chi_1(\text{term } 3) \ \leq \ \chi_1(\text{term } 1)\,,$$

with inequality unless all basepoints inside chord b are basepoints of left-outgoing arcs with respect to b (i. e., their left basepoint is left from b and their right basepoint is within b).

I. e., each time we can find a and b of type I so that not all basepoints of b belong to left-outgoing arcs (with respect to b) we have a *simplifying* 4T relation. We can also exclude the case len$(b) = 1$, as then term 2 & 3 cancel.

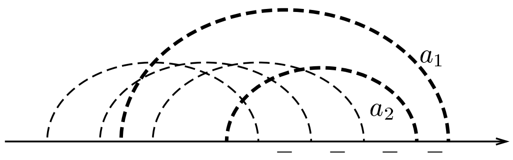
Assume we have a 4T relation to apply but no simplifying one (i.e., each type I looks as in figure 2).

Figure 2. A non-simplifying 4T relation I

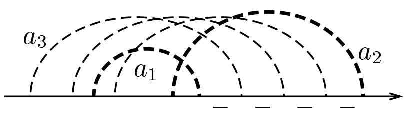


Now look at the left-most type I interval. All arcs having an endpoint within b are left-outgoing with respect to b and all mutually intersect. I. e., we have something like figure 3, which we will call A.

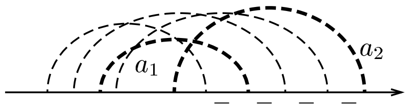
Figure 3. A non-simplifying 4T relation II



Apply in A a 4T relation at a_1, a_2. Term 2 and 4 are simpler as above. Look at term 3, which we will call A_1.



We have $\chi_1(A_1) = \chi_1(A)$. Now the chord a_1 in A_1 has a right-outgoing arc, i. e. 4T(a_1, a_3) is simplifying, and term 3 and 4 have a χ_1 less than $\chi_1(A_1) = \chi_1(A)$ (we will just see that in case there is no a_3 the step we perform is unnecessary). There remains term 2, which we will call A_2.



Using the same argument, repeat sliding the right basepoint of a_1 again to the right (a_2 remains always right-outgoing for a_1) until a_1 intersects all chords intersecting a_2. The resulting LCD has χ_1 equal and χ_2 higher than A, since now a_1 intersects all chords it was intersecting in A, and additionally at least a_2. (Here we would have immediately arrived if we had no a_3). Now we are done. $\square$

From the proof one can see, that one can use only $-\chi_1$ to prove that $\mathcal{A}$ is generated modulo 4T by LCD's without type III.

Now comes the hard combinatorial part – the enumeration of regular linearized chord diagrams.

5.2 Regular linearized chord diagrams

We will number in an LCD the chords by the order (from left to right) of their left basepoints.

Let's start with a simple observation: The number of LCD's of degree D without one of the types I, II or III is $D!$.

To see this, for type I and III place the chords in their reverse order and show that there are exactly $D + 1 - k$ ways to place chord k. For type II apply mirroring of the LCD.

This way we obtain with the observation, that LCD's without type III are generating, a simple proof of the bound $D!$.

Let

$$\Xi_D \quad := \quad \{ \text{ regular LCD's of deg } D \,\} \qquad \xi_D \quad := \quad \#\Xi_D$$

We would like to calculate the numbers ξ_D of regular LCD's of degree D and compare them with Ng's bound $(D-2)!/2$ [Ng].

Unfortunately the enumeration of regular LCD's is much harder. Let us start.

We will consider a certain map

$$m_D \,:\, \Xi_D \longrightarrow \Xi_{D-1}\,.$$

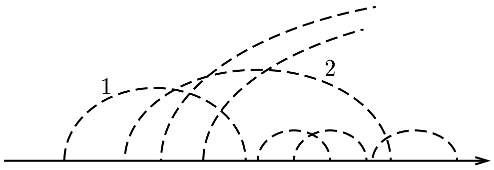
The image of a regular LCD L with D chords under m_D is obtained as follows:

1. Remove the first chord (i. e., the one, whose left basepoint is the leftmost one).

2. The resulting LCD might not be regular. But if not it has exactly one type II interval. Note, that arcs with left basepoint between the left basepoint of chord 2 and the right basepoint of chord 1 are right-outgoing with respect to chord 2, so removing chord 1 we do not get a new type I interval.

3. Now mark all the chords whose left basepoints are between the left basepoint of chord 1 and the first right basepoint of a chord on the right of the right basepoint of

chord 1. E. g., in an LCD beginning like this



the marked chords will be 2 – 6.

4. Separate the marked chords into a left and a right part depending on whether their left basepoint was left or right from the right basepoint of chord 1, e. g. in the above picture the left part would consist of chords 2 – 4, and the right one – of chords 5 and 6. Within both parts we have the property, that each two of the chords intersect, i. e. their right basepoints lie in the same order as the left ones.

5. Now apply a shuffle permutation[16] on the left basepoints of both parts, so that *all* marked chords mutually intersect. In our example this will be the $(3, 2)$-shuffle 34512. The resulting LCD of degree $D - 1$ is regular. This is by definition $m_D(L)$.

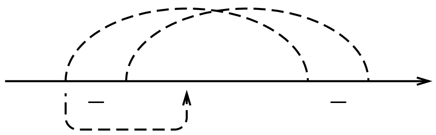
Consider

$$\Xi_{D,k} \quad := \quad \big\{ \text{ regular LCD's of deg } D \text{ with len(chord 1) } = k \,\big\} \qquad \xi_{D,k} \quad := \quad \#\Xi_{D,k}$$

We can construct all preimages under m_{D+1} of $L \in \Xi_{D,k}$ by *re*shuffling the left basepoints of the leftmost k chords with an (α, β)-shuffle (for some $\alpha + \beta = k$) and putting a new $(D + 1)^{\text{st}}$ chord, which becomes in the resulting LCD chord number 1, of length $\alpha + 1$.

Now comes the unpleasant detail. Not for all LCD's in $\Xi_{D,k}$ and all (α, β)-shuffles this procedure gives a regular LCD. The problem is that the reshuffling can produce type I intervals, if both the left and right basepoints of two marked arcs were neighbored in L. In this case it will be forbidden to permute the left basepoint of the lefter arc to the right of the left basepoint of its right neighbor (see figure 4).

Figure 4. A forbidden move.



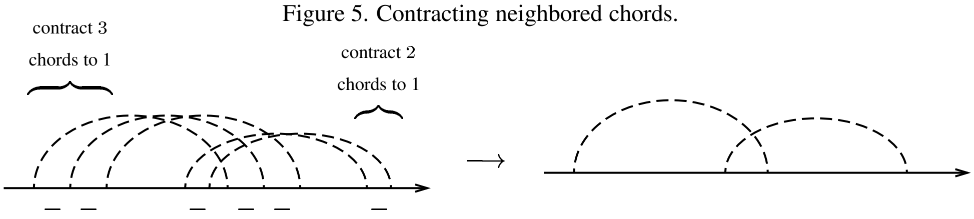
Define between the first k arcs in L a (de)composition (a representation as sum of ordered numbers) by setting chords l and $l + 1$ to belong to the same part iff their right basepoints are also neighbored, i. e. if no right basepoints are neighbored we get the decomposition of k into k parts of length 1, whereas in the case that all right basepoints are neighbored we get the decomposition of k into 1 single part of length k. We will call this decomposition the *induced (de)composition* of L.

Definition 5.4 *A composition P' is* rougher *than P if it can be obtained from P by a sequence of additions of neighbored parts of P. Conversely, P' is* finer *than P. E. g., the composition* $(1, 2, 1, 1, 1, 3, 2, 1)$ *of 12 is finer than* $(1, 2{+}1, 1, 1{+}3{+}2, 1) = (1, 3, 1, 6, 1)$. *The length len$(P)$ of a composition P is the number of numbers appeing in it.*

[16]For the definition and algebraic properties of shuffles see e. g. [St3, Lo, Re, BN].

For a fixed decomposition P of k the number $\hat{\xi}_{D,k,l}$ of diagrams in $\Xi_{D,k}$ whose induced decomposition is exactly P, depends only on $l = \text{len}(P)$ and can be computed using the combinatorial inclusion-exclusion principle and the simple idea, that each diagram T in $\Xi_{D,k}$ with induced decomposition rougher than one fixed decomposition P' of k of length l bijectively corresponds to a diagram in $\Xi_{D+l-k,l}$ by contracting chords in T belonging to the same component of the decomposition to one (see figure 5).

Figure 5. Contracting neighbored chords.



The result is

$$\hat{\xi}_{D,k,l} = 0 \text{ for } l \leq 0 \text{ or } l > k$$
$$\hat{\xi}_{D,k,l} = \sum_{i=0}^{l-1} \binom{l-1}{i} (-1)^i \xi_{D+l-k-i,l-i} ,$$

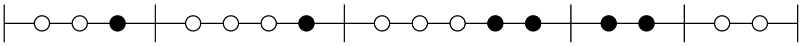
where D is the degree, k is the length of chord 1 and l is the length of the induced decomposition.

Now we determine the number of applicable shuffles for a fixed $L \in \Xi_{D,k}$. Note that to have a preimage of L in $\Xi_{D+1,k'}$ we must have $k \geq k' - 1$ (since the new to be installed chord 1 must intersect the previous chord 1). In each such case we apply the inverse of a $(k'-1, k-k'+1)$-shuffle on the first k basepoints of L. A $(k'-1, k-k'+1)$-shuffle corresponds to a coloring of k chords by $k - k' + 1$ black (to be reshuffled to the right) and $k' - 1$ white (to be reshuffled to the left) chords. The restriction is that within each component of the induced decomposition there must be no black chord followed to the right by a white one. Since the number of LCD's with induced composition P depends only on $p = \text{len}(P)$, let us sum over all compositions of length p and compute the number

$$\eta_{l,k,p} := \# \left\{ \begin{array}{l} \text{choices of } k \text{ out of a decomposition } P \text{ of } l \\ \text{elements of length } p\text{, such that in each com-} \\ \text{ponent of } P \text{ all chosen elements are to the} \\ \text{left} \end{array} \right\} .$$

Note that in $\eta_{l,k,p}$ a fixed choice is counted sometimes multiple times for different decompositions!

Here is an example (where chords are replaced for simplicity by beads):



$$(l = 16,\ P = 3 + 4 + 5 + 2 + 2,\ p = 5,\ k = 10)$$

We see that such a picture can be described uniquely by the following procedure:

1. Choose l' components out of p, where we will have at least one white mark.
2. Build a composition of k of length l' with the white marks and a composition with zeros of the same length (of some number d) with the black marks appearing in the l' chosen components.

3. Join the compositions of the white and black marks, i. e. add parts at the same position.
4. Complete the composition of l by a composition of $l-k-d$ of $l-l'$ parts.

In the above example

- The composition of $k=10$ is $2+3+3+2$,
- the choice of $l'=4$ components out of $p=5$ is $\{1,2,3,5\}$,
- the composition with zeros of $d=4$ is $1+1+2+0$,
- the composition of $l-k-d=16-10-4=2$ of length $l-l'=5-4=1$ parts is 2.

The number $P_{k,l}$ of compositions of k elements into l parts is

$$P_{k,l}=\binom{k-1}{l-1}.$$

It makes sense to set

$$P_{0,0}:=1 \quad \text{and} \quad P_{0,\cdot}:=0 \ \text{ else}.$$

The number of compositions with zeros of k elements into l parts is $P_{k+l,l}$.

So we get

$$\eta_{l,k,p}=\sum_{l'=0}^{k}\sum_{d=0}^{l-k}P_{k,l'}\cdot P_{d+l',l'}\cdot P_{l-k-d,p-l'}\binom{p}{l'}.$$

Remark 5.1 Using some combinatorial formulas the expression for the $\eta_{l,k,p}$'s can be a little bit simplified:

$$\eta_{l,k,p}\;=\;\begin{cases}P_{l,p} & k=0\\ \displaystyle\sum_{l'=0}^{p}\binom{k-1}{l'-1}\binom{l-k+l'-1}{p-1}\binom{p}{l'} & k>0\end{cases}.$$

With some effort one can find an alternative characterization of these numbers (which is unimprtant in our context).

$$\eta_{l,k,p}=\left[\left(\sum_{n=1}^{l}\frac{1-x^{n+1}}{1-x}\,y^n\right)^p\right]_{x^k\cdot y^l},$$

where $[\text{polynomial}]_{\text{monomial}}$ denotes the coefficient of 'monomial' in 'polynomial', and we obtain

$$\sum_{l,k,p=0}^{\infty}\eta_{l,k,p}\,x^k\,y^l\,z^p\;=\;f(x,y,z)\;:=\;\frac{1-x}{1-x-z\left(\dfrac{y}{1-y}\;-\;\dfrac{x^2y}{1-xy}\right)},$$

i. e., the numbers $\eta_{l,k,p}$ appear as the TAYLOR coefficients of $f(x,y,z)$ at $(0,0,0)$.

With the previous preparations we are ready to write down the recursive formula for $\xi_{D,k}$

$$\xi_{1,1}=1\,,$$

$$\xi_{D,k}=\sum_{l=k-1}^{D-1}\sum_{p=1}^{l}\hat{\xi}_{D-1,l,p}\eta_{l,k-1,p}\,. \tag{5.1}$$

Finally, what we seek is

$$\xi_D = \sum_{k=1}^{D} \xi_{D,k} \,.$$

5.3 Connected regular LCD's

With some more effort one can attack a further combinatorial task to give a way to compute the number of *connected* regular LCD's. By the facts recalled in section 1 this gives a bound for the primitive (additive) VI's.

The first observation we make is the following: if in a regular LCD there is one connected component enclosing another one, then the latter is attached at a type III interval to the first one.

So first we shall count regular LCD's by their number of type III intervals. For the later calculation it will turn out more useful to use the *characteristics*

$$\chi(L) \;:=\; \#\{\text{ type III intervals of } L\,\} \,+\, 1$$

instead. So let

$$\begin{aligned}
\Xi^2_{D,n} &:= \{\text{ regular LCD's } L \text{ of deg } D \text{ with } \chi(L)=n\,\} & \xi^2_{D,n} &:= \#\Xi^2_{D,n}\\
\Xi^2_{D,k,n} &:= \left\{\begin{array}{c}\text{regular LCD's } L \text{ of deg } D \text{ with}\\ \text{len(chord 1) } = k \text{ and } \chi(L)=n\end{array}\right\} & \xi^2_{D,k,n} &:= \#\Xi^2_{D,k,n}
\end{aligned}$$

The counting ia analogous to that of $\xi_D, \xi_{D,k}$ once one has noticed that the reattachment of chord 1 done in the construction of the various preimages in $\Xi_{D+1,k'}$ of a regular LCD $L \in \Xi_{D,k}$ under m_{D+1} increases the number of type III intervals of the LCD by 1 unless in the case $k'-1=k$.

Then the formulas for $\xi^2_{D,k,n}$ follow directly from those of $\xi_{D,k}$.

We obtain

$$\xi^2_{D,k,n} = 0 \text{ for } k \le 0 \text{ or } k > D \text{ or } n < 0$$

and else

$$\xi^2_{D,k,n} \;=\; \sum_{l=k}^{D-1} \sum_{p=1}^{l} \hat\xi^2_{D-1,l,p,n-1}\eta_{l,k-1,p} \;+\xi^2_{D-1,k-1,n}\,,$$

where

$$\begin{aligned}
\hat\xi^2_{D,k,l,n} &= 0 \text{ for } l \le 0 \text{ or } l > k \text{ or } n < 0, \text{ and else}\\
\hat\xi^2_{D,k,l,n} &= \sum_{i=0}^{l-1} \binom{l-1}{i} (-1)^i \xi^2_{D+l-k-i,l-i,n}
\end{aligned}$$

with $\hat\xi_{D,k,l,n}$ being explained similarly to $\hat\xi_{D,k,l}$.

In order to make the recursive computation start correctly, it makes sense to set for $D=0$

$$\begin{aligned}
\xi^2_{0,0,1} &:= 1 \quad\text{and}\\
\xi^2_{0,k,n} &:= 0 \text{ for } (k,n) \neq (0,1)\,.
\end{aligned}$$

Finally,

$$\xi^2_{D,n} \;=\; \sum_{k=0}^{D} \xi^2_{D,k,n} \;.$$

Now consider

$$\Lambda_{D,n,l} \quad := \quad \left\{ \begin{array}{c} \text{regular LCD's } L \text{ of deg } D \text{ with } l \text{ conn. comp.} \\ \text{and } \chi(L) = n \end{array} \right\} \qquad \lambda_{D,n,l} \quad := \quad \#\Lambda_{D,n,l}$$

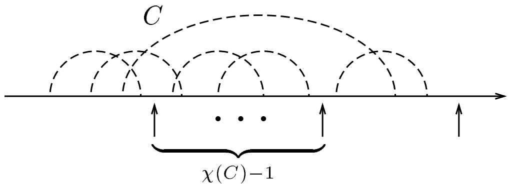
Let

$$\lambda^{\mathrm{c}}_{D,n} \;:=\; \lambda_{D,n,1} \,.$$

For the calculation of these numbers we will proceed followingly. First we have

$$\lambda_{D,n,1} \;=\; \xi^2_{D,n} - \sum_{l=2}^{D} \lambda_{D,n,l} \;.$$

For the non-connected case look at the following picture



Choose the connected component C of L containing chord 1. Then all remaining components are attached either to the right of C or at type III intervals within C, i. e. we have exactly $\chi(C)$ positions of attachment. From this we find the following recursive formula

$$\lambda_{D,n,l} \;=\; \sum_{d'=1}^{D-1} \; \sum_{n'=1}^{\min(d',n)} \; \lambda^{\mathrm{c}}_{d',n'} \cdot \Big[P_\lambda(x,y,z)^{n'} \Big]_{x^{D-d'} y^{l-1} z^{n-n'}} \qquad \text{for } l > 1 \tag{5.2}$$

with

$$P_\lambda(x,y,z) \;:=\; \sum_{D=0}^{\infty} \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \; \lambda_{D,n,l} \cdot x^D \cdot y^l \cdot z^n \;.$$

Here d' and n' are the degree and characteristics of C. We use for this formula the observation that attaching (where ever!) an LCD with $\chi = k$ the total number of type III intervals is augumented exactly by k. There is however one important exception – the attachment of the empty LCD does not augument this number. So it makes sense to set for $D = 0$

$$\lambda_{0,0,0} := 1 \qquad \text{and} \qquad \lambda_{0,n,l} := 0 \text{ for } (n,l) \neq (0,0)$$

in order for the recursive formula (5.2) to work correctly.

The number of connected regular LCD's of degree D is then

$$\lambda^{\mathrm{c}}_{D} \;=\; \sum_{n=0}^{D} \; \lambda^{\mathrm{c}}_{D,n} \,.$$

5.4 Numerical and asymptotical results

After all the considerations it is now possible to calculate ξ_D. Let us compare it with the bound $(D-2)!/2$ obtained by Ng [Ng] for $D \leq 20$ (see table 1)[17].

[17] A MATHEMATICA™ package with all formulas will soon be available on my web page.

We see that ξ_D does not grow as fast as $(D-2)!/2$ and our bound is better for $D \geq 18$. It appears obvious that this is true in general. At the end of this section we will give a (non-straight forward) proof for this fact.

Using the bound λ^c_D for the primitive VI's, we get a sharper bound β_D for $\mathcal{A}$ and β^r_D for $\mathcal{A}^r$ than ξ_D by summing over partitions of the degree of products of various λ^c_D's and omitting symmetry in factors of equal degree[18]. But the situation with asymptotical behaviour is even more complicated. However,

Remark 5.2 It is very likely that

$$\frac{\lambda^c_D}{\xi_D} \quad \xrightarrow[D\to\infty]{} \quad \frac{1}{e},$$

where e is the EULER number $2.71828\ldots$ (although a proof appears very messy). This limit appears in some other enumeration problems of chord diagrams, see [St2, St9]. (So, I don't expect a big qualitative improvement of the bound in this way.)

Remark 5.3 The computation of λ^c_D is the hardest one (λ^c_{30} took ≈ 30 h total time with C++ on a Sparc10). For $D \leq 14$ the bound is still worse than Ng's, but for $D = 15$ we already have an improvement in both the framed and non-framed case.

The following table 1 compares our numbers with Ng's bound χ_D for primitive VI in [Ng, theorem 4.2] and the resulting bounds α_D and α^r_D for $\mathcal{A}$ and $\mathcal{A}^r$.

Here is a further challenge:

Problem. Is $\mathcal{A}$ generated by diagrams which do not posess any chord exclosing another one?

Note that for such diagrams the recursion formula (5.1) would change to

$$\bar{\xi}_{D,k} \;=\; \sum_{l\geq k-1} \bar{\xi}_{D-1,l}$$

(since there is only one way to (re)shuffle, i. e. the identity), and one easily shows by induction

$$\bar{\xi}_{D,k} \;\leq\; \binom{2D-k}{D}$$

and hence

$$\bar{\xi}_D \;\leq\; \binom{2D+1}{D+1} \;<\; 2^{2D+1},$$

i. e., we would have an *exponential* bound.

Conversely, it is possible to show that ξ_D grows stronger than any exponential.

Theorem 5.5 *We have*

$$\xi_D \;\geq\; B(D),$$

where the $B(D)$'s are the BELL *numbers [Ri], satisfying the recursive formula*

$$B(0) = 1 \quad \textit{and} \quad B(d) := \sum_{k=1}^{d} \binom{d-1}{k-1} B(d-k).$$

[18]This relation is extensively treated in [St10].

Table 1. Numerical values for the various bounds.

D	$(D-2)!/2$	χ_D	α^r_D	α_D	ξ_D	λ^c_D	β^r_D	β_D
1	0	1	0	1	1	1	0	1
2	$1/2$	1	1	2	2	1	1	2
3	$1/2$	1	1	3	5	2	2	4
4	1	2	3	6	15	5	6	10
5	3	4	5	11	53	16	18	28
6	12	14	18	29	217	63	72	100
7	60	54	61	90	$1,014$	293	321	421
8	360	332	357	447	$5,335$	$1,561$	$1,680$	$2,101$
9	$2,520$	$2,246$	$2,330$	$2,777$	$31,240$	$9,321$	$9,852$	$11,953$
10	$20,160$	$18,264$	$18,715$	$21,492$	$201,608$	$61,436$	$64,168$	$76,121$
11	$181,440$	$164,950$	$167,783$	$189,275$	$1,422,074$	$442,134$	$457,659$	$533,780$
12	$1,814,400$	$1,664,354$	$1,686,327$	$1,875,602$	$10,886,503$	$3,446,077$	$3,543,785$	$4,077,565$
13	$19,958,400$	$18,423,144$	$18,615,873$	$20,491,475$	$89,903,100$	$28,905,485$	$29,578,097$	$33,655,662$
14	$239,500,800$	$222,406,776$	$224,310,263$	$244,801,738$	$796,713,190$	$259,585,900$	$264,623,046$	$298,278,708$
15	$3,113,510,400$	$2,905,943,328$	$2,926,679,435$	$3,171,481,173$	$7,541,889,195$	$2,485,120,780$	$2,525,897,073$	$2,824,175,781$
16	$43,589,145,600$	$40,865,005,494$	$41,112,186,396$	$44,283,667,569$	$75,955,177,642$	$25,267,283,367$	$25,622,275,565$	$28,446,451,346$
17	$653,837,184,000$	$615,376,173,184$	$618,573,033,369$	$662,856,700,938$	$810,925,547,354$	$271,949,606,805$	$275,257,714,175$	$303,704,165,521$
18	$1.046,139,4 \cdot 10^{13}$	$9.880,209,2 \cdot 10^{12}$	$9.924,788,8 \cdot 10^{12}$	$1.058,764,5 \cdot 10^{13}$	$9.148,832,1 \cdot 10^{12}$	$3.089,330,1 \cdot 10^{12}$	$3.122,190,8 \cdot 10^{12}$	$3.425,895,0 \cdot 10^{12}$
19	$1.778,437,1 \cdot 10^{14}$	$1.684,835,1 \cdot 10^{14}$	$1.691,501,2 \cdot 10^{14}$	$1.797,377,7 \cdot 10^{14}$	$1.087,597,5 \cdot 10^{14}$	$3.694,347,7 \cdot 10^{13}$	$3.729,011,4 \cdot 10^{13}$	$4.071,600,9 \cdot 10^{13}$
20	$3.201,186,8 \cdot 10^{15}$	$3.041,127,5 \cdot 10^{15}$	$3.051,766,5 \cdot 10^{15}$	$3.231,504,3 \cdot 10^{15}$	$1.358,836,1 \cdot 10^{15}$	$4.639,430,0 \cdot 10^{14}$	$4.678,128,6 \cdot 10^{14}$	$5.085,288,7 \cdot 10^{14}$
25	$1.292,600,8 \cdot 10^{22}$	$1.240,896,8 \cdot 10^{22}$	$1.243,475,7 \cdot 10^{22}$	$1.300,029,2 \cdot 10^{22}$	$8.090,291,0 \cdot 10^{20}$	$2.812,171,4 \cdot 10^{20}$	$2.826,415,8 \cdot 10^{20}$	$3.021,075,6 \cdot 10^{20}$
30	$1.524,441,7 \cdot 10^{29}$	$1.473,626,9 \cdot 10^{29}$	$1.475,655,8 \cdot 10^{29}$	$1.530,326,3 \cdot 10^{29}$	$1.263,341,6 \cdot 10^{27}$	$4.439,023,6 \cdot 10^{26}$	$4.454,162,5 \cdot 10^{26}$	$4.707,888,4 \cdot 10^{26}$

It is possible to show using the recursive formula of the Bell numbers that they grow stronger than any exponential. E. g., since

$$B(d) \geq (d-1)\, B(d-2)$$

we have

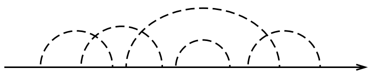
$$B(d) \geq (d-1)!!\,.$$

More generally one has asymptotically $B(d) > (d!)^{1-\epsilon}$ for each $\epsilon > 0$.

Theorem 5.5 follows from the following

Theorem 5.6 *The Bell number $B(D)$ is the number of regular LCD's L of degree D with the following property:*
Consider the sequence $L_0 = L, L_1, L_2, \ldots$ of LCD's, where L_{i+1} is obtained from L_i by removing its first chord and all chords intersecting or enclosed by it. Then all L_i are regular.

Not all regular LCD's have this property. The most simple example of an LCD without this property is



Proof. Denote by d the degree of an LCD with the above property. If k is the number of marked chords then the right endpoints of the last $k-1$ marked chords can be attached at arbitrary positions in the remaining regular LCD of degree $d-k$ except at intervals whose left end is a right basepoint of one of the $d-k$ non-marked chords. So there are $d-k+1$ positions of attachment (the position of the right end of chord 1 is determined – left from all non-marked chords). Since the order of the right basepoints of the marked chords is equal to the order of their left basepoints and at an interval you can put more than one new right basepoint, we have to count ordered choices with repetitions of $k-1$ out of $d-k+1$ elements. Since this number is

$$\binom{(d-k+1)+(k-1)-1}{k-1} = \binom{d-1}{k-1},$$

in this way we obtain exactly the recursion formula for the Bell numbers. □

Here is an upper bound for ξ_D which shows the improvement we have achieved.

Theorem 5.7 *ξ_D grows more slowly than $D!$/any polynomial in D.*

Proof. Fix a k with $1 \leq k \leq D$ and consider a regular LCD L of degree D. This means especially that L has no type I interval at all and no type II intervals among the first k intervals (i. e., between the $k+1$ leftmost basepoints). Make the following case distinction for L.

Case 1. Among the first $k+1$ chords there exist two consecutive chords with neighbored right basepoints. The number of such LCD's L is bounded by the contraction described in section 5.2 (and explained in figure 5) by[19] $k\xi_{D-1}$.

Case 2. There exist two chords which have both their left and right basepoints among the first $k+1$. Then chord 1 is one of these chords. Since, applying m_D, we shuffle maximally k basepoints and the number of shuffles with k elements is 2^k, we obtain each LCD in Ξ_{D-1} maximally 2^k times, i. e., the number of such LCD's L is bounded by $2^k\xi_{D-1}$.

[19] Here is the only time we use the non-existence of *any* type II intervals in L; else we will use their non-existence only among the first k intervals.

Case 3. There exists no chord with both left and right basepoint among the first $k+1$ (i. e., all are left and all chords mutually intersect). In this case the symmetric group S_{k+1} acts freely on the first $k+1$ basepoints and all LCD's obtained have (by exclusion of case 1) no type I intervals. Since the number of LCD's with no type I is $D!$, the number of these LCD's L is bounded by $D!/(k+1)!$.

Case 4. There exists exactly one chord with right basepoint among the first $k+1$. Then this chord has to be chord 1. The right basepoint of chord 1 separates the left basepoint of the $k-1$ remaining chords starting at the first $k+1$ basepoints into two parts, where within each part all chords mutually intersect. Now apply the argument in case 3 for $D-1$ instead of D and each of the two parts. For $\operatorname{len}(\text{ chord } 1) = l$ you get the bound

$$\frac{(D-1)!}{(l-1)!\,(k-l)!}\,.$$

Summing over l you get

$$\frac{(D-1)!}{(k-1)!}\,2^{k-1}$$

as a bound for this case.

Putting it all together, we find that

$$\xi_D \;\leq\; (k+2^k)\,\xi_{D-1} \;+\; \frac{D!}{k!}\,\left(2^{k-1}+1\right) \tag{5.3}$$

for each $1 \leq k \leq D$.

Fix some polynomial $P(D)$ and assume that we find arbitrary larde D with

$$\xi_D \;\geq\; \frac{D!}{P(D)} \tag{5.4}$$

Choose k_D minimal with $k_D! > 2(2^{k_D-1}+1)P(D)$. This k_D grows very slowly (more slowly than $\log_C D$ for any $C > 1$). Then

$$\xi_D \;\leq\; (k_D+2^{k_D})\,\xi_{D-1} \;+\; \frac{1}{2}\xi_D$$

and

$$\xi_D \;\leq\; 2(k_D+2^{k_D})\,\xi_{D-1}\,. \tag{5.5}$$

Note, that the expression $k_D + 2^{k_D}$ grows less fast than any positive power of D. Furthermore, we do *not* use (5.4) for $D-1$ instead of D. This will be crucial later.

To finish the proof we consider two cases.

Case 1. For almost all D we have (5.4). By (5.5) you get for sufficiently large D, say,

$$\frac{\xi_D}{\xi_{D-1}} \;<\; \frac{D}{2}\,,$$

which gives an asymptotical contradiction.

Case 2. There exists a sequence $\{D_i\}$ such that

$$\xi_{D_i} \;\leq\; \frac{D_i!}{P(D_i)}\,.$$

W.l.o.g. you can choose $\{D_i\}$ so that additionally

$$\xi_{D_i+1} \;\geq\; \frac{(D_i+1)!}{P(D_i+1)}$$

(if this is not possible, you are done). Then by (5.5) you get once again for sufficiently large i

$$\xi_{D_i+1} \leq 2\left(k_{D_i+1} + 2^{k_{D_i+1}}\right) \xi_{D_i} .$$

But by the choice of $\{D_i\}$ you get on the other hand for each $\epsilon > 0$

$$\frac{\xi_{D_i+1}}{\xi_{D_i}} > (D_i+1)(1-\epsilon)$$

for sufficiently large i and in this way a contradiction. □

Remark 5.4 The maximal result we can achieve with our proof is a little better. Fix some $\epsilon' > 0$. Then

$$\xi_D \leq \delta_D := 2\,\frac{D!}{k_D!}\left(2^{k_D-1}+1\right)$$

with

$$k_D = \lfloor \log_2 D - \epsilon - 1 \rfloor$$

The proof works as above using

$$\frac{\delta_{D+1}}{(D+1)\,\delta_D} \xrightarrow[D\to\infty]{} 1$$

and

$$2\left(k_D + 2^{k_D}\right) < D(1-\epsilon')$$

for some $\epsilon' > 0$ and sufficiently high D.

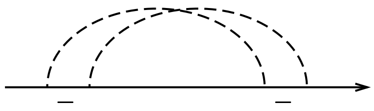
5.5 A further improvement

It appears that our asymptotical result is very weak. The numerical computation suggests as an estimate something like $D!/1.5^D$. I did not manage to obtain this, but achieved a qualitatively similar result.

Theorem 5.8 *ξ_D is bounded by $D!$ over an exponential in D.*

The basic idea for our further improvement is to consider *reduced* regular LCD's.

Definition 5.9 *An LCD is called* reduced *if there is no pair of contractable chords (that is, intersecting chords with both left and right basepoints neighbored).*



Recall the STIRLING formula

$$n! \asymp \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad n\to\infty\,,$$

basically saying that asymptotically $n! \asymp \left(\frac{n}{e}\right)^n$ modulo a polynomial in n.

Definition 5.10 *Define the* path *of an LCD L of degree D as the sequence of pairs $\{(i,l_i)\}_{i=1}^{D}$, where l_i is the length of chord 1 in L_i, and L_i is obtained from L by removing chords $1,\dots,D-i$.*

If L is regular we have by the previous discussion $l_{i+1} \leq l_i + 1$. Call

$$\alpha(L) := \#\{\, i \,:\, l_{i+1} = l_i + 1 \,\}$$

the *ascend* of L.

Recall the map m_D, which mapped regular LCD's of degree D to those of degree $D-1$ by removing chord 1 and reshuffling the leftmost left basepoints. We carefully studied the number of applicable shuffles for each image under m_D. If we simply take *all* possible shuffles, we obtain the inequality

$$\xi_{D,k} \;\leq\; \sum_{l \geq k-1} \binom{l}{k-1} \xi_{D-1,l}\,. \tag{5.6}$$

Applied this way, this does not give much because it is an expression of forgetting the type II condition, and therefore we get $D!$ as bound. In fact, the numbers satisfying the equality in (5.6) are the STIRLING numbers (see [Ri]).

So we will use this formula with a certain constraint.

Proof of theorem 5.8. Fix some $0 \leq \delta \leq 1$ and distinguish two cases.

Case 1. Consider first LCD's L with $\alpha(L) < \delta D$. Let $k = \text{len}(\text{ chord } 1\text{ })$ in L. Call $\xi^{\delta}_{D,k}$ the number of such LCD's. Expanding (5.6) recursively we obtain

$$\xi^{\delta}_{D,k} \;\leq\; \sum_{\substack{\text{paths } \{(i,k_i)\}_{i=1}^{D} \text{ of} \\ L \text{ with } \alpha(L) < \delta D}} \;\prod_{j=2}^{D} \binom{k_{j-1}}{k_j - 1}\,.$$

We always have $k_1 = 1$ and $k_D = k$.

Expanding the binomial coefficients we obtain for the above product

$$\prod_{j=2}^{D} \frac{(k_{j-1})!\,k_j}{k_j!\,(k_{j-1} - k_j + 1)!} \;=\; \frac{1}{k!} \prod_{j=2}^{D} \frac{k_j}{(k_{j-1} - k_j + 1)!}\,. \tag{5.7}$$

Set $k'_j := k_{j-1} - k_j + 1$ for $j = 2, \dots, n$. We have $k'_j \geq 0$ with $\#\{\, j \,:\, k'_j = 0 \,\} = \alpha(L) < \delta D$ and

$$k_j \;=\; j - \sum_{l=2}^{j} k'_l \;\geq\; 0 \qquad \forall j = 2, \dots, D \tag{5.8}$$

So the term in (5.7) is

$$\prod_{j=2}^{D} \frac{j - \sum_{l=2}^{j} k'_l}{k'_j!}\,. \tag{5.9}$$

Since we have that more than $(1-\delta)D$ of the k'_j are at least equal to 1, you see that no numerators in (5.9) can be greater than δD, while the first δD ones are less than or equal to their indexing number. So as an estimate of the product of numerators in (5.9) we obtain[20]

$$(\delta D)^{(1-\delta)D}\,(\delta D)!$$

which is by Stirling formula less than

$$(\delta D)^{D}\, e^{-\delta D}\, P(D)$$

[20]The reader might forgive us that we write down factorials of non-(necessarily) integral numbers, but as this can be corrected by a polynomial(ly bounded) term in D and polynomials will not matter to us, this does not spoil anything.

for a certain polynomial P in D.

As an estimate for the r.h.s. of (5.9) we obtain

$$\frac{D!}{k!}\sum_{(k'_i)}\prod_{l=2}^{D}\frac{1}{k'_l!}\cdot\frac{(\delta D)^D}{D!\,e^{\delta D}}\,P(D)\,,$$

where the summation is over all (k'_i) with $k'_i\geq 0$, $\sum k'_i = D-k$ and $\#\left\{\,j\,:\,k'_j=0\,\right\} = \alpha(L)$. Noticing that

$$\sum_{l=2}^{D} k'_l \,=\, D-k$$

(which is (5.8) for $j=D$) and $\frac{1}{k'_l!}=\left[e^x\right]_{k'_l}$ we have

$$\sum_{(k'_i)}\prod_{l=2}^{D}\frac{1}{k'_l!}\,\leq\,\sum_{(l'_i)}\prod_{l=2}^{D}\frac{1}{l'_l!}\,=\,\left[(e^x)^D\right]_{D-k}\,=\,\frac{D^{D-k}}{(D-k)!}\,.$$

where the summation in the second term is over all (l'_i) with $l'_i\geq 0$ and $\sum l'_i = D-k$. So (5.9) is less than

$$\binom{D}{k}\,D^{D-k}\,\frac{(\delta D)^D}{D!\,e^{\delta D}}\,P(D)\,.$$

We obtain a bound for ξ^{δ}_D by summing over $k=1,\ldots,D$.

$$\xi^{\delta}_D\,\leq\,(D+1)^D\,\frac{(\delta D)^D}{D!\,e^{\delta D}}\,P(D)\,. \tag{5.10}$$

The r.h.s. is by Stirling formula less than or equal to

$$\delta^D\,e^{(2-\delta)D}\,D!\,P'(D)$$

for a certain polynomial P' in D.

Case 2. For $\alpha(L)>\delta D$ consider first only reduced regular LCD's with path l. Let $\mathring{\xi}^{\bar\delta}_D$ be this number.

By artificially putting to the left of such a LCD a chord of length 1 we obtain a reduced regular LCD of degree $D+1$ with len(chord 1) $=1$ and with a path with ascend $\geq\delta D$. The ascend of this LCD is equal to its *total descend*

$$\sum_{i\,:\,k_{i+1}\leq k_i}(k_i-k_{i+1})\,\geq\,\delta D\,.$$

Each descend of k'_i-1 means letting a segment of k'_i left neighbored basepoints to the right of the right basepoint of the newly placed chord (which is preserved by the addition of all remaining chords) by applying m_i^{-1}. Let the symmetric groups of corresponding order act on such segments simultaneously. The orbits are free and, since we consider reduced LCD's, all LCD's in the orbits have no type I. As we noted, the number of LCD's with no type I is $D!$. The lowest cardinality of the direct product of symmetric groups is in the case where we have 1-descends only, which is $\geq 2^{\delta D}$. So the total number of regular reduced LCD's L with $\alpha(L)\geq\delta D$ is

$$\mathring{\xi}^{\bar\delta}_D\,\leq\,\frac{D!}{2^{\delta D}}\,.$$

Combining both cases we find that for reduced regular LCD's their number $\mathring{\xi}_D$ is

$$\begin{aligned}\mathring{\xi}_D &\leq P''(D)\left(\delta^D\, e^{(2-\delta)D}\, D! + \frac{D!}{2^{\delta D}}\right) \\ &\leq P''(D)\,\frac{D!}{\lambda(\delta)^D}, \qquad \lambda(\delta) := \frac{1}{\max(\delta e^{2-\delta}, \frac{1}{2^\delta})}\,.\end{aligned}$$

Now choose δ so that $\delta < \frac{1}{e^2}$ and you are done for reduced LCD's.

The best choice of δ would be one for which

$$\delta e^{2-\delta} = \frac{1}{2^\delta}\,.$$

Call δ_0 this number and $\lambda_0 := \lambda(\delta_0)$.

However, our choice of δ will be somewhat worse in order to remove all polynomial mess in D by the exponential correction.

The result for all regular LCD's is now easy. We have

$$\begin{aligned}\xi_D &= \sum_{i=0}^{D-1}\binom{D-1}{i}\mathring{\xi}_{D-i} \\ &\leq P(D)\cdot\sum_i \frac{\lambda_0^i}{i\,!}\,\frac{D!}{\lambda_0^D} \\ &\leq P(D)\cdot e^{\lambda_0}\,\frac{D!}{\lambda_0^D}\,.\end{aligned}$$

Again, kill the polynomial by perturbing λ_0. □

Numerically we have $\delta_0 \approx 0.141334$, where $\lambda_0 = 2^{\delta_0} \approx 1.10292$. So we have

Theorem 5.11 *The number ξ_D (and therefore the number of linearly independent Vassiliev invariants of order D) is asymptotically bounded by*

$$\xi_D < \frac{D!}{1.1^D}\,.$$

5.6 The segment length inequality

After having estabilished our result it is perhaps worth saying a word about some possibilities left open in the proof of our bound.

The observation made in case 2 of the proof can be generalized somewhat.

Definition 5.12 *Call a* segment *of an LCD a maximal piece of the solid line containing left basepoints only and its length the number of such basepoints.*

Then by the argument above we have the

Theorem 5.13 (Segment length inequality)

$$\sum_{\substack{\text{reduced reg.}\\ \text{LCD's } L \text{ of deg } D}}\ \prod_{\text{segments of } L}\big(\,\textit{length of segment}\,\big)\,! \quad\leq D!$$

Basically our proof was that we bounded the L's with $\leq \delta D$ factors equal to one in the product and used that the rest appears with multiplicity at least $2^{\delta D}$ in the sum. However, many reduced regular LCD's appear with much higher factors and if one were able to control their number (which probably requires much labour) this would improve the base in the denominator we obtain in case 2 (and the total one for all regular LCD's). One might even hope that one can achieve *each* base in case 2 (and therefore as well for the total bound). But, to put an end to our dreams, recall that we will never be able to prove $(D!)^{1-\epsilon}$ (for some $\epsilon > 0$) this way!

6 The dimension of a graded commutative algebra and asymptotics of Vassiliev invariants

Here we discuss the relation between the dimension of a symmetric algebra (with the induced grading) over a graded vector space (latter called henceforth the primitive part of the algebra), and apply it to deduce a lower bound for the number of all Vassiliev invariants.

One of the combinatorial aspects of a commutative graded algebra (CGA) A is the relation between the asymptotical behaviour of its gradeded pieces A_D depending on their primitive parts P_D (d and D will denote the degree). We will make two assumptions on such an algebra:

1) $\deg(a \cdot b) = \deg(a) + \deg(b) \quad \forall a, b \in A,$
2) prime factorization is unique in A.

Consider the commutative graded Hopf algebra $A = \mathcal{A}$ of chord diagrams.

Recently, Chmutov and Duzhin [CD2] obtained the following result for $\mathcal{A}$.

Theorem 6.1 *The dimension of primitive elements in A_D has the asymptotical lower bound $D^{\log_{4+\varepsilon} D}$ for each $\varepsilon > 0$.*

As it was not explained by the authors which base of the logarithm we can choose, we should do this here.

Proof. We have by [CD2, theorem 2.5] the lower bound

$$\sum_{\substack{n=1 \\ n \text{ even}}}^{d} \frac{n^{d-n}}{(d-n)! \cdot 2^{\frac{(d-n+1)(d-n)}{2}} 3^{d-n}}.$$

Set $n := d - \log_C d$ for fixed $C > 1$. The summand we consider is (temporarily omitting C for simplicity)

$$\frac{(d-\log d)^{\log d}}{(\log d)! \cdot \sqrt{2}^{\log^2 d} \left(3\sqrt{2}\right)^{\log d}}.$$

By Stirling formula this is asymptotically equivalent to

$$\left(\frac{e\,(d-\log d)}{3\sqrt{2} \cdot \log d \cdot d^{\log \sqrt{2}}}\right)^{\log d} \cdot \frac{1}{2\pi\sqrt{\log d}}$$
$$\geq \quad \left(d^{1-\log\sqrt{2}-\varepsilon}\right)^{\log d} \qquad \forall \varepsilon > 0$$

$$= \quad d^{\ln d\left(\underbrace{\frac{1-\frac{\ln\sqrt{2}}{\ln C}}{\ln C}}_{f(\ln C)}-\varepsilon\right)} \,. \tag{6.1}$$

Now vary C and try to maximize $f(x)$ over $x := \ln C \in (0,\infty)$. We have

$$f'(x) = -\frac{1}{x^2} + \frac{2\ln\sqrt{2}}{x^3}$$

with the zero $x_0 = \ln 2$. We find

$$f(x_0) = \frac{1}{2\ln 2}\,,$$

so the best expression in (6.1) is

$$d^{\ln d\left(\frac{1}{\ln 4}-\varepsilon\right)} = d^{\log_{4+\varepsilon} d} \qquad \square$$

Remark 6.1 As all estimates were sharp and we took the maximum it is very likely that this is the best we can do.

Such a result opens the question which lower bound it implies for the dimension of the space of *all* chord diagrams of degree D, isomorphic [BN2] to the factor space of Vassiliev invariants of degree D modulo such of degree $\leq D-1$.

Generally, the relation between $p_d := \dim P_d$ and $a_d := \dim A_d$ is

$$a_D \;=\; \sum_{\substack{(k_1,\ldots,k_D)\,\geq\,0 \\ \sum_i i\cdot k_i \,=\, D}} \prod_{j=1}^{D} \binom{p_j+k_j-1}{k_j} \tag{6.2}$$

where the D-tuple $k = (k_1,\ldots,k_D)$ corresponds to products of k_i factors of degree i. Denote the number of such products by con_k.

Remark 6.2 In terms of the generating functions $a(x)$ of (a_d) and $p(x)$ of (p_d), equation (6.2) can be rewritten as

$$a(x) \;=\; \prod_{d=1}^{\infty} \frac{1}{(1-x^d)^{p_d}} = \exp\left(p(x) + \frac{1}{2}p(x^2) + \frac{1}{3}p(x^3) + \cdots\right)\,. \tag{6.3}$$

This relation is well-known in combinatorics. It appears explicitly in Cayley's counting of rooted trees [Ca].

Using this relation, Chmutov and Duzhin gave a lower bound for a_D without discussing details.

The aim of this section is to elucidate a little more the relation between p_d and a_d in general, and to apply it to the special case of chord diagrams, explaining and motivating one possible approach to such sort of problems.

6.1 The dominating partition

In the following we discuss the search of partitions of D which give the maximal contribution in (6.2). This maximal contribution we denote by $dom(D)$. For each D we choose one special such partition (if there are several) and consider the sequence of these partitions.

In general, the asymptotic contribution in (6.2) of a sequence of partitions of D as $D \to \infty$ roughly depends on the asymptotic behaviour of the number of parts d in the partition as function of d and D. So here is a possible strategy: take partitions with different asymptotics of number of parts d and calculate the corresponding contributions.

Ideally, it would be nice to have a result telling us *which* asymptotics of number of parts we have to take to find asymptotically the *dominating* partition in (6.2), that is, the one k giving the summand with the highest contribution con_k, which we will denote by $dom(D)$, but neither I could deduce nor I know of any such (really ingenious) result.

Below we will make an ansatz to come close to the dominating partition. We will use the number of parts of a fixed degree d in the partition to be constant, while $D \to \infty$, except finitely many D where it is 0. We will fix this constant a priori for each length and for given D take so many lengths, until D is exhausted (this of course will not work exactly in general; we will forget about the small rest, or add it to the part in degree one). That is, we consider partitions $((d_{i,D})_{i=1}^{l_D})_{D=1}^{\infty}$ with $d_{i,D} = d_{i,D+1}$ for $1 < i \leq l_D$ and $l_D \leq l_{D+1}$. This turns out to be a good ansatz, i.e. a sequence of partitions asymptotically producing the dominating contribution can always be chosen (by possibly losing a minimum on the quality of the asymptotics) to have this property (note, that some small variations of the partitions in such a sequence will produce asymptotically equal contributions). We will explain this at the end of the subsection.

In the following I shall only briefly discuss three main cases.

1) The case of 1 primitive element per degree. The numbers a_d are the so-called *partition numbers* $p(d)$ [An, Ri], which are known to have the asymptotical behaviour

$$p(d) \asymp \frac{1}{4\sqrt{3}\,d} \cdot \left(e^{\sqrt{2/3}\,\pi}\right)^{\sqrt{d}} .$$

   See [An, page 70]. Here the asymptotics of a_d comes from the abundance of summands on the r.h.s. of (6.2), not from their single contributions.

2) The case of polynomially many primitive elements per degree $p_d = d^i$, $i \in [0, \infty)$ fixed.

   I guess (without having a proof) that the dominating asymptotical contribution comes from partitions of D, where the number of parts equal to d is d^i, $1 \leq d \leq d_0(D)$ for some $d_0(D) \in \mathbf{N}$. This contribution is between $C_{1,2}^{D^{(i+1)/(i+2)}}$ for two constants $C_{1,2} > 1$, which can be chosen arbitrarily close to each other. This is the maximal contribution from all partitions with d^p parts d, where p varies over $[0, \infty)$.

   Note, that already in the case of polynomial behaviour the question of asymptotics of a_d is basically equal to the one of finding the dominating partition in (6.2), since even multiplication by $p(d)$ (and adding $\sqrt{d}$ to the exponent) doesn't give any significant improvement anymore.

3) Asymptotically $p_d > d^i$ for all $i \in \mathbf{N}$. By 2) you obtain $a_D \geq e^{D^{1-\varepsilon}}$ for each $\varepsilon > 0$.

So why was that a good ansatz? Here is one justification: For the sequence of dominating partitions either the number of parts equal to d is bounded for all d, or grows beyond any limit for all d as $D \to \infty$ (unless $p_d = 1$ or $p_d = 0$), so that the ratio between the number

k_{d_1} of parts equal to d_1 and the number k_{d_2} of parts equal to d_2 converges to the ratio of $(p_{d_1} - 1)/d_1$ and $(p_{d_2} - 1)/d_2$.

In the case $p_d = 1$ we will always have in our sequence of dominating partitions $k_d = 0$ (unless all $p_d = 1$, which is not interesting), and in the case $p_d = 0$ we set $k_d = 0$.

Here is how fast this number can grow for $D \to \infty$. Let us assume that $p_2 > 1$. If not, replace the '2' by some other 'i' with $p_i > 1$.

Theorem 6.2 *In the sequence of dominating partitions the number $k_2(D)$ of parts equal to 2 satisfies for all D the relation*

$$\sum_{l=1}^{k_2(D)-1} \frac{(p_{2l} - 1) \cdot 2l}{\left(\frac{k_2(D)}{k_2(D)-l}\right)^{p_2-1} - 1} - 2l \leq D\,. \tag{6.4}$$

Proof. The last l factors in the expansion

$$\binom{p_d + k_d - 1}{k_d} = \frac{p_d}{1} \cdot \frac{p_d + 1}{2} \cdot \ldots \cdot \frac{p_d + k_d - 1}{k_d}\,. \tag{6.5}$$

of the binomial coefficient $\binom{p_d + k_d - 1}{k_d}$ in degree $d = 2$ are

$$\left(1 + \frac{p_2 - 1}{k_2 + 1 - l}\right) \cdot \ldots \cdot \left(1 + \frac{p_2 - 1}{k_2}\right)\,. \tag{6.6}$$

The factor coming to the right of (6.5) for $d = 2l$ when augmenting k_{2l} by one is

$$1 + \frac{p_{2l} - 1}{k_{2l} + 1}\,. \tag{6.7}$$

Then we must have $(6.6) > (6.7)$. Otherwise by removing the summands equal to 2 and taking one more summand equal to $2l$ we get a larger contribution. We have for all k_2 and $l < k_2$

$$\begin{aligned}
(6.6) \quad &\leq \quad \exp\left((p_2 - 1)\left(\frac{1}{k_2 + 1 - l} + \ldots + \frac{1}{k_2}\right)\right)\\
&\leq \quad \exp\big(\,(p_2 - 1)(\ln k_2 - \ln(k_2 - l))\,\big)\\
&= \quad \left(\frac{k_2}{k_2 - l}\right)^{p_2-1}\,.
\end{aligned}$$

Thus

$$k_{2l} \geq \frac{p_{2l} - 1}{\left(\frac{k_2}{k_2-l}\right)^{p_2-1} - 1} - 1$$

for all $l < k_2$. The assertion is immediate, since (k_i) is a partition of D. □

Corollary 6.1 *There exist constants C', C'' so that*

$$\sum_{l=1}^{C' \cdot k_2(D)} 2l\left(\frac{p_{2l} - 1}{C''} - 1\right) \leq D\,.$$

Very roughly, replacing the sum by an integral, you see that $k_2(D)$ must be bounded above (modulo constants) by something like $F^{-1}(D)$, where $F(l) = p_{2l} \cdot l^2$.

Definition 6.3 An asymptotic is an equivalence class of sequences of naturals modulo the equivalence $(a_i) \asymp (b_i) \iff \lim_{i\to\infty} a_i/b_i = 1$. The asymptotic $[(a_i)]$ is higher than the asymptotic $[(b_i)]$ if $\liminf_{i\to\infty} a_i/b_i > 1$. This gives a partial ordering among all asymptotics.

Definition 6.4 *Denote by $dom_b(D)$ the highest asymptotic of contributions of a sequence of partitions in (6.2) with bounded number of parts d as $D \to \infty$.*

Note that $dom_b(D)$ is not defined as a sequence itself, only its asymptotical behaviour is determined. Furthermore note, that $dom_b(D)$ is a maximal element in a partial ordering, so it does not need to exist (not even by Zorn's lemma)! The following discussion is under the (naive) assumption that it does really exist.

Corollary 6.2 *Let $dom_b(D)$ grow less fast than any exponential in D. Denote here by $dom_b(D)$ a* special representant *of its asymptotic. Then*

$$dom(D) < \left(dom_b\left(\frac{D}{B(D)}\right)\right)^{B(D)} \tag{6.8}$$

for any sequence $B(D)$ with $B(D) \geq k_2(D)$.

Proof. Take a sequence of partitions of $D/B(D)$ with $k_d(D)/B(D)$ parts d, use the observations directly before theorem 6.2 and note that, expanding the binomial coefficients as in (6.5), the remaining $k_d(D) - k_d(D)/B(D)$ factors in each degree are lower than the first $k_d(D)/B(D)$. □

For example, if you assume the bound in 2) on 50 is the best one, you get by corrollary 6.2

$$dom(D) < C^{D^{\frac{i+1}{i+2}+\frac{1}{(i+1)(i+2)}}},$$

using $B(D) := D^{\frac{1}{i+1}}$, where $B(D)$ can be chosen from corrollary 6.1 ignoring C', C''. For large i the additional term is small. If the bound is larger, we get less of an improvement. In 3) on 50 the additional term would be compensated by the choice of ε.

Let us come back to our original ansatz to fix the number of parts equal to d as $D \to \infty$. It is sufficient to consider only the asymptotic behaviour of the number of parts k_d equal to d as $d \to \infty$, not their sequence $(k_d)_{d=1}^{\infty}$ itself. More precizely, we have

Proposition 6.1 Take two sequences of partitions with k_d, k'_d parts equal to d where k_d, k'_d are constant in D. Let $k_d \asymp k'_d$ as $d \to \infty$. Then asymptotically as $D \to \infty$ we have

$$con_k^{1-\varepsilon} \leq con_{k'} \leq con_k^{1+\varepsilon}$$

for all $\varepsilon > 0$.

Proof. Use a similar argument to the one in the proof of corollary 6.2.

6.2 A lower bound for the number of all Vassiliev invariants

We will now follow the strategy in our ansatz and consider the case $p_d = d^{\log_{4+\varepsilon} d}$, which is the relevant bound for Vassiliev invariants. It is my reproduction of the result in [CD2, Appendix] with a small correction.

Theorem 6.5 *For the dimension a_d of the part in degree d of the commutative graded algebra of chord diagrams, we have asymptotically*

$$a_d \geq C^{d/(4+\varepsilon)^{\sqrt{\log_4 d}}}$$

for each constant $C > 1$ and each $\varepsilon > 0$.

Note that the variation of ε makes the choice of C unimportant.

Proof. Look at partitions into equal parts and vary their length: set (in the notation of [CD2, Appendix]) $n \cdot p_n := (C + \varepsilon') \cdot d$. You have as a lower bound the expression

$$f(n) := \left(\frac{n \cdot p_n}{d}\right)^{d/n} .$$

Then use the fact that for each $a > 1, \varepsilon' > 0$

$$\frac{d}{a^{\sqrt{\log_a (Cd)}}} \geq (1 - \varepsilon') \frac{d}{a^{\sqrt{\log_a (d)}}} \tag{6.9}$$

and you see that for $a = 4 + \varepsilon$ by a reparametrization of ε you can transform the denominators in (6.9) to the one in the theorem. □

Remark 6.3 Theorem 6.5 was suggested to Chmutov and Duzhin by myself. It can be obtained from our ansatz, but the proof presented here is basically due to Chmutov and Duzhin and is much more elegant.

Note that the proof holds also for the unframed case, as $n > 1$.

Remark 6.4 By some technical calculation you can find that, by being able to vary C by an ε, no improvement by corollary 6.2 would be possible (if this were the dominating asymptotical contribution). E.g., take (using corollary 6.1) $B(D) := (4+\varepsilon)^{\sqrt{\log_{4+\varepsilon} D}}$. So, whatever the dominating contribution of constant number of parts is, any improvement due to considering an unbounded number of parts will be gobbled up by adding any $\varepsilon > 0$ to the base.

In particular, the ansatz of Chmutov and Duzhin can only be (if at all) negligably better than mine. More precisely, one can prove that, if

$$\text{r.h.s. of (6.8)} \; < \; dom_b(D)^{1+\varepsilon} \tag{6.10}$$

and

$$\frac{\log dom'_b(D)}{\log dom_b(D)}$$

is monotonously growing, then (6.10) is also true for dom'_b instead of dom_b.

To see this, take the logarithm on both sides of (6.10) for dom_b and dom'_b.

Remark 6.5 If you go to the bother of taking the derivate of f and find its maximal value, the lower bound for a_d is a little better, but only for fixed ε, in the quality of

$$C^{d \cdot \sqrt{\log_{4+\varepsilon} d}/(4+\varepsilon)^{\sqrt{\log_{4+\varepsilon} d}}} \tag{6.11}$$

for some constant $C > 1$.

My proof of theorem 6.5 works with $k_d := \varepsilon' \cdot d^{\log_{4+\varepsilon} d}$, letting $\varepsilon' \to 0$. The proof suggests that taking a partition with the number of parts d growing somewhat weaker than $\varepsilon' \cdot d^{\log_{4+\varepsilon} d}$ will produce a further improvement. Actually, you can obtain (6.11) by setting $k_d := d^{\log_{4+\varepsilon} d - C'}$ and a lot of highly technical arguments, which we preferred to omit here. In fact, it is a strong challenge to find (and prove!) the asymptotics of the number k_d of parts equal to d in our ansatz, producing the dominating contribution.

On the other hand, however, in the end such attempts wont give much, since taking more awkward asymptotics of the number of parts will make the expressions fairly unwielding (as you can see in remark 6.5) and whatever we try, the improvement will stay small – it turns out that we will never be able to remove that "almost" before the exponential, unless we manage to do the same already with the bound for the primitive invariants.

6.3 The exponential barrier

More precisely, this fact can be formulated as follows.

Theorem 6.6 *If p_d grows less fast than any exponential, then so does a_d.*

Here "grows weaker than any exponential" means that the sequence does not contain a subsequence admitting a lower exponential bound (so we do not need to restrict ourselves to monotone sequences).

Remark 6.6 Looking at the second equality in (6.3), this is just the statement that if the radius of convergence of $p(z)$ is 1, then so it is for $a(z)$. This might have been noticed or implicitly conceived already by Pólya in his celebrated paper [Po]. Such arguments are used in the asymptotical analysis of graphical trees. See, e.g., [HP]. Here we present a proof without use of Pólya theory.

Proof. Note, that we may assume (making if necessary p_d a little bigger) that $p_d = \hat{C}^{d/f(d)}$ with some constant $\hat{C} > 1$ and a monotonous f with $f(1) = 1$, $f(d) < d^{1-\varepsilon}$ for some $\varepsilon > 0$.

First we shall establish that for this p_d the sequence a_d is bounded *above* by an exponential in d. Use

$$\binom{n+k-1}{k} < n^k$$

on the r.h.s. of (6.2) and observe that the maximal contribution of a partition (=summand) therein is the one from the partition of d into d parts '1', which is an exponential in d. The multiplication with the number of partitions $\sim C^{\sqrt{d}}$ does not change anything essential.

Now assume, that from the growth of p_d primitive elements you would also obtain an exponential *lower* bound $a_d > C'^d$ with a $C' > 1$. Consider now $p'_d := \hat{C}^{d/\sqrt{f(d)}}$ and note that p'_d grows faster than any power of p_d. For p'_d you would have by the above argument an asymptotical exponential upper bound for the corresponding a'_d. But now the contradiction follows from the following lemma.

Lemma 6.7 *If p_d primitive elements produce a_d total elements, then p_d^2 primitive elements produce $\geq \frac{a_d^2}{2}$ total elements. As a consequence, $p_d^{2^k}$ primitive elements produce $\geq a_d^{2^k - \varepsilon''}$ total elements for all $\varepsilon'' > 0$.*

Proof of lemma. Expand the binomial coefficients on the r.h.s. of (6.2) as a product of k_j factors as in (6.5). Now you see that replacing $p_d \mapsto p_d^2$ each factor at least squares itself.

It only remains to apply the standard inequality

$$2\sum_{i\vdash d} a_i^2 \;\geq\; \left(\sum_{i\vdash d} a_i\right)^2,$$

with a_i being the contributions from the partitions of the degree. $\square$

Of course, for the k we have to choose in the proof of theorem 6.6, when applying lemma 6.7 we have

$$\hat{C}^{d/\sqrt{f(d)}} \;\geq\; \left(\hat{C}^{d/f(d)}\right)^{2^k}$$

only for *almost* all d, say for $d \geq d_0$, but omitting factors of the first d_0 degrees divides the contribution by a term bounded above by a polynomial in d, e.g. you can take

$$d^{\sum_{i=1}^{d_0} \hat{C}^{i/f(i)}},$$

which can be compensated by choosing ε'' in the lemma a little bigger (since $d/f(d)$ grows faster than d^{ε} for some $\varepsilon > 0$), so the argument still works. $\square$

We see that the exponential growth is a very strong "barrier". If the growth of p_d is less than exponential, then there is a qualitative difference between the growth of the total dimension and the one of the primitive part.

And, once it is broken, the primitive elements become dominating in each degree, so the asymptotics of a_D and p_D is (up to a negligable factor) equal.

In view of all this the decisive question is

Question. Is the exponential asymptotics a lower or an upper bound for Vassiliev invariants?

Answering this question will surely be hard. We saw why for the lower bound it will be so – we are much further away from the exponential bound than theorem 6.5 suggests. On the other hand, for an upper bound the best we can offer at present is something like $D!/1.1^D$ [St6]. Thus also in this case hard work is in store for us …

7 The braid index and the growth of Vassiliev invariants

In this section, we use the new approach of braiding sequences to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots.

Diagrams refer henceforth to knot diagrams (and not to chord diagrams).

7.1 Braiding sequences

Recall the basic definitions in the context of braiding sequences from §1.7.

Definition 7.1 *For some odd $k \in \mathbf{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 6). A braiding sequence (associated to a numbered set P of crossings in a diagram D; all crossings by default) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1, \dots, x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done.*

Any Vassiliev invariant v of degree at most k behaves on a braiding sequence as a polynomial of degree at most k in $x_1, \dots, x_{|P|}$ (see [St4] and [Tr]), and this polynomial is called braiding polynomial of v on this braiding sequence.

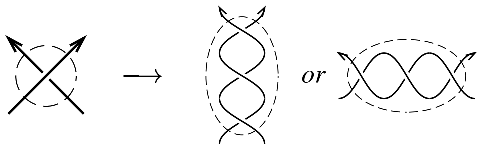


Figure 6. Two ways to do a -3-braiding at a crossing.

Let C be a class of knots and $v : C \longrightarrow \mathbf{Q}$ a map. Extend v to singular knots as described in section 1, equation (1.1). This extension is well-defined on those singular knots, all of whose resolutions result in knots from C.

Definition 7.2 *Let C be a knot class. Then a Vassiliev invariant of deg $\le m$ on C is a map $v : C \longrightarrow \mathbf{Q}$ with v vanishing on all $(m+1)$-singular knots, whose all 2^{m+1} possible resolutions give knots in C.*

Question. Is any Vassiliev invariant on some knot class C the restriction of a Vassiliev invariant (on all knots) to C? Or, in other words, does any Vassiliev invariant on C admit an extension to a Vassiliev invariant (on all knots)?

Remark 7.1 It is maybe a question of personal taste to which degree this question is interesting. However, while the classical approach fails to give any statement about counterexample invariants to question 7.1 (if such exist), such counterexample invariants would be naturally incorporated into the braiding sequence approach. Therefore, until the non-existence of such counterexamples is not proved, it is not clear, that the finite-dimensionality results of [St4] are consequences of [Ko, Dr, BN2]. However, the braiding sequence approach fails to give any indication if (or for which classes) such examples exist.

Definition 7.3 *Let C be a class of knots. The set of Vassiliev invariants on C is called finitely-determined, if for all $n \in \mathbf{N}$ there exists a constructible finite set $C_n \subset C$, such that any Vassiliev invariant on C of degree $\le n$ is uniquely determined by its values on C_n.*

In [St4] I proved that Vassiliev invariants are finitely-determined on arborescent knots (and connected sums thereof) and closed 3 braids.

Let us briefly recall the idea in [St4, §10], how braiding sequences can be used to prove finite-determination and upper bounds for the number of Vassiliev invariants on some knot classes. We covered such a class C by braiding sequences and worked inductively over the length/weight (in some specified sense, which depends on the context, see lemma 7.7 and definition 7.13) of these braiding sequences. Assume that for a certain set $C' \subset C$ a Vassiliev invariant v of degree p is 0 and use induction. Special cases of a braiding sequence B give shorter braiding sequences, on which v is 0 by induction, and each such special case on the level of braiding polynomials gives a *simplifying* or recursive relation (an equality between special values of a braiding polynomial P and values of a braiding polynomial of a shorter braiding sequence, which are 0 by induction). That is, a simplifying or recursive relation is an equality of the kind

$$P_B\big|_{x_{i_1}:=a_{i_1},\dots,x_{i_n}:=a_{i_n}} \;=\; P_{B'}\big|_{x_{i'_1}:=b_{i'_1},\dots,x_{i'_m}:=b_{i'_m}}$$

for some $m, n, i_j, i'_j \in \mathbf{N}$ and $a_{i_j}, b_{i'_j} \in \mathbf{Z}$, and $P_B, P_{B'}$ being the braiding polynomials of some fixed Vassiliev invariant v on braiding sequences B and B' with $\mathrm{len}(B') < \mathrm{len}(B)$.

A simplifying or recursive relation augments the degree of a non-trivial solution for the braiding polynomial P of v on B. This happens, because if P turns to zero setting $x_{i_j} := a_{i_j}$ for some $i_j, j = 1, \dots, k, a_{i_j} \in \mathbf{Z}$, then any top degree monomial of P contains at least one of the x_{i_j}'s. If the degree of P is pushed higher than the degree of v by finding sufficiently many such (disjoint) sets $\{x_{i_j}\}$, only the trivial solution for the braiding polynomial P of v on B is possible, and so the induction step shows that v is 0 on the whole class C. Then Vassiliev invariants are finite-determined on C by setting $C_p := C'$, and in particular, the number of linearly independent Vassiliev invariants of degree $\leq p$ on C is maximally $|C'|$.

7.2 Arborescent knots

Rational (2-bridge) knots are a nice example of such a class which can be dealt with by the above idea – the braiding sequences are given by (the parities of) the coefficients in the Conway notation.

The Conway notation [Ad, Co] can be formally considered as a map of formal expressions built up of the letter set $\mathbf{Z} \cup \{\infty\}$ and binary operations '$\cdot$' (called product and often omitted and replaced by concatenating the factors) and ',' (called sum) to tangle diagrams, as shown on figure 7.

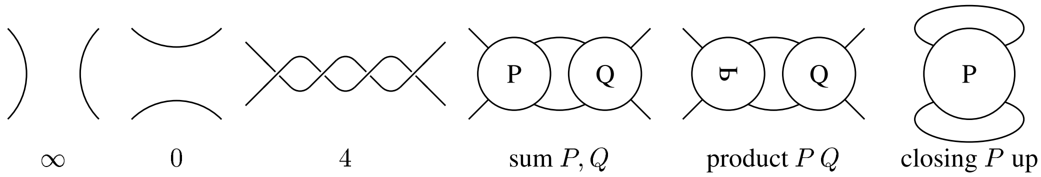


Figure 7. Operations with arborescent tangles

Such tangles are called arborescent. Note, that the product operation is not associative. The omission of parentheses in the notation means left parenthesation, that is $abcd$ is treated as $((ab)c)d$. Rational tangles are arborescent tangles, whose notation does not contain parentheses and sum operations. Rational/arborescent knots are closed rational/arborescent tangles, as shown to the right of figure 7.

Now any summation can be replaced by a product, where the first factor (tangle) has been mirrored with respect to its NW-SE axis. More generally, it is possible to rotate a tangle through an angle of $k \cdot \pi/2, k \in \mathbf{Z}$ or mirror it in the plane or in projection direction producing another tangle. Such an operation we will call $flip$. Replacing in the Conway notation sums by products and flips, we obtain the *notation with flip*.

Now, in all the Conway notations or notations with flip varying some coefficient by ± 2 corresponds to a braiding and so any notation is a member of a braiding sequence given by the parities ('$+$' for even parity and '$-$' for odd parity) of its coefficients, e. g.,

$$(2,3) \cdot (5,6) \in (+,-) \cdot (-,+)\,.$$

Using this method, in [St4] we proved:

Theorem 7.4 *([St4]) Vassiliev invariants on rational knots are exponentially bounded in the degree, that is, there exists some $C > 1$ with*

$$\dim \{ \textit{ Vassiliev invariants of degree} \leq k \textit{ on rational knots } \} \leq C^k\,.$$

We also proved in the same way:

Theorem 7.5 *([St4]) Vassiliev invariants on arborescent knots with all Conway coefficients even are exponentially bounded in the degree.*

Our first aim is the obvious extension both of these theorems.

Theorem 7.6 *Vassiliev invariants on arborescent knots are exponentially bounded in the degree.*

Proof. We split it into two lemmas. The first one is [St4, lemma 8.1], but we include the proof here.

Lemma 7.7

$$\#\{\, \textit{simplifying relations} \,\} \geq \frac{\sqrt{n}}{2}, \quad n = \textit{len}(\, \textit{notation} \,)\,,$$

where the length of a braiding sequence is the number of integers (or rather their parities) in the notation with flip, and such that each variable appears maximally in one such relation.

Proof of lemma 7.7. Start induction with $n = 2$ and $n = 3$.
For $n = 2$ there are 4 choices: $--, -+, +-, ++$, with possible 'flip's applied to some signs. In all cases one of the following relations applies (note, that $\infty = \text{flip}\,(0)$)

$$\begin{array}{c} 1 \cdot 1 = 2 \\ 1 \cdot 0 = 1 \\ 0 \cdot 1 = \infty \\ 0 \cdot 0 = \infty \end{array}$$

and analogously for ∞. So there is always 1 simplifying relation. In the same way argue for $n = 3$.

Now do the induction step.

Assume the inequality of the lemma is true for all notations with flip of len $< n$ and consider a notation A with len $= n$. There are 4 choices for A

$$(ab)c \quad \text{flip}\,(ab)c \quad a(bc) \quad a\text{flip}\,(bc) \tag{7.1}$$

or 'flip's of these 4 expressions, but it is unnecessary to consider $A = \text{flip}\,(\text{ something })$ as all the simplifications we can achieve in 'something' carry over after flips.

Denote by $l_i, i \in \{a, b, c\}$ the lengths of the subexpressions a, b, c in (7.1).

Case 1. Assume, that maximally one of a, b, c has length 1. Then

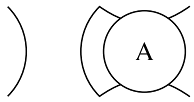
$$\#\{\text{ simplifying relations }\} \geq \frac{\sqrt{l_a}}{2} + \frac{\sqrt{l_b}}{2} + \frac{\sqrt{l_c}}{2} - \frac{1}{2}\,,$$

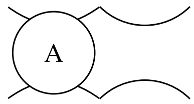
latter term standing to equilibrate a possible uncorrect contribution from one of l_a, l_b, l_c being 1. So, as $l_a + l_b + l_c = n$,

$$\frac{\sqrt{l_a} + \sqrt{l_b} + \sqrt{l_c}}{2} - \frac{1}{2} \geq \frac{\sqrt{n}}{2} \text{ for } n \geq 3\,.$$

Case 2. So two of a, b, c must be of length 1.

Assume, one of these two would be a '+' (or a 'flip' thereof). Then set 0 into the Conway notation. There are basically 2 possibilities (any 'flip's of the factors do not change qualitatively the picture):

2.1) $0 \cdot A$ 

2.2) $A \cdot 0$ 

In case 1 your diagram decomposes. You obtain

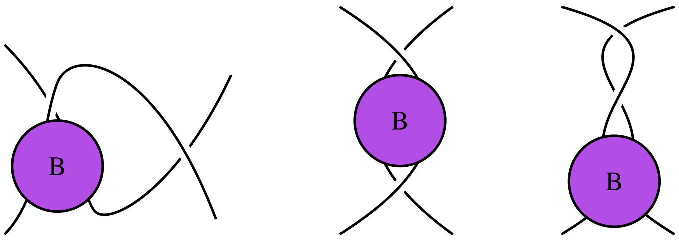
$$\ldots(0 \cdot A)\ldots \;=\; \ldots 0 \ldots \,\#\, \text{flip}\,(A)\,.$$

By the additivity of v under '$\#$', it suffices to consider the (possibly trivial) factors separately. But both factors have a shorter notation and are hence dealt with by induction premise.

In case 2 we just have $A \cdot 0 = \text{flip}\,(A)$, which is also simplifying.

So in both cases there are $\geq \frac{\sqrt{n-2}}{2} + 1 \geq \frac{\sqrt{n}}{2}$ $(n \geq 3)$ recursive relations.

Case 3. Two of a, b, c are of length 1 and both are '$-$' (or flip $(-)$). Then by inserting appropriately 1 and -1 into these '$-$'es, you obtain modulo flips one of the following tangles



where B is some flip of the remaining tangle of len > 1 in the notation (7.1). But, after performing a flype, in all 3 cases you can simplify the tangles to ones, having as a notation with flip this of B and only *one* additional number (with some 'flip's performed on subexpressions). But this notation is again simpler, so it produces a relation, and you have

$$\geq \frac{\sqrt{n-2}}{2} + 1 \geq \frac{\sqrt{n}}{2}$$

simplifying relations.

Now the case distinction and the proof of the lemma are complete (note, that in our inductive procedure we never involved an entry into two recursive relations). □

The main point now is to prove the following improved

Lemma 7.8

$$\#\{\, \textit{simplifying relations} \,\} \geq C \cdot \textit{len}(\, \textit{notation}\,), \quad C > 0\,.$$

Then by the argument at the end of subsection 7.1, any Vassiliev invariant v of degree n on arborescent knots is uniquely determined by its values on braiding sequences of length $\leq {}^{1}\!/_{C} \cdot n$, and the number of such braiding sequences is exponentially bounded in the degree. Finally, the number of knots in each braiding sequence which suffice to determine v on it is exponentially bounded in n as well, and theorem 7.6 is proved. □

Proof of lemma. Let us start with a definition.

Definition 7.9 *A critical entry in a notation with flip C is a '$-$', which appears as one of a, b, c in one subexpression of C of the form*

$$(ab)c \quad \text{flip}\,(ab)c \quad a(bc) \quad a\text{flip}\,(bc) \tag{7.2}$$

such that the other 2 expressions have length > 1 (where length is the number of signs '$+$' and '$-$').

We proved in lemma 8.1 of [St4], that every pair of signs of non-critical entries gives rise to a relation, so that each variable appears maximally in one such relation.

Therefore, to prove the lemma 7.8, it suffices to see that

$$\#\{\text{ critical entries }\} \le C \cdot \text{len}(\text{ notation }), \quad C < 1\,.$$

This, however, follows from the observation, that in any notation with flip, we have

$$\#\{\text{ critical entries }\} \le \#\{\text{ non-critical entries }\} - 1\,,$$

which can be proved straightforwardly by induction over the construction of the notation, and this completes the proof of lemma 7.8. □

7.3 Bounds for braid representations

In the following we will use closed braid representations of knots combined with the braiding sequences approach to Vassiliev invariants to give upper bounds for the number of Vassiliev invariants on classes of knots with special braid representations. Although it is my feeling (because of the special shape of diagrams it restricts us to) that this is not the best attack for solving the question of finite-determination of Vassiliev invariants on all knots, they give rise to a considerable generalization of the results on closed 3 braids in [St4].

Theorem 7.10 *For all $k \in \mathbf{N}$ the space of Vassiliev invariants of degree $\le n$ on knots with braid index $\le k$ is finite-dimensional and exponentially bounded in n.*

Although a little restrictive, I hope this statement provides a flavour of the capabilities of the new approach and we hope it proves to be useful in a later attempt to settle the question of finite-determination of Vassiliev invariants on all knots using the closed braid approach.

For its proof we need to introduce a specific, but very appealing notation for braid words.

Definition 7.11 *A braid scheme is a checkerboard diagram with integers put on the black fields, e. g.*

$$\begin{array}{ccccccc}
 & & -5 & & & & 6 \\
 & -1 & & -2 & & 3 & \\
1 & & 3 & & 2 & & 4 \\
 & & & & & & \\
\sigma_1 & & \multicolumn{3}{c}{\dots\dots\dots\dots} & & \sigma_7
\end{array}\,.$$

If integers are omitted, they are assumed to be 0. The braid word corresponding to the diagram is the concatenation of powers of interchangingly with the row of the scheme (rows are numbered from bottom to top) odd and even index (Artin) generators, the powers given by the entries in the scheme. E. g., the braid word corresponding to the above scheme is

$$(\sigma_1\sigma_3^3\sigma_5^2\sigma_7^4)\,(\sigma_2^{-1}\sigma_4^{-2}\sigma_6^3)\,(\sigma_3^{-5}\sigma_7^6)\,.$$

Definition 7.12 *A reducing move in a braid scheme is the "pulling down" of an entry (in row ≥ 3) two rows below, if one row below its 2 neighbors (or its 1 neighbor, if it is the first or last generator), are equal to* 0*,*

$$\begin{array}{ccc} & x & \\ 0 & & 0 \\ & y & \end{array} \longrightarrow \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & x+y & \end{array} ,$$

adding it to the below entry (if it is not 0*).*

Example 7.1

$$\begin{array}{cccccc} & 3 & & -2 & \\ 0 & & 0 & & -4 \\ & 1 & & 0 & \\ 1 & & 2 & & 3 \end{array} \longrightarrow \begin{array}{cccccc} & 4 & & -2 & \\ 1 & & 2 & & -1 \end{array} .$$

Definition 7.13 *The weight of a scheme is the sum of all generator indices in the corresponding braid word (i. e.,* $(1+3+5+7)+(2+4+6)+(3+7)=38$ *in the example of definition 7.11).*

Definition 7.14 *A scheme is called reduced if it does not admit a reducing move. Any scheme can be reduced by finitely many reducing moves, not augmenting its weight.*

Proof of theorem 7.10. The idea is always the same as used for all the previous theorems: to any braid scheme there corresponds a braiding sequence according to the parity of its entries. Then prove, that inserting special values into this scheme at sufficiently many different positions gives braids, admitting a scheme representation of smaller weight.

So the key point is as always before the following

Lemma 7.15

$$\#\{\text{ recursive relations }\} \geq C \cdot \frac{1}{k^3}, \quad \text{weight(scheme)}, \quad C > 0 .$$

Using this lemma, we know, that to determine a Vassiliev invariant of deg $\leq n$ on knots with braid index $\leq k$, we only need to consider braiding sequences of weight $\leq C' \cdot k^3 \cdot n$ (for some constant $C' > 0$). As such schemes have $\leq C' \cdot k^3 \cdot n$ entries and each entry of the braiding sequence (or rather the corresponding sequence of braid words) is given by the index of the generator and the parity of the exponent, we have $\leq \big(2(k-1)\big)^{C' \cdot k^3 \cdot n}$ braiding sequences to consider. The number of monomials of the braiding polynomials of all such braiding sequences is at most $\binom{C' \cdot k^3 \cdot n + n}{n}$, which is exponentially bounded in n, and so the theorem follows. □

Proof of lemma. We will show, that we have at least in every $2k$-th row of each reduced scheme of a braiding sequence (that is, a braid scheme with $\pm$ signs instead of numbers) a situation like this

$$\begin{array}{ccc} & x & \\ x & & 0 \\ & x & \end{array} , \qquad \text{where all } x \in \{+,-\},$$

(and the x's are not necessarily equal).

Such setting always gives rise to a recursive relation: if one of the x's is '$+$', then set 0 and the entry disappears. If all x's are '$-$', set into the upper two entries 1 and apply a

Reidemeister III move (or Yang-Baxter relation) to slide the lower generator to the top, reducing its index by 1 (and so also the weight of the scheme). Then eventually bring the resulting scheme into its reduced form, which does not augment its weight.

Consider the function, assigning to each row in the reduced scheme the maximal index of its entries (i. e., the position of the right-most entry in this row of the scheme). This function has odd values on odd rows and even values on even rows. Each k rows it has an ascent. As the scheme is reduced, this ascent is by 1 (in row ≥ 3).

Then you have a picture like this

$$\begin{array}{ccccc} & & x & & \\ & x & & 0 & \\ ① & & ② & & \end{array}$$

If ② is not empty, then you are done. If it is, then ① cannot be empty as the scheme is reduced. Therefore, you obtain the same picture one row below and with the index of the generators decreased by 1. If you repeat this argument, at least after k steps, when your index of the generator has become 1 and ① does not exist, you must be able to find the desired picture.

As the contribution (index sum) to the weight of the scheme, coming from $2k$ rows, is cubically bounded in k, the lemma follows. □

Evidently, the problem to extend the theorem to arbitrary braid index is the possibility to have large parts in the scheme, consisting of '$-$'es only, so that there are no empty entries (see [St11, remark 3.4] for an example).

Whenever we can avoid this, we are almost immediately done. Here is an example of a theorem which does not involve a bound on the braid index. It can easily be proven in the same way. The details are always the same and therefore I leave them to the reader.

Theorem 7.16 *Vassiliev invariants are finitely-determined and exponentially bounded on all knots, which are closed braids β with*

$$\beta = \prod_{i=1}^{r} \sigma_{p_i}^{r_i}\,,$$

$p_i > 0, r_i \in \mathbf{Z}$*, such that even (index) generators appear with exactly one odd power (that is, for each even $p \leq \max_{j=1}^{r} p_i$ $\exists!i$ with $p_i = p$ and r_i odd).*

Remark 7.2 Any generator must appear with at least one odd power, $\hat{\beta}$ to be a knot.

Proof sketch. Consider formal connected sums of reduced schemes (i. e., expressions like

$$<\text{ scheme 1 }>\#<\text{ scheme 2 }>\#<\text{ scheme 3 }>,$$

where "scheme i" stands for some scheme) with the weight defined by the number of all entries in all schemes.

It easily follows from the premise and the reducedness of the scheme that

$$\frac{\#\{\text{ odd entries }\} - \#\{\text{ generators }\}}{2} \leq \#\{\text{ even entries }\}\,,$$

and isolated (odd) occurences of a generator in a scheme can be made into isolated crossings in the braid diagram by setting ± 1, and then turning the entry into a '#', which is also simplifying.

This shows, that

$$\#\{\text{ recursive relations }\} \geq 1/9 \cdot \text{ weight(scheme sum) }.$$

So for degree $\leq n$ Vassiliev invariants you only need to consider schemes with $k \leq 9n$ entries. Now for each such scheme/braiding sequence, the braiding polynomial of deg $\leq n$ is uniquely determined by its values on tuples $(x_1, \ldots, x_k)$ with $0 \leq x_i$, $\sum_{i=1}^{k} x_i \leq n$ (see remark 7.3 below), and all braids corresponding to such $(x_i)_{i=1}^{k}$ have $\leq 11n$ crossings. To finish with, use the result of D. Welsh [We], as quoted in [Ad, p. 49], that the number of knots with n crossings is exponentially bounded (above) in n. □

Remark 7.3 The fact, that a polynomial P of deg $\leq n$ in k variables $x_1, \ldots, x_k$ is uniquely determined by its values on $(x_1, \ldots, x_k)$ with $0 \leq x_i$, $\sum_{i=1}^{k} x_i \leq n$, can be observed from the multivariable Newton formula (see [Tr] and *loc. cit.*) by remarking, that $\Delta_{x_1}^{a_1} \ldots \Delta_{x_k}^{a_k} P(0)$ can be computed out of $P(x_1, \ldots, x_k)$ with $0 \leq x_i \leq a_i$, and only such difference sequences appear in the formula, where $\sum_{i=1}^{k} a_i \leq n$. (Here $\Delta_{x_i}^{a_i}$ means taking the a_i-th difference sequence of P with respect to x_i.)

Remark 7.4 As quoted in [Ad, p. 49], D. Welsh proved the exponential upper bound only for prime knots. From this, however, it follows for composite knots as well. The key point in this argument is to notice, that it suffices to see it for alternating composite knots. This is, because the number of crossing changes is exponentially bounded, and it follows from the Tait flyping conjecture [MT] and the fact, that crossing changes commute with flypes, that you can achieve a minimal diagram of any knot by crossing changes in alternating diagrams, such that for each alternating knot you just take *one* alternating diagram. The exponential upper bound for alternating knots follows from the fact for alternating prime knots, which is proved by Welsh, using the additivity of the crossing number for alternating knots under connected sum (which follows from results of Menasco [Me] on the one hand and Kauffman [Ka3], Murasugi [Mu] and Thistlethwaite [Th] on the other hand, see [Ad, §6.2] for details), and some standard combinatorial arguments (e. g., that the partition function [An, p. 70] is subexponentially growing; see e. g. [St10] for details).

Remark 7.5 Theorem 7.10 can also be proved via the Fundamental theorem for braids [Hu]. However, the way via (proving first) the Fundamental theorem is of course more tedious, and theorem 7.16 shows that our arguments appear more universally applicable even for braids.

Question. Which knots have braid representations as in theorem 7.16? Maybe all knots?

7.4 The growth of the number of knots and Vassiliev invariants

Finally, we will try to outline a possible strategy to prove the finite-determination of Vassiliev invariants on all knots, relating their growth to the growth of the number of knots.

Here is a project, which will lead to a very simple proof of the finite-determination of Vassiliev invariants on all knots using braiding sequences.

Conjecture 7.17 *There exists a function $f : \mathbf{N} \to \mathbf{R}$ with*

1) $\lim_{n\to\infty} f(n) = \infty$, $f(n+1) > f(n)$.

2) *In any minimal diagram of a knot of n crossings, there exists a Gousarov scheme [G, Ng, NS] (a family of disjoint sets of crossings in the diagram) N with $\#N \geq f(n)$, such that $\forall S \in N$, after changing of a set of crossings $S' \subset S$ in the diagram, the resulting diagram can be (crossing) reduced by Reidemeister moves, involving crossings of S only.*

Theorem 7.18 *Assume conjecture 7.17 is true. Then Vassiliev invariants are finite-determined on all knots. More exactly, any Vassiliev invariant of degree* $\leq n$ *is uniquely determined by its values on alternating knots with* $\leq f^{-1}(n)+2n$ *crossings. In particular, if in conjecture 7.17 we can choose* $f(n) \geq C \cdot n$ *for some constant* $C > 0$*, then Vassiliev invariants are exponentially bounded in the degree.*

Proof. Consider the braiding sequences, associated to the minimal knot diagrams ordered by the number of crossings. Then choose for a fixed diagram (and its associated braiding sequence) the scheme N, and inserting special values into the variables corresponding to crossings in any set in $S \in N$, you obtain a recursive relation (we need the additional condition to the simplification by Reidemeister moves to ensure that the recursive relation holds for *all* values of the remaining variables!). The rest of the argument is as usual.

Now note, that we only need to consider braiding sequences of diagrams with $\leq f^{-1}(n)$ crossings. Combining the observations in the proof of theorem 7.16 with the one that appropriate sign choice of the arguments of the braiding polynomial corresponds to alternating diagrams only, we deduce the second assertion. For the third assertion, use again the above cited result of D. Welsh. □

One motivatation for this conjecture is to note, that all our previous proofs were basically proofs of special cases of this conjecture, with the difference that we distinguished for any considered knot a set of diagrams and proved that simplification of a distinguished diagram gives again a distinguished diagram, such that the simplification commutes with the other braidings (i. e., the insertions of the braids σ_1^{2k+1} into the distinguished rooms of the braiding sequence, see figure 1). This way we can relax our conjecture.

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Abstract

The subject of the present thesis are combinatorics of chord diagrams and asymptotics of Vassiliev invariants.

In sections 3 and 4 we will derive some (purely) enumerative results on special kinds of chord diagrams. Although not directly related to Vassiliev invariants, these results provide a glimpse of the combinatorial complexity of chord diagrams – already for easily to define properties the enumeration is rather hard and requires additional ideas.

In section 5 we will use combinatorial techniques to relate enumeration of special chord diagrams to a context of Vassiliev invariants and will prove the asymptotical upper bound $D!/1.1^D$ for the number of Vassiliev invariants in the degree D.

In section 6 we will use the techniques of section 5 and the result of Chmutov and Duzhin [CD2] to deduce a lower bound for the number of all Vassiliev invariants and discuss the relation between the asymptotics of prime and all Vassiliev invariants. Parallely, we give a summary on what we know about the asymptotics of Vassiliev invariants.

Finally, in section 7 we use the rahter different approach of braiding sequences to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots.

Parts of this work can be found in several papers of mine [St2, St6, St8, St9, St10].

Zusammenfassung

Der Gegenstand dieser Arbeit ist die Kombinatorik von Sehnendiagrammen und Asymptotik von Vassiliev-Invarianten.

In den Abschnitten 3 und 4 werden wir einige (reine) Abzählresultate über Sehnendiagramme herleiten. Obwohl nicht direkt in Beziehung zu Vassiliev-Invarianten, verdeutlichen sie die kombinatorische Komplexität der Sehnendiagramme – schon für einfache Eigenschaften wird die Abzählung kompliziert und erfordert zusätzliche Ideen.

Im Abschnitt 5 werden wir kombinatorische Techniken benutzen, um Abzählung bestimmter Sehnendiagramme mit Vassiliev-Invarianten in Verbindung zu bringen, und werden eine obere Abschätzung der Anzahl der Vassiliev-Invarianten in Abhängigkeit vom Grad herleiten.

Im Abschnitt 6 werden wir mit Hilfe der Techniken aus Abschnitt 5 und dem Resultat von Chmutov und Duzhin [CD2] eine untere Abschätzung der Anzahl aller Vassiliev-Invarianten herleiten und die Beziehung zwischen der Anzahl der primitiven und aller Vassiliev-Invarianten diskutieren. Parallel dazu werden wir alles, was über Asymptotik von Vassiliev-Invarianten bekannt ist, zusammenfassen.

Im Abschnitt 7 werden wir schließlich mit Hilfe der Methode der Verzopfungsreihen exponentielle obere Schranken für die Anzahl der Vassiliev-Invarianten auf Knoten von beschränktem Zopfindex und arboreszenten Knoten herleiten.

Teile dieser Dissertation können in mehreren Arbeiten von mir [St2, St6, St8, St9, St10] gefunden werden.

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