On the effective cone of algebraic surfaces

Georg Hein

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Abstract

The effective cone of an algebraic surface X may contain infinitely many curves of negative self intersection. We show that at most $\max\{\rho(X)-1, 2\rho(X)-4\}$ of these curves can have trivial intersection with a given nontrivial nef line bundle L. Here $\rho(X)$ denotes the Picard number of X, i.e., the rank of the Néron-Severi group $\mathrm{NS}(X)$.

We end this article with upper bounds (see 3.2 and 3.3) for the number of reducible fibers of surface fibrations over curves.

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1 Introduction

1.1 Notations. Let X be a smooth algebraic surface over an algebraically closed field. We will denote the Néron-Severi group of X by $\mathrm{NS}(X)$, and set $\mathrm{NS}_{\mathbb{Q}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. A Cartier divisor D defines a point in $\mathrm{NS}_{\mathbb{Q}}(X)$. In the same way, a curve C in X defines a point in that vector space. The dimension $\rho(X)$ of the $\mathbb{Q}$-vector space $\mathrm{NS}_{\mathbb{Q}}(X)$ is the Picard number of X. We define the effective cone $\mathrm{NE}(X)$ to be those vectors in $\mathrm{NS}_{\mathbb{Q}}(X)$ which can be written in the form $\sum_i a_i C_i$, where the a_i are nonnegative rational numbers, and the C_i are irreducible curves in X.
The intersection pairing $\mathrm{NS}_{\mathbb{Q}}(X) \times \mathrm{NS}_{\mathbb{Q}}(X) \to \mathbb{Q}$ is of index $(1, \rho(X)-1)$ by the Hodge index theorem. We write $C.D$ for the intersection pairing of C and D, where C and D are two points in $\mathrm{NS}_{\mathbb{Q}}(X)$. Let H be an ample line bundle on X. By definition we have $H.D \geq 0$ for all $D \in \mathrm{NE}(X)$. On the other hand, a line bundle L satisfying $L.D \geq 0$ for all $D \in \mathrm{NE}(X)$ is called a nef line bundle.

1.2 Curves with negative self intersection. If we consider $\mathrm{NS}_{\mathbb{Q}}(X)$ as a topological vector space with the standard topology, then we can show that effective curves C with positive self intersection are in the interior of $\mathrm{NE}(X)$, whereas irreducible curves C with $C^2 \leq 0$ are in the boundary of $\mathrm{NE}(X)$ (cf. 4.4 and 4.5 in [2]). Furthermore, if C is irreducible and of negative self intersection, then it is an edge of $\mathrm{NE}(X)$.
To explain the term edge we consider the affine hyperplane $V_1 := \{D \in NS_{\mathbb{Q}}(X) | D.H = 1\}$ and the convex subset $\mathrm{NE}_1(X) := \mathrm{NE}(X) \cap V_1$. C is an edge means that the point $\frac{1}{C.H}C \in NE_1(X)$ is an edge of the convex set $\mathrm{NE}_1(X)$.
There are surfaces X with infinitely many curves C of negative self intersection. Thus, $\mathrm{NE}(X)$ may have infinitely many edges. The following theorem, which is the main result

of this article, tells us that only finitely many edges can be in a linear boundary component of $\mathrm{NE}(X)$.

1.3 Theorem. *Let (X, H) be a polarized projective surface. If L is a nef line bundle on X which is not numerically trivial, then the number of irreducible curves C satisfying $C^2 < 0$, and $C.L = 0$ cannot exceed* $\max\{\rho(X) - 1, 2\rho(X) - 4\}$.

1.4 The proof of this theorem is given in section 2. Some applications to fibrations of algebraic surfaces are provided in section 3. In corollaries 3.2 and 3.3 we give upper bounds for the number of fiber components and the number of reducible fibers of a fibration $f : X \to Y$ of a surface over a curve. Eventually, we interpret these results in terms of intersection numbers on the moduli space of stable maps (see 3.6).

2 Proof of Theorem 1.3

We will divide the proof into steps. First let us fix the notation. Since L is nontrivial and nef, the intersection $\mathrm{NE}_{L=0}(X) = \{D \in \mathrm{NE}(X) \mid D.L = 0\}$ of the effective cone $\mathrm{NE}(X)$ with the orthogonal hyperplane to L is contained in the boundary $\partial\mathrm{NE}(X)$ of the effective cone. Thus, let $V = \{C_1, C_2, \ldots\}$ be the (possibly infinite) set of irreducible curves on X satisfying $C_i^2 < 0$ and $C.L = 0$. By definition these are points in $\mathrm{NE}_{L=0}(X)$.
Step 1: Non-positive self intersections. Any nontrivial linear combination $D = \sum_{i=1}^{l} \alpha_i C_i$ with $\alpha_i \geq 0$ is contained in $\mathrm{NE}_{L=0}(X)$ and satisfies $D.H > 0$. Thus, if D^2 were positive, then the divisor D would be in the interior of $\mathrm{NE}(X)$ (see Lemma 4.3 in [2]). Since this is a contradiction, we have $D^2 \leq 0$.
Step 2: Linear relations. If we take $\rho(X)$ (or more) different curves, say $C_1, \ldots, C_{\rho(X)}$, of V, then there exists a nontrivial linear relation. This relation may (after permuting indices) be written as $A := \sum_{i=1}^{l} \alpha_i C_i = \sum_{i=l+1}^{\rho(X)-m} \alpha_i C_i =: B$ with all $\alpha_i > 0$, and $m \geq 0$ the number of zero coefficients in the linear relation. We have seen in the first step that $A^2 \leq 0$. On the other hand, the support of A and B consists of different divisors, which implies $A.B \geq 0$. This implies $A.B = A^2 = B^2 = 0$ and eventually the following three statements:
(1) $C_i.C_j = 0$ whenever $i \leq l$, and $l + 1 \leq j \leq \rho(X) - m$;
(2) There exists a pair (i, j) of integers with $1 \leq i < j \leq l$ such that $C_i.C_j > 0$;
(3) There exists a pair (i, j) of integers with $l+1 \leq i < j \leq \rho(X)-m$ such that $C_i.C_j > 0$;
Let now $k > \rho(X) - m$, i.e., C_k is a curve not contained in the support of the divisors A and B. We remark that k can be greater than $\rho(X)$. From step 1 we know that
$(A + \varepsilon \cdot C_k)^2 = 2\varepsilon \sum_{i=1}^{l} \alpha_i C_i.C_k + \varepsilon^2 C_k^2$ has to be less than or equal to zero for all $\varepsilon \geq 0$. This implies that $C_i.C_k = 0$, for all $i = 1, \ldots, l$. An analogous consideration of $(B+\varepsilon \cdot C_k)^2$ yields the fourth statement:
(4) $C_i.C_j = 0$ whenever $i \leq \rho(X) - m$, and $j > \rho(X) - m$.
Step 3: Translation to a problem in graph theory. We consider the graph Γ with vertex set V. Two vertices C_i and C_j are adjacent, iff $C_i.C_j > 0$. We have seen in the second step that this graph fulfills the following condition $(*)$:
Any subset $M \subset V$ with $\rho(X)$ elements can be separated into three disjoint subsets $M = M' \cup M'' \cup M'''$ such that:
(1) There exists no adjacency between the three sets M', M'', and $V \setminus (M' \cup M'')$;
(2) There exists an adjacent pair of vertices in M', as well as in M''.
Step 4: Any graph Γ satisfying $(*)$ has at most $\max\{\rho(X)-1, 2\rho(X)-4\}$ vertices.
Condition $(*)$ is trivially satisfied for graphs with less than $\rho(X)$ vertices. So let us assume

that Γ has at least $\rho(X)$ vertices. We consider the connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_l$ of the Graph Γ. We denote the number of vertices of Γ_i by γ_i. We assume that the γ_i form a decreasing sequence. Let k be the biggest integer, such that $\gamma_k \geq 2$. Condition $(*)$ can be interpreted in the form that each subset of V containing $\rho(X)$ elements contains at least two of the Γ_i with $i \leq k$. We consider the set M_1 of all vertices of Γ_0 and one vertex of each of the remaining connected components. Since M_1 does not contain two connected components we follow from our condition $(*)$ that its cardinality is at most $\rho(X) - 1$.

$$\gamma_0 + l \quad \leq \quad \rho(X) - 1 \tag{1}$$

Now we consider the set M_2 of all vertices of Γ_0 and $\gamma_i - 1$ vertices of Γ_i for $i = 1, \ldots, k$. As before, by construction M_2 does not contain two of the components Γ_i for $i = 0, \ldots, k$ which gives

$$\gamma_0 + \sum_{i=1}^{k} (\gamma_i - 1) \quad \leq \quad \rho(X) - 1 \tag{2}$$

If γ denotes the cardinality of V, then we obtain from (1) and (2)

$$\gamma_0 + \gamma \leq 2\rho(X) - 2\,.$$

Having in mind that $\gamma_0 \geq 2$ this yields the claimed inequality $\gamma \leq 2\rho(X) - 4$. $\square$

3 Applications to fibrations of surfaces

3.1 Reducible fibers in fibrations. Let $f : X \to Y$ be a fibration of a smooth projective surface X over a smooth algebraic curve Y. After passing to the Stein factorization, we may assume that f has connected fibers. Let L be the pull back of an ample line bundle on Y. L is a nontrivial nef line bundle on X. An irreducible curve $C \subset X$ which satisfies $C^2 < 0$ and $C.L = 0$ is a connected component of a reduced fiber which is not irreducible (cf. §III Lemma 8.2 in [1]). Therefore, we call these curves C proper fiber components. As a direct consequence of theorem 1.3 we obtain:

3.2 Corollary. *Using the notations of 3.1, we have that the number of proper fiber components of the fibration $f : X \to Y$ is at most $2\rho(X) - 4$.*

Proof: Since $2\rho(X) - 4 = \max\{\rho(X) - 1, 2\rho(X) - 4\}$ unless $\rho(X) \in \{1, 2\}$ we have to consider only these two cases by theorem 1.3. If $\rho(X) = 1$, then we have no fibration. If $\rho(X) = 2$, then all fibers are obviously irreducible. $\square$

Taking into account that for each point $y \in Y$ such that X_y^{red} is reducible we have at least two proper fiber components we obtain the next result.

3.3 Corollary. *Let $f : X \to Y$ be a fibration of a smooth projective surface X over a smooth algebraic curve Y with connected fibers. The number of points $y \in Y$ such that X_y^{red} is reducible is at most $\rho(X) - 2$.*

3.4 Remark. Analogously, we may deduce that each fiber of f has less than $\rho(X)$ proper fiber components. However, this might be directly deduced from Zariski's lemma (III.8.2 in [1]).

It should be remarked that condition $(*)$ also excludes graphs with less than $2\rho(X) - 4$ vertices. So we can deduce, for example, that a surface X fibered over a curve Y with m fibers with at least n irreducible components, has Picard number $\rho(X) \geq m \cdot n - m + 2$.

3.5 Notations. Corollary 3.2 can be interpreted as an intersection result on Kontsevich's moduli space $\bar{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps of algebraic curves of arithmetic genus g with n marked points to a target space X. The universal curve over $\bar{\mathcal{M}}_{g,n}(X,\beta)$ is denoted by $\mathcal{C}_{g,n}$. Moreover we have natural morphisms $\bar{\mathcal{M}}_{g,n}(X,\beta) \xleftarrow{q} \mathcal{C}_{g,n} \xrightarrow{p} X$. See [5] and [3] for the definition of $\bar{\mathcal{M}}_{g,n}(X,\beta)$ and $\mathcal{C}_{g,n}$.
We consider the divisor $D_{\text{reducible}} \subset \bar{\mathcal{M}}_{g,n}(X,\beta)$ of stable maps of reducible curves to X. If $g=0$, then $D_{\text{reducible}}$ is the boundary divisor $\big(\bar{\mathcal{M}}_{g,n}(X,\beta) \setminus \mathcal{M}_{g,n}(X,\beta)\big)$ of stable maps of singular curves to X.

3.6 Corollary. *Let $\psi : Y \to \bar{\mathcal{M}}_{g,n}(X,\beta)$ be morphism of smooth projective algebraic curve to $\mathcal{M}_{g,n}(X,\beta)$ which is transversal to $D_{\text{reducible}}$. If $Z := Y \times_{\mathcal{M}_{g,n}(X,\beta)} \mathcal{C}_{g,n}$ is an algebraic surface, then we have* $\deg(\psi^* D_{\text{reducible}}) \leq \rho(Z) - 2$.

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Georg Hein, Freie Universität Berlin, Institut für Mathematik II, Arnimallee 3, D-14195 Berlin, Germany, ghein@math.fu-berlin.de