Interlace Polynomials

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1. Introduction

In a recent paper Arratia, Bollobás and Sorkin discussed a graph polynomial which they called the *interlace polynomial* $q(G)$. To define $q(G)$ we need the *switching operation along* an edge $uv \in E(G)$. Let $A, B, C \subseteq V(G) \setminus \{u, v\}$ be the sets of vertices adjacent to u but not to v, to v but not to u, and to both u and v, respectively (see figure 1). Then $G^{(uv)}$ is the graph obtained

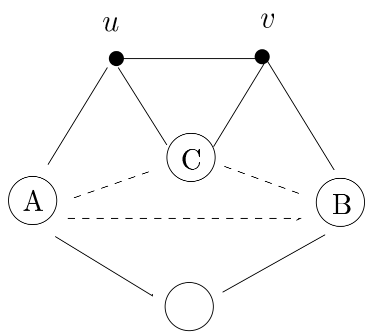

Figure 1

from G by exchanging edges and non–edges between any two different sets from A, B, C, keeping the rest of the graph unchanged (including the edges within A, B and C). The following figure 2 shows an example:

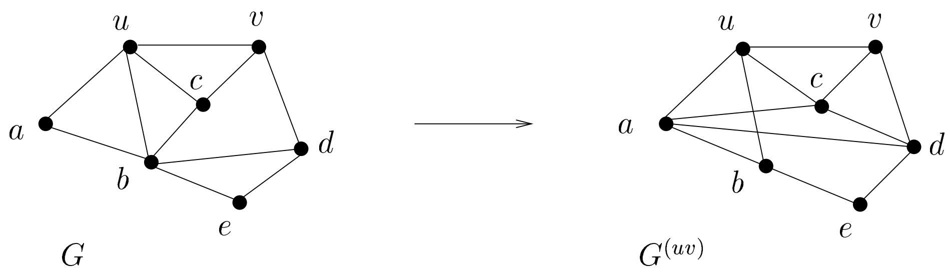

Figure 2

The interlace polynomial $q(G)$ of a *simple* graph G is now defined recursively as follows:

$$\begin{aligned} &\text{i)}\ \ q(E_n, x) = x^n \text{ where } E_n \text{ is the edgeless graph on } n \text{ vertices} \\ &\text{ii)}\ \ q(G, x) = q(G \setminus u, x) + q(G^{(uv)} \setminus v, x) \text{ for } uv \in E(G). \end{aligned} \tag{1}$$

Arratia, Bollobás and Sorkin show in [1] that $q(G, x)$ is well–defined, that is, it is independent of the sequence of edge–removals. They give several interesting results on $q(G, x)$ with applications to the Alexander polynomial, and state the conjecture that $|q(G, -1)|$ is always a power of 2.

Examples. i) $q(K_n, x) = 2^{n-1}x$, ii) $q(K_{1,n}, x) = 2x + x^2 + \ldots + x^n$, and more generally, $q(K_{m,n}, x) = (1 + x + \ldots + x^{m-1})(1 + x + \ldots + x^{n-1}) + x^m + x^n - 1$.

We derive in section 2 a formula for $q(G, x)$ in terms of the adjacency matrix of G, thereby reproving the independence of the order of removal of edges. Then we look at the evaluation of $q(G, x)$ at $x = 1$ and $x = -1$, proving the conjecture for $q(G, -1)$, and discuss some further results for trees. In section 3 we show that the interlace polynomial $q(G)$ of a bipartite graph equals the symmetric Tutte polynomial of a certain binary matroid associated with G. In section 4 we discuss another interlace polynomial $Q(G, x)$. Section 5 shows how a linear algebra approach can be used to provide further interesting evaluations which are difficult to obtain (if at all) by the basic recursion. Finally, in section 6 we show that $q(G, x)$ and $Q(G, x)$ arise as the Martin polynomials $m(\mathcal{S}, x)$ and $M(\mathcal{S}, x)$ of a certain isotropic system $\mathcal{S}$ (as introduced by Bouchet [2]). Thus the results of [1] and in the present paper can be found within the theory outlined by Bouchet in several papers [2,3,4]. It appears, however, worthwhile to look at these polynomials via the recursive definition (1) and via the approach in section 5, since many of the proofs become simpler and more transparent.

2. The interlace polynomial $q(G, x)$

Let $G = (V, E)$ be a simple graph on $V = \{1, \ldots, n\}$, A the adjacency matrix of G, and I_n the identity matrix. Henceforth all matrices will be considered as matrices over $GF(2)$. Let L be the $(n \times 2n)$–matrix

$$L = (\underset{1\ldots n}{A} \mid \underset{\overline{1}\ldots\overline{n}}{I_n})$$

where we label the rows by $1, \ldots, n$ and the columns by $1, \ldots, n; \overline{1}, \ldots, \overline{n}$. We say that a column–set S is *admissible* if $|S \cap \{i, \overline{i}\}| = 1$ for all i, thus $|S| = n$. Let L_S be the $(n \times n)$–submatrix of L with column–set S. We denote by rk$M$ the *rank* of a matrix M.

Theorem 1. *We have*

$$q(G, x) = \sum_S (x-1)^{co(L_S)} \tag{2}$$

where the sum extends over all admissible column–sets S, and $\operatorname{co}(L_S)$ *is the corank of L_S.*

Proof. For $G = E_n$ we have $L = (O_n | I_n)$ and so $\operatorname{co}(L_S) = |S \cap \{1, \ldots, n\}|$. Hence $\sum\limits_S (x-1)^{\operatorname{co}(L_S)} = \sum\limits_{k=0}^{n} \binom{n}{k} (x-1)^k = x^n$. Suppose w.l.o.g. $(n-1)n \in E(G)$, then we have to verify the recursion (1) for the right–hand side of (2).

Case i) $\overline{n} \in S$. Let $S' = S \setminus \overline{n}$, then the matrix L_S looks as follows:

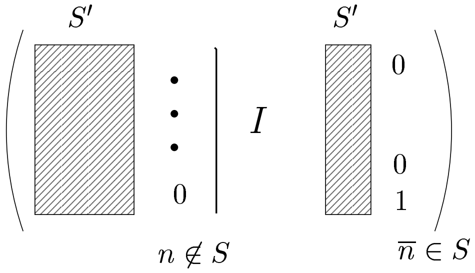

Figure 3

Clearly, $\operatorname{rk}(L_S) = \operatorname{rk}(L_{S'}) + 1$, where $L_{S'}$ is the submatrix with the n–th row removed. Thus $\operatorname{co}(L_{S'}) = \operatorname{co}(L_S)$ and we obtain by induction:

$$\sum_{S:\overline{n} \in S} (x-1)^{\operatorname{co}(L_S)} = \sum_{S'} (x-1)^{\operatorname{co}(L_{S'})} = q(G \setminus n, x).$$

Case ii) $n \in S$. We write L as

$$\begin{array}{cc} L = & \left(\begin{array}{ccc|ccc} B & c_1 & c_2 & I & 0 & 0 \\ c_1^T & 0 & 1 & 0^T & 1 & 0 \\ c_2^T & 1 & 0 & 0^T & 0 & 1 \end{array} \right) \\ & \begin{array}{cccccc} & n-1 & n & & \overline{n-1} & \overline{n} \end{array} \end{array}$$

and multiply L from the left by the matrix C

$$C = \begin{pmatrix} I & c_2 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since C is *non–singular*, all coranks are preserved, and we obtain

$$CL = \begin{array}{c} \left(\begin{array}{ccc|ccc} B + c_1 c_2^T + c_2 c_1^T & 0 & 0 & I & c_2 & c_1 \\ c_1^T & 0 & 1 & 0 & 1 & 0 \\ c_2^T & 1 & 0 & 0 & 0 & 1 \end{array}\right) \\ \begin{array}{cccccc} & n-1 & n & & \overline{n-1} & \overline{n} \end{array} \end{array}.$$

It is easily seen that $B + c_1 c_2^T + c_2 c_1^T$ is precisely the adjacency matrix of $G^{(n-1,n)}$ on $V \setminus \{n-1, n\}$. Interchanging columns $n-1 \longleftrightarrow \overline{n-1}$, $n \longleftrightarrow \overline{n}$ and rows $n-1 \longleftrightarrow n$ yields

$$\begin{array}{c} \left(\begin{array}{ccc|ccc} B + c_1 c_2^T + c_2 c_1^T & c_2 & c_1 & I & 0 & 0 \\ c_2^T & 0 & 1 & 0 & 1 & 0 \\ c_1^T & 1 & 0 & 0 & 0 & 1 \end{array}\right) \\ \begin{array}{cccccc} & \overline{n-1} & \overline{n} & & n-1 & n \end{array} \end{array}.$$

Hence by the same argument as in case i) we find

$$\sum_{S: n \in S} (x-1)^{\mathrm{co}(L_S)} = q(G^{(n-1,n)} \setminus (n-1), x),$$

and the proof is complete, $\square$

Formula (2) can be rewritten in a more convenient way. We clearly have $\mathrm{co}(A_T) = \mathrm{co}(L_S)$ where $T = S \cap \{1, \dots, n\}$, and A_T is the corresponding $T \times T$–submatrix of A. Since A_T is a principal submatrix of A we conclude:

Corollary 1. *We have*

$$q(G,x) = \sum_{T\subseteq\{1,\dots,n\}} (x-1)^{co(A_T)}, \tag{3}$$

where A_T is the adjacency matrix of the subgraph induced by T, and where we define $\mathrm{co}(\varnothing) = 0$.

The Corollary permits some interesting evaluations of $q(G,x)$. Let us first look at $x = 1$. By (3)

$$q(G,1) = \#\{T \subseteq \{1,\dots,n\} : \mathrm{rk}(A_T) = |T|\}$$

or equivalently

$$q(G,1) = \#\{T : \det A_T = 1\}. \tag{4}$$

Let H be any graph and B its adjacency matrix. It is well–known that $\det B = 0$ if H has an odd number of vertices (always over $GF(2)$). On the other hand, if $|V(H)|$ is even, then the *Pfaffian* $\mathrm{Pf}(B)$ counts the number of perfect matchings in H, and we have (see e.g. [6]):

$$\det B = (\mathrm{Pf}(B))^2.$$

Hence we obtain the following corollary:

Corollary 2. *We have*

$q(G,1) =$ *# induced subgraphs of G with an odd number of perfect matchings (including the empty set).*

Since a forest has at most one perfect matching, this yields in particular:

Corollary 3. *For a forest G, $q(G,1)$ counts the number of matchings in G (including the empty matching).*

Example. i) $q(K_n,x) = 2^{n-1}x$, $q(K_n,1) = 2^{n-1}$, and it is precisely the 2^{n-1} (complete) subgraphs K_{2h} on an even number of vertices that have an odd number $1\cdot 3\cdot 5\cdots(2h-1)$ of perfect matchings.
ii) $q(K_{m,n},1) = mn+1$. Since $K_{h,h}$ has $h!$ perfect matchings, we see that only the mn individual edges have an odd number of perfect matchings (plus $\varnothing$).

We come next to the evaluation at $x = -1$. Another proof will be given in section 5.

Theorem 2. *We have*

$$q(G, -1) = (-1)^n (-2)^{co(A+I_n)} \tag{5}$$

where A is the adjacency matrix of G and I_n is the identity matrix.

Proof. Set $M = A + I_n$. For the edgeless graph $G = E_n$ we have $M = I_n$, thus $\mathrm{co}(M) = 0$ in agreement with $x^n|_{x=-1} = (-1)^n$.
Suppose $(n-1)n \in E(G)$. By recursion (1)

$$q(G, -1) = q(G \setminus n, -1) + q(G^{(n-1,n)} \setminus (n-1), -1).$$

Using the same notation as in the proof of Theorem 1 with $C = B + I_{n-2}$, we have to consider the matrices

$$M = \begin{pmatrix} C & c_1 & c_2 \\ c_1^T & 1 & 1 \\ c_2^T & 1 & 1 \end{pmatrix}, \quad M' = \begin{pmatrix} C & c_1 \\ c_1^T & 1 \end{pmatrix}, \quad M'' = \begin{pmatrix} C + c_1 c_2^T + c_2 c_1^T & c_2 \\ c_2^T & 1 \end{pmatrix}.$$

It is easily checked that for these matrices exactly one of the following cases holds:

i) $\mathrm{co}(M) = \mathrm{co}(M') + 1 = \mathrm{co}(M'') + 1$

ii) $\mathrm{co}(M) = \mathrm{co}(M') = \mathrm{co}(M'') - 1$

iii) $\mathrm{co}(M) = \mathrm{co}(M'') = \mathrm{co}(M') - 1$.

Now by induction $q(G \setminus n, -1) = (-1)^{n-1}(-2)^{\mathrm{co}(M')}$,

$$q(G^{(n-1,n)} \setminus (n-1), -1) = (-1)^{n-1}(-2)^{\mathrm{co}(M'')}.$$

Hence in case i) we obtain

$$q(G, -1) = 2(-1)^{n-1}(-2)^{\mathrm{co}(M)-1} = (-1)^n(-2)^{\mathrm{co}(M)}$$

and in cases ii) or iii)

$$q(G, -1) = (-1)^{n-1}[(-2)^{\mathrm{co}(M)} + (-2)^{\mathrm{co}(M)+1}] = (-1)^n(-2)^{\mathrm{co}(M)},$$

as claimed. $\square$

Let us make a few general remarks about the interlace polynomial.

1. Since $G^{(uv)(uv)} = G$, it follows from (1) that

$$q(G^{(uv)}, x) = q(G, x) \quad \text{for all edges} \quad uv. \tag{6}$$

Let us say that two graphs G and H are *equivalent*, $G \approx H$, if one can be obtained from the other by a sequence of edge–switchings. Let $[G] = \{H : H \approx G\}$ be the switch–class of G. Thus $q(H, x) = q(G, x)$ for all $H \in [G]$.

2. If G is not connected, then $q(G, x) = \prod_{i=1}^{t} q(G_i, x)$ where G_i are the components of G. Furthermore, all coefficients q_i of $q(G, x)$ are non–negative integers, and $t = \#$ components is the smallest index h with $q_h > 0$. Again this is clear from (1).

3. Next we look at the degree of $q(G, x)$. Denote by $\alpha(G)$ the independence number of G.

Proposition 1. *We have*

$$\deg q(G, x) = \max_{H \in [G]} \alpha(H). \tag{7}$$

Proof. Suppose $H \in [G]$ and U is a maximum independent set, $|U| = \alpha(H)$. Since the adjacency matrix A_U is the all–zero matrix, we have $\mathrm{co}(A_U) = \alpha(H)$, and hence $\deg q(H, x) \geq \alpha(H)$, for all $H \in [G]$. Now choose $H \in [G]$ with $\alpha(H) \geq \alpha(K)$ for all $K \in [G]$. If $n = |V(H)| = 1$ or 2, then (7) is true. We proceed by induction on n. We have to show $\alpha(H) \geq \deg q(G, x) = \deg q(H, x)$, where we may assume H to be connected. Choose an edge $uv \in E(H)$, then

$$q(H, x) = q(H \setminus u, x) + q(H^{(uv)} \setminus v, x).$$

By induction, $\deg q(H \setminus u, x) = \alpha(K \setminus u)$ with $K \approx H$, and hence by the maximality of $\alpha(H)$

$$\deg q(H \setminus u, x) = \alpha(K \setminus u) \leq \alpha(K) \leq \alpha(H).$$

Similarly, we find $\deg q(H^{(uv)} \setminus v, x) \leq \alpha(H)$, and thus $\deg q(H, x) \leq \alpha(H)$. $\square$

Proposition 2. *If G is a forest, then* $\deg q(G,x) = \alpha(G)$.

Proof. We may assume that G is a tree. Let u be a leaf of G and v its neighbor. Then $G^{(uv)} = G$ and thus

$$q(G,x) = q(G \setminus u, x) + q(G \setminus v, x).$$

The following figure explains the proof:

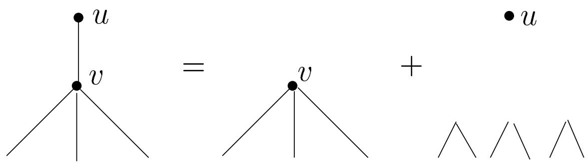

Figure 4

We have $\alpha(G \setminus u)$, $\alpha(G \setminus v) \leq \alpha(G)$. Any maximum independent set U of G contains either u or v. If $v \in U$, then $\alpha(G \setminus u) = \alpha(G)$ and hence by induction $\deg q(G \setminus u, x) = \alpha(G)$. If, on the other hand, $u \in U$, then $\alpha(G \setminus v) = \alpha(G)$, and we find $\deg q(G \setminus v, x) = \alpha(G)$. $\square$

Note that we have also proved on the way that for forests G the highest coefficient q_α equals the *number* of maximum independent sets.

4. Let $q(G,x) = q_1 x + q_2 x^2 + \ldots + q_d x^d$. In general, it appears to be difficult to say something substantial about the coefficients, except that $q_t > 0$, $q_{t+1} > 0, \ldots, q_d > 0$ where $t = \#$ components. But for trees we can say more. Looking at figure 4 we find with a little work

$q_1 =$	2	$(n \geq 2)$
$q_2 =$	$2i - 1$, where i is the number of non–leaves	$(n \geq 3)$
$q_3 =$	$i^2 - 2d_2' - 4d_2''$, where d_2' is the number of vertices of degree 2 which are adjacent to some leaf, and d_2'' the number of the remaining vertices of degree 2.	$(n \geq 4)$

5. If H is an *induced* subgraph of G, then Theorem 1 immediately implies that $q(H,x) \leq q(G,x)$ meaning that for corresponding coefficients $q_i(H) \leq q_i(G)$ holds.

As an example, suppose that G is connected and non–bipartite. Then G contains an induced odd cycle C_k of length $k \geq 3$. Since any odd cycle can be tranformed by a series of edge–switchings into a graph containing triangles (see figure 5), it follows that

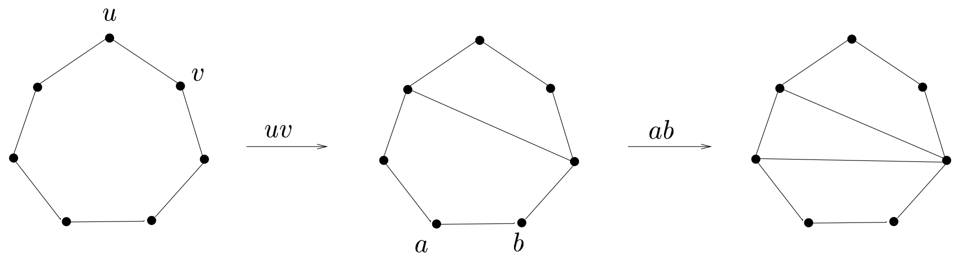

Figure 5

$q_1 \geq 4$ since $q(K_3, x) = 4x$.

Now (1) implies by induction that for any connected graph with at least two vertices, the linear coefficient q_1 is always even. Hence $q_1 = 2$ implies that G is *connected* and *bipartite*.

6. Another interesting question concerns invariants of a switching class $[G]$. Apart form $|V(G)|$ and $q(G, x)$ we have seen that the number of components is one such invariant. A more interesting invariant is *bipartiteness*. More precisely, if G is connected and bipartite with color classes containing r resp. s vertices, then any graph $H \in [G]$ has the same property, since all switching operations occur between the color classes.

This settles also the natural question: "Are two trees with the *same* interlace polynomial equivalent?" in the negative. Figure 6 shows the smallest example:

Figure 6

Both trees have interlace polynomial $2x + 7x^2 + 8x^3 + 6x^4 + 2x^5$. But they cannot be equivalent since the bipartition of the first tree is $(4, 4)$, while it is $(5, 3)$ for the second tree.

3. Bipartite graphs and the Tutte polynomial

The last remark in the previous section suggests that bipartite graphs play a special role, and this is indeed the case. Suppose the bipartite graph G has color–classes $R = \{1, \ldots, r\}$, $S = \{r+1, \ldots, r+s\}$. We consider the so–called shortened adjacency matrix

$$A = (a_{ij}) = \begin{array}{c c} & \begin{array}{ccc} r+1 & \ldots & r+s \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ r \end{array} & \left(\begin{array}{ccc} 1 & & \\ & & 1 \\ & 0 & \end{array} \right) \end{array}$$

where $a_{ij} = 1$ or 0 depending on whether i is adjacent to j or not. We associate to A the binary matroid $\mathcal{M}$ generated by the rows of the $r \times (r+s)$–matrix N

$$N = \left(\underset{1 \,\ldots\, r}{I_r} \;\middle|\; \underset{r+1 \ldots r+s}{A} \right)$$

Theorem 3. *If G is bipartite on $R = \{1, \ldots, r\}$, $S = \{r+1, \ldots, r+s\}$, then*

$$q(G, x) = T_{\mathcal{M}}(x, x), \tag{8}$$

where $T_{\mathcal{M}}$ is the Tutte polynomial of $\mathcal{M}$.

Proof. Let $e_1, e_2, \ldots, e_{r+s}$ be the elements of $\mathcal{M}$ corresponding to the columns of N. If G has no edges, then $N = (\,I \,|\, O\,)$. Every e_i is a loop or coloop, whence $T_{\mathcal{M}}(x, x) = x^{r+s}$. Now suppose $1, (r+1) \in E(G)$, then we have to verify recursion (1) for $T_{\mathcal{M}}(x, x)$. We write N in the form

$$N = \begin{array}{c} \begin{array}{ccccc} 1 & \ldots & r & & r+1 \ldots r+s \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0^T & 1 & b_1^T \\ 0 & I_{r-1} & b_2 & B \end{array} \right). \end{array}$$

Clearly, e_1 is not a loop of $\mathcal{M}$, and it is also not a coloop since the first column is the sum of column $r+1$ and an appropriate subset of the columns $2, \ldots, r$. For the Tutte polynomial we have the recursion

$$T_{\mathcal{M}}(x,x) = T_{\mathcal{M}/e_1}(x,x) + T_{\mathcal{M}\setminus e_1}(x,x),$$

and we now verify $q(G \setminus 1, x) = T_{\mathcal{M}/e_1}(x,x)$, $q(G^{(1,r+1)} \setminus (r+1), x) = T_{\mathcal{M}\setminus e_1}(x,x)$, which will prove our result.

The contraction $\mathcal{M}/e_1$ is generated by the matrix

$$N' = (\, I_{r-1} \,|\, b_2 \;\, B),$$

whence $q(G \setminus 1, x) = T_{\mathcal{M}/e_1}(x,x)$ by induction.

To treat the deletion $\mathcal{M} \setminus e_1$, we multiply N on the left by the *non–singular* matrix $\left(\begin{smallmatrix} 1 & 0^T \\ b_2 & I_{r-1} \end{smallmatrix}\right)$. This gives

$$N'' = \left(\begin{array}{cc|cc} 1 & 0^T & 1 & b_1^T \\ b_2 & I_{r-1} & 0 & B + b_2 b_1^T \end{array} \right),$$

and it is immediately verified that $B + b_2 b_1^T$ corresponds to $G^{(1,r+1)}$ on $\{2, \ldots, r; r+2, \ldots, r+s\}$. $\mathcal{M}$ is again generated by the matrix N''. Now $\mathcal{M} \setminus e_1$ is generated by the matrix

$$\left(\begin{array}{cc|c} 1 & 0^T & b_1^T \\ 0 & I_{r-1} & B + b_2 b_1^T \end{array} \right)$$

(after moving column $r+1$ to the front), and we obtain by induction $q(G^{(1,r+1)} \setminus (r+1), x) = T_{\mathcal{M}\setminus e_1}(x,x)$. $\square$

The results of section 2 can be reproved quite easily for bipartite graphs using Theorem 3. On the other hand, section 2 can be used to provide insights into the Tutte polynomial $T_{\mathcal{M}}(x,x)$. As an example consider the evaluation at $x = -1$. The matrix M of section 2 is $M = \left(\begin{smallmatrix} I & A \\ A^T & I \end{smallmatrix}\right)$. Hence $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ is in the nullspace of M if and only if $a = Ab$, $b = A^T a$. On the other hand, considering the matrix N we find $a^T N = (a^T, a^T A = b^T)$, and $a = Ab$ is equivalent to $N \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = 0$. Thus $\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ is in the nullspace of M if and only if (a^T, b^T) is in the *bicycle space* $\mathcal{C} \cap \mathcal{C}^{\perp}$ of $\mathcal{M}$, where $\mathcal{C}$ is the row space (cycle space) of $\mathcal{M}$ and $\mathcal{C}^\perp$ the cocycle space. This gives the theorem of Read–Rosenstiehl [11]: $T_{\mathcal{M}}(-1,-1) = (-1)^n(-2)^{\dim(\mathcal{C}\cap\mathcal{C}^\perp)}$.

4. The interlace polynomial $\mathbf{Q(G, x)}$

Let us first recall the *switch operation* at a vertex u. The graph $G * u$ is obtained from G by interchanging edges $\longleftrightarrow$ non–edges in the neighborhood of u. We trivially have $G * u * u = G$. If u and v are not adjacent, then $G*u*v = G*v*u$. Furthermore, it is well–known that for adjacent vertices u, v (see [6])

$$G * u * v * u = G * v * u * v, \tag{9}$$

where the operations are always read from left to right. If H is a graph, then H_{uv} is the graph obtained by swapping the labels $u \longleftrightarrow v$. It is now easily checked that

$$G * u * v * u = (G^{(uv)})_{uv}, \quad uv \in E(G). \tag{10}$$

Recursion (1) becomes therefore

$$q(G, x) = q(G \setminus u, x) + q(G * u * v * u \setminus u, x).$$

Now we define the polynomial $Q(G, x)$ by a 3–term recursion:

i) $Q(E_n, x) = x^n$

ii) $Q(G, x) = Q(G \setminus u, x) + Q(G * u \setminus u, x) + Q(G * u * v * u \setminus u, x), \quad (11)$
if $uv \in E(G)$, or equivalently

$$Q(G, x) = Q(G \setminus u, x) + Q(G * u \setminus u, x) + Q(G^{(uv)} \setminus v, x).$$

Example. $Q(K_2, x) = 3x$, $Q(K_3, x) = 6x + x^2$, $Q(K_4, x) = 12x + 2x^2 + x^3$.

We want to show that $Q(G, x)$ is independent of the order of removal of edges. Let A be the adjacency matrix of G and consider the $(n \times 3n)$–matrix L

$$L = (\ A \mid I_n \mid A + I_n\ ),$$

where the columns are indexed by $1, \dots, n; \overline{1}, \dots, \overline{n}; \bar{\bar{1}}, \dots, \bar{\bar{n}}$. A column–set S is *admissible* if $|S \cap \{i, \bar{i}, \bar{\bar{i}}\}| = 1$ for all i, thus $|S| = n$. Let us denote by L_S the $n \times n$–submatrix of L with column–set S. The following proof proceeds along the same lines as that of Theorem 1 and is omitted.

Theorem 4. *We have*

$$Q(G, x) = \sum_S (x-2)^{co(L_S)},$$

where S extends over all admissible column–sets.

Corollary 4. *We have*

i) $Q(G * u, x) = Q(G, x)$ *for all* $u \in V(G)$

ii) $Q(G^{(uv)}, x) = Q(G, x)$ *for all* $uv \in E(G)$.

Proof. If u is an isolated vertex, then $G * u = G$, and there is nothing to show. Otherwise, let $uv \in E(G)$. By (9) and (11)

$$Q(G, x) = Q(G \setminus u, x) + Q(G * u \setminus u, x) + Q(G * v * u * v \setminus u, x).$$

On the other hand,

$$Q(G * u, x) = Q(G * u \setminus u, x) + Q(G \setminus u, x) + Q(G * u * u * v * u \setminus u, x).$$

Since by induction

$$Q(G * v * u \setminus u, x) = Q((G * v * u \setminus u) * v, x),$$

and clearly $G * v * u * v \setminus u = (G * v * u \setminus u) * v$, claim i) follows. The proof of ii) is similar. $\square$

Let us look at the evaluation of $Q(G, x)$ at $x = 2$. By Theorem 4

$$Q(G, 2) = \#\{S : \det L_S = 1\}.$$

The result of Corollary 1 now carries over with the following modification. Let $H = G[T]$ be the induced subgraph on T. With H we consider all subgraphs with possible loops attached to the vertices, and call all these subgraphs induced. So if $|T| = k$, there are altogether 2^k (general) induced subgraphs on T. A (general) *perfect matching* of H is now a perfect matching where we also allow loops to be part of the matching. The following result is now again proved by considering the Pfaffian.

Corollary 5. *We have*

$Q(G, 2) =$ *# (general) induced subgraphs with an odd number of (general) perfect matchings.*

Example. $Q(K_4, x) = 12x + 2x^2 + x^3$, thus $Q(K_4, 2) = 40$. The induced subgraphs with an odd number of perfect matchings are:

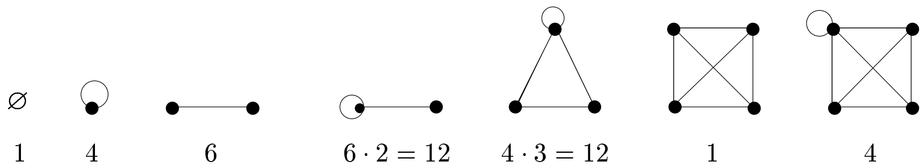

Figure 7

Another interesting evaluation which can be shown using (11) and the switch operation occurs at $x = 4$. It will be proved in the next section. An *Eulerian graph* is one in which all degrees are even.

Theorem 5. *We have*

$$Q(G, 4) = 2^n \cdot \text{ (\# induced Eulerian subgraphs).}$$

In particular, if G is a forest, then the induced Eulerian subgraphs are just the independent sets.

Corollary 6. *If G is a forest, then*

$$Q(G, 4) = 2^n \cdot \text{ (\# independent sets).}$$

Example. Let P_n be the path on n vertices. Solving recursion (11) we find

$$\begin{aligned} Q(P_{2m}, x) &= \sum_{k=0}^{m-1} \frac{2m+4k+1}{m+k} 2^{2k} \binom{m+k}{2k+1} x^{m-k} \\ Q(P_{2m+1}, x) &= \sum_{k=0}^{m} \frac{m+2k}{m+k} 2^{2k} \binom{m+k}{2k} x^{m+1-k}. \end{aligned}$$

It is easily seen that the number of independent sets in P_n is precisely the Fibonacci number F_{n+2}. Hence Corollary 6 gives the formulae

$$\left.\begin{aligned} F_{2m} &= \sum_{k=0}^{m-2} \tfrac{2m+4k-1}{m+k-1}\tbinom{m+k-1}{2k+1} \\ F_{2m+1} &= \sum_{k=0}^{m-1} \tfrac{2m+4k-2}{m+k-1}\tbinom{m+k-1}{2k} \end{aligned}\right\} \quad (m \geq 2).$$

5. A linear algebra look at the interlace polynomials

Let us consider the interlace polynomial $q(G, x)$ in the form (3) of Corollary 1. Set $V = \{1, 2, \ldots, n\}$, then

$$q(G, x) = \sum_{T \subseteq V} (x-1)^{\mathrm{co} A_T}.$$

We will see how easy matrix manipulations (as always over $GF(2)$) yield some further interesting evaluations of $q(G, x)$. We begin with some simple observations.

1. *Let B be a symmetric matrix, then* $\mathrm{Im}B = (\mathrm{Ker}B)^{\perp}$, $\mathrm{Ker}B = (\mathrm{Im}B)^{\perp}$.

If $y \in \mathrm{Ker}B$, $z = Bw \in \mathrm{Im}B$, then $z^T y = w^T B y = 0$, and hence $\mathrm{Im}B \subseteq (\mathrm{Ker}B)^{\perp}$. But since the subspaces $\mathrm{Im}B$ and $(\mathrm{Ker}B)^{\perp}$ have the same dimension, they are, in fact, equal.

For $y \in GF(2)^n$, denote by $\|y\|$ the *support* of y, that is $\|y\| = \{i \in V : y_i = 1\}$.

2. *If A is the adjacency matrix of any graph G, then $y^T A y = 0$ for all y.*

Let $R = \|y\|$, then $y^T A y = \sum\limits_{i,j \in R} a_{ij} = 2 \cdot (\#\ \text{edges in}\ \ G[R])$, hence $y^T A y = 0$.

3. It follows from 2) that $y^T (A + I) y = y^T y$, and thus $y^T y = 0$ if $y \in \mathrm{Ker}(A + I)$.

4. *We have* $\mathbf{1} \in \mathrm{Im}(A + I)$ *where* $\mathbf{1}$ *is the all–ones vector.*

By 3), $\mathbf{1}$ is orthogonal to all $y \in \mathrm{Ker}(A+I)$, and hence $\mathbf{1} \in \mathrm{Im}(A+I)$ by 1).

Lemma 1. *We have* $\mathrm{co}A_T \equiv |T| \pmod 2$ *for any* $T \subseteq V$.

Proof. If $|T| = 0$, then $\mathrm{co}A_{\varnothing} = 0$, and for $|T| = 1$ we have $\mathrm{co}A_T = 1$. Now we proceed by induction on $|T|$. Consider $N = A_T$ and

$$N' = \begin{pmatrix} 0 & c^T \\ c & N \end{pmatrix},$$

thus $\mathrm{rk}N' \in \{\mathrm{rk}N, \mathrm{rk}N + 1, \mathrm{rk}N + 2\}$.

Case i). $c \notin \mathrm{Im}N$. This means $\mathrm{rk}N' = \mathrm{rk}N + 2$, and hence $\mathrm{co}N' = \mathrm{co}N - 1$.

Case ii). $c \in \mathrm{Im}N$, $c = Ny$. Then $y^T c = y^T N y = 0$ by 2), and thus $y^T(c, N) = (0, c^T)$, implying $\mathrm{rk}N' = \mathrm{rk}N$, that is, $\mathrm{co}N' = \mathrm{co}N + 1$. □

Remark. Lemma 1 shows, in particular, that $\mathrm{co}A = 0$ implies $|V| \equiv 0 \pmod 2$. In other words, if $|N|$ is odd, then A is a singular matrix. Furthermore, we obtain the evaluation

$$q(G, 0) = \sum_{T \subseteq V} (-1)^{\mathrm{co}A_T} = \sum_{T \subseteq V} (-1)^{|T|} = 0 \quad \text{for} \quad |V| \geq 1.$$

Lemma 2. *If $(A + I)y = \mathbf{1}$, then $\big|\|y\|\big| \equiv rk(A + I) \pmod 2$.*

Proof. This is shown by an induction argument as in Lemma 1. □

Now we come to the central definition.

Definition. A vector y is *Eulerian* if the subgraph $G[R]$ induced by the support $R = \|y\|$ is an Eulerian graph.

A moment's thought shows that Eulerian vectors are characterized in the following manner.

Lemma 3. *The vector y is Eulerian if and only if $\|y\| \cap \|Ay\| = \varnothing$.*

If y is Eulerian, then we say that y *spans* $F = \|y\| \cup \|Ay\|$. Let E_F denote the set of Eulerian vectors which span F, and set $e_F = |E_F|$. We call $F \subseteq V$ *proper* if $e_F \neq 0$.

The following result is immediate.

Lemma 4. *The Eulerian vector y spans F if and only if $\|y\| \subseteq F$, $\|Ay\| \subseteq F$ and $\|(A + I)y\| = F$. In particular, the whole set V is proper (since $\mathbf{1} \in \mathrm{Im}(A + I)$ by 4)), and $y \in E_V$ if and only if $(A + I)y = \mathbf{1}$.*

The next two propositions are the key results.

Proposition 3. *Let $T \subseteq V$. Then*

$$|\mathrm{Ker} A_T| = \#\{y : \|y\| \subseteq T, \quad \|Ay\| \subseteq V \smallsetminus T\}.$$

Proof. Let y_T be the restriction of y to T. Then clearly

$$\|y\| \subseteq T, \|Ay\| \subseteq V \smallsetminus T \iff \|y\| \subseteq T \quad \text{and} \quad A_T y_T = 0,$$

and the result follows. $\square$

Proposition 4. *Let $F \subseteq V$ be proper. Then*

$$e_F = \#\{y : \|y\| \subseteq F, \quad y \in \mathrm{Ker}(A+I)\}.$$

Proof. Denote the set on the right–hand side by $\widetilde{E}_F$, and let $u \in E_F$. We claim that $\widetilde{E}_F = u + E_F$ which will prove the result. Suppose $z \in E_F$, then by Lemma 4 $\|u+z\| \subseteq F$ and further $\|(A+I)u\| = F = \|(A+I)z\|$ which implies $\|(A+I)(u+z)\| = \varnothing$, i.e. $u+z \in \mathrm{Ker}(A+I)$. Hence $u + E_F \subseteq \widetilde{E}_F$. The converse inclusion $u + \widetilde{E}_F \subseteq E_F$ is just as easily established. $\square$

We can bring the result of Proposition 4 into the following succinct form. Denote by $P(F)$ the subspace of all vectors y with $\|y\| \subseteq F$. Clearly,

$$\dim P(F) = |F| \quad \text{and} \quad P(F)^{\perp} = P(V \smallsetminus F). \tag{12}$$

With this notation, $\widetilde{E}_F = P(F) \cap \mathrm{Ker}(A+I)$, and in particular $\widetilde{E}_V = \mathrm{Ker}(A+I)$. In summary, we note that for a proper set F

$$e_F = 2^{\dim(P(F) \cap \mathrm{Ker}(A+I))} \tag{13}$$

and

$$e_V = 2^{\dim \mathrm{Ker}(A+I)}. \tag{14}$$

We come to the main results. First we reprove Theorem 2.

Proof of Theorem 2. For $x = -1$ we have

$$q(G,-1) = \sum_T (-2)^{\operatorname{co} A_T} = \sum_T 2^{\operatorname{co} A_T}(-1)^{\operatorname{co} A_T},$$

which is by Proposition 3

$$= \sum_T \sum_{\substack{y:\|y\|\subseteq T\\ \|Ay\|\subseteq V\smallsetminus T}} (-1)^{\operatorname{co} A_T}$$

$$= \sum_{y \text{ Eulerian}} \sum_{\|y\|\subseteq T\subseteq V\smallsetminus\|Ay\|} (-1)^{\operatorname{co} A_T}.$$

Now if $\|y\| \cup \|Ay\| \neq V$, then the inner sum is by Lemma 1

$$\sum_{\|y\|\subseteq T\subseteq V\smallsetminus\|Ay\|} (-1)^{|T|} = 0.$$

Hence we obtain from Lemmas 1 and 2 and(14)

$$\begin{aligned} q(G,-1) &= \sum_{y\in E_V} (-1)^{\left|\|y\|\right|} = (-1)^{\operatorname{rk}(A+I)} e_V \\ &= (-1)^{\operatorname{rk}(A+I)} 2^{\dim \operatorname{Ker}(A+I)} = (-1)^n (-2)^{\operatorname{co}(A+I)}. \qquad \square \end{aligned}$$

The following result was proved by Las Vergnas [9] for the Tutte polynomial $T(x,x)$ of a graph and generalized by Jaeger to binary matroids [7]. Our proof is an adaption of their arguments.

Theorem 6. *For any graph, $q(G,3)$ is divisible by $q(G,-1)$, and the quotient is an odd integer.*

Proof. We have by Proposition 3

$$\begin{aligned} q(G,3) &= \sum_T 2^{\operatorname{co} A_T} = \sum_T \sum_{\substack{y:\|y\|\subseteq T\\ \|Ay\|\subseteq V\smallsetminus T}} 1 \\ &= \sum_{y \text{ Eulerian}} \sum_{\|y\|\subseteq T\subseteq V\smallsetminus\|Ay\|} 1 = \sum_{y \text{ Eulerian}} 2^{|V|-\left|\|y\|\cup\|Ay\|\right|} \\ &= \sum_{F \text{ proper}} \sum_{y\in E_F} 2^{|V|-|F|} = \sum_{F \text{ proper}} 2^{|V|-|F|} e_F. \end{aligned}$$

Claim. *Let* $F \neq V$ *be proper, then*

$$2^{\dim \operatorname{Ker}(A+I)+1} \text{ divides } 2^{|V|-|F|}e_F.$$

Using (13) and (12) we find

$$\begin{aligned}
2^{|V|-|F|}e_F &= 2^{|V|-|F|}2^{\dim(P(F)\cap \operatorname{Ker}(A+I))} \\
&= 2^{|V|-|F|}2^{\dim P(F)+\dim \operatorname{Ker}(A+I)-\dim(P(F)+\operatorname{Ker}(A+I))} \\
&= 2^{\dim \operatorname{Ker}(A+I)+\dim(P(F)+\operatorname{Ker}(A+I))^{\perp}} \\
&= 2^{\dim \operatorname{Ker}(A+I)+\dim(P(V\smallsetminus F)\cap \operatorname{Im}(A+I))},
\end{aligned}$$

and it remains to show that $P(V\smallsetminus F)\cap \operatorname{Im}(A+I) \neq \{0\}$.

Let u span F, then by Lemma 4, $\|(A+I)u\| = F$, and hence $y = \mathbf{1}+(A+I)u$ is in $P(V\smallsetminus F)\cap \operatorname{Im}(A+I)$, since $\mathbf{1} \in \operatorname{Im}(A+I)$. Finally, we note that $y \neq 0$, since $F \neq V$. $\square$

To finish the proof we have by the claim and (14)

$$\begin{aligned}
q(G,3) &= e_V + \sum_{F\neq V \text{ proper}} 2^{|V|-|F|}e_F \\
&= 2^{\dim \operatorname{Ker}(A+I)} + 2 \sum_{F\neq V \text{ proper}} 2^{\dim \operatorname{Ker}(A+I)}p_F \\
&= 2^{\dim \operatorname{Ker}(A+I)}[1+2\textstyle\sum p_F],
\end{aligned}$$

where the p_F are integers, and the proof is complete by Theorem 2. $\square$

We finally come to the proof of Theorem 5. As in section 4 we consider the matrix $L = (A|I|A+I)$, and admissible subsets S. Let $T = S\cap\{1,\dots,n\}$, $T_1 = S\cap\{\overline{1},\dots,\overline{n}\}$, $T_2 = S\cap\{\overline{\overline{1}},\dots,\overline{\overline{n}}\}$, and denote by $L_{T\cup T_2}$ the submatrix of L_S with rows and columns from $T\cup T_2$. Thus

$$L_{T\cup T_2} = \underbrace{\begin{array}{|c|}\hline \\ \quad A \quad \\ \\ \hline\end{array}}_{T}\ \underbrace{\begin{array}{|c|}\hline A \\ \hline \begin{matrix}1&&\\&\ddots&\\&&1\end{matrix} \\ \hline\end{array}}_{T_2} \begin{array}{l}\left.\vphantom{A}\right\}T \\ \left.\vphantom{\begin{matrix}1\\ \ddots\\1\end{matrix}}\right\}T_2\end{array}$$

We clearly have $\mathrm{co}L_S = \mathrm{co}L_{T\cup T_2}$. Furthermore, we note

$$z = \begin{pmatrix} z_T \\ z_{T_2} \end{pmatrix} \in \mathrm{Ker}L_{T\cup T_2} \Longrightarrow z_{T_2}^T z_{T_2} = 0. \tag{15}$$

Indeed, by 2) we have

$$0 = z^T L_{T\cup T_2} z = z^T (A_{T\cup T_2}) z + z_{T_2}^T z_{T_2} = z_{T_2}^T z_{T_2}.$$

We say that an Eulerian vector y *fits* the admissible set S if $\|y\| \subseteq T \cup T_2$, $\|Ay\| \subseteq T_1 \cup T_2$, $T_2 \subseteq \|(A+I)y\|$. Let Fit_S be the set of Eulerian vectors that fit S.

Lemma 5. *For any Eulerian vector y there are precisely 2^n admissible sets for which y fits.*

Proof. If $i \in \|y\|$, then i may belong to T or T_2. Similarly $i \in \|Ay\|$ may belong to T_1 or T_2, and $i \in V \smallsetminus (\|y\| \cup \|Ay\|)$ may belong to T or T_1. $\square$

Proposition 5. i) *An Eulerian vector y fits S if and only if $y_{T_1} = 0$ and $L_{T\cup T_2}\binom{y_T}{y_{T_2}} = \binom{\mathbf{0}_T}{\mathbf{1}_{T_2}}$.*
ii) $|\mathrm{Fit}_S| = 2^{\mathrm{co}L_{T\cup T_2}} = 2^{\mathrm{co}L_S}$ *for any admissible set S.*

Proof. i) is proved by an analogous argument as in Proposition 3. To verify ii) we need only show $\mathrm{Fit}_S \neq \varnothing$ for any S, since then Fit_S corresponds to a coset of $\mathrm{Ker}L_{T\cup T_2}$ by i). Now

$$\mathrm{Fit}_S \neq \varnothing \Longleftrightarrow \begin{pmatrix} \mathbf{0}_T \\ \mathbf{1}_{T_2} \end{pmatrix} \in \mathrm{Im}L_{T\cup T_2} \Longleftrightarrow \begin{pmatrix} \mathbf{0}_T \\ \mathbf{1}_{T_2} \end{pmatrix} \perp \begin{pmatrix} y_T \\ y_{T_2} \end{pmatrix} \text{ for all } \begin{pmatrix} y_T \\ y_{T_2} \end{pmatrix} \in \mathrm{Ker}L_{T\cup T_2}.$$

By (15) we find $y_{T_2}^T y_{T_2} = 0$, and hence

$$(0_T^T, \mathbf{1}_{T_2}^T)\begin{pmatrix} y_T \\ y_{T_2} \end{pmatrix} = \mathbf{1}_{T_2}^T y_{T_2} = y_{T_2}^T y_{T_2} = 0. \qquad \square$$

Proof of Theorem 5. The interlace polynomial $Q(G,x)$ is given by $Q(G,x) = \sum_S (x-2)^{\mathrm{co}L_S}$. According to Lemma 5 and and Proposition 5 we find

$$\begin{aligned} 2^n \cdot (\#\text{Eulerian vectors}) &= \sum_{y \text{ Eulerian}} \sum_{S: y\in \mathrm{Fit}_S} 1 = \sum_S \sum_{y \in \mathrm{Fit}_S} 1 \\ &= \sum_S 2^{\mathrm{co}L_S} = Q(G,4). \qquad \square \end{aligned}$$

6. Isotropic systems

Isotropic systems were introduced by Bouchet in a series of papers to unify certain properties of binary matroids and transition systems of 4–regular graphs. For convenience we recall the definition.

Let $V = \{1, 2, \dots, n\}$ be the ground–set. We consider the vector space $\mathcal{V} = (GF(2))^{2n}$ where the coordinates are indexed by $\{1, \dots, n; \overline{1}, \dots, \overline{n}\}$. On the set $GF(2)^2 = \{(0,0), (1,0), (0,1), (1,1)\}$ we consider the bilinear form

$$\langle (x,y), (x',y') \rangle = \begin{cases} 1 & \text{if } (0,0) \neq (x,y) \neq (x',y') \neq (0,0) \\ 0 & \text{otherwise,} \end{cases}$$

and extend this by linearity to $\mathcal{V}$ (over $GF(2)$)

$$\langle X, Y \rangle = \sum_{v \in V} \langle (X_v, X_{\overline{v}}), (Y_v, Y_{\overline{v}}) \rangle. \tag{16}$$

A subset $\mathcal{L} \subseteq \mathcal{V}$ is called *totally isotropic* if $\langle X, Y \rangle = 0$ for any $X, Y \in \mathcal{L}$.

Definition. $\mathcal{S} = (V, \mathcal{L}) \subseteq GF(2)^{2n}$ is called an *isotropic system* if

i) $\mathcal{L}$ is a totally isotropic subspace

ii) $\dim \mathcal{L} = n$.

Example. Let $C \in \mathcal{V}$ with $(C_v, C_{\overline{v}}) \neq (0,0)$ for all v. By $C(P)$ we denote the restriction to $P \subseteq V$, that is

$$(C(P)_v, C(P)_{\overline{v}}) = \begin{cases} (C_v, C_{\overline{v}}) & \text{if } v \in P \\ (0,0) & \text{if } v \notin P. \end{cases}$$

Clearly, $\widehat{C} = \{C(P) : P \subseteq V\}$ is an isotropic system.

For our purposes we are interested in the following isotropic system (see [3]). Let $G = (V, E)$ be a simple graph and A its adjacency matrix. Consider the matrix

$$L = (\underset{1 \dots n}{A} \mid \underset{\overline{1} \dots \overline{n}}{I_n})$$

as in Theorem 1, indexed by $1, \dots, n; \overline{1}, \dots, \overline{n}$.

Claim. $\mathcal{S}_G = (V, \mathcal{L}_G)$ where $\mathcal{L}_G$ is the row space of L is an isotropic system. We obviously have $\dim \mathcal{L}_G = n$. Consider two rows C and D of L, corresponding to the vertices c and d, respectively. For $v \neq c, d$ we have $C_{\overline{v}} = D_{\overline{v}} = 0$, and hence

$$\langle (C_v, C_{\overline{v}}), (D_v, D_{\overline{v}}) \rangle = 0 \quad \text{by (16)}.$$

If $cd \notin E(G)$, then $(D_c, D_{\overline{c}}) = (0,0)$ and $(C_d, C_{\overline{d}}) = (0,0)$. On the other hand, if $cd \in E(G)$, then

$$\begin{aligned} (C_c, C_{\overline{c}}) &= (0,1), \quad (D_c, D_{\overline{c}}) = (1,0) \\ (C_d, C_{\overline{d}}) &= (1,0), \quad (D_d, D_{\overline{d}}) = (0,1), \end{aligned}$$

and we conclude $\langle C, D \rangle = 1 + 1 = 0$ by (16). Thus in all cases $\langle C, D \rangle = 0$, and by linearity $\mathcal{L}_G$ is totally isotropic.

Bouchet defines in [4] the *Martin polynomials* $m(\mathcal{S}, x)$ of an arbitrary isotropic system $\mathcal{S}$ (relative to a complete vector), which in our case reduces to

$$m(\mathcal{S}_G, x) = \sum_C (x-1)^{\dim(\mathcal{L}_G \cap \widehat{C})},$$

where the sum is extended over all vectors C with $(C_v, C_{\overline{v}}) \neq (0,0), (1,1)$.

Now it is readily verified that for $\mathcal{S}_G$

$$\dim(\mathcal{L}_G \cap \widehat{C}) = \operatorname{co}(L_S)$$

in the notation of Theorem 1 where $v \in S \iff C_v = 0$ and thus $\overline{v} \in S \iff C_v = 1$. From this follows

Theorem 7. *We have*

$$q(G, x) = m(\mathcal{S}_G, x).$$

The second interlace polynomial $Q(G, x)$ can also be found within the context of isotropic systems. Call a vector $C \in GF(2)^{2n}$ *complete* if $(C_v, C_{\overline{v}}) \neq (0,0)$ for all $v \in V$. Then the *Martin polynomial* $M(\mathcal{S}, x)$ of an isotropic system $\mathcal{S} = (V, \mathcal{L})$ is defined as

$$M(\mathcal{S}, x) = \sum_C (x-2)^{\dim(\mathcal{L}_G \cap \widehat{C})},$$

where C runs through all complete vectors.

It can again be shown with the notation of Theorem 4 that for the system $\mathcal{S}_G$ considered above

$$\dim(\mathcal{L}_G \cap \widehat{C}) = \mathrm{co}(L_S),$$

where $v \in S \iff (C_v, C_{\overline{v}}) = (0,1)$, $\overline{v} \in S \iff (C_v, C_{\overline{v}}) = (1,0)$, $\bar{\bar{v}} \in S \iff (C_v, C_{\overline{v}}) = (1,1)$. From this follows

Theorem 8. *We have*

$$Q(G,x) = M(\mathcal{S}_G, x).$$

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