TOPOLOGICALLY GENERATED AND UNIFORMLY GENERATED SPACES

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Abstract. Though the construct **SUConv** of semiuniform convergence spaces is a strong topological universe, its subconstructs of topological and uniform spaces are not even cartesian closed. Better behaved subconstructs are obtained by forming their (bi)coreflective hulls where these hulls can be described by means of suitable axioms.

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Introduction

Neither topological nor uniform spaces form a cartesian closed topological construct and hence not a topological universe, i.e. a topological construct which is a quasitopos in the sense of M. J. Penon [5]. Recently, in the realm of Convenient Topology the author was successful in embedding uniform spaces and symmetric topological spaces in the topological universe **SUConv** of semiuniform convergence spaces (cf. [6] or [7]). Thus the concepts "topological" and "uniform" appear as properties of semiuniform convergence spaces. Since in the framework of topological spaces structural improvements have been obtained by forming coreflective hulls, e.g. the topological construct of {compact Hausdorff}-generated spaces and the coreflective hull of all metrizable spaces, whose objects are called sequential spaces, have natural function spaces, the question arises whether by forming such hulls in **SUConv** better behaved subconstructs containing topological or uniform spaces will result.

Indeed, topologically generated spaces form a topological construct with natural function spaces and in which quotients are productive, i.e. they form a strongly cartesian closed topological construct. They are exactly the symmetric Kent convergence spaces, i.e. they can be described by means of suitable axioms. Furthermore, they are essential, because every semiuniform convergence space has an underlying (symmetric) Kent convergence space. Kent convergence spaces are needed for studying compactness, i.e. the convergence of ultrafilters. Uniformly generated spaces form a topological universe and can also be described by means of suitable axioms.

The terminology of this article corresponds to [1] and [7].

0. Preliminaries

For the convenience of the reader some definitions and results are repeated.

0.1 Proposition. *Let $\mathcal{A}$ be a full subconstruct of a topological construct $\mathcal{C}$ such that $|\mathcal{A}|$ contains at least one object with non-empty underlying set, and let $\mathcal{B}$ be the coreflective (=bicoreflective) hull of $\mathcal{A}$ in $\mathcal{C}$. Then the following are equivalent:*
(1) $X \in |\mathcal{B}|$.
(2) *X is a quotient object of a coproduct of $\mathcal{A}$-objects in $\mathcal{C}$.*
(3) *There is a final epi-sink $(f_i : X_i \to X)_{i\in I}$ in $\mathcal{C}$ with $X_i \in |\mathcal{A}|$ for each $i \in I$.*
(4) *There is a final sink $(f_i : X_i \to X)_{i\in I}$ in $\mathcal{C}$ with $X_i \in |\mathcal{A}|$ for each $i \in I$.*
(5) *The total sink $(f_i : X_i \to X)_{i\in I}$ with $X_i \in |\mathcal{A}|$ for each $i \in I$ is final.*

0.2 Definitions. 1) A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$ where X is a set and $\mathcal{J}_X$ a subset of the set $F(X \times X)$ of all filters on $X \times X$ such that the following are satisfied:
UC$_1$) The filter $\dot{x} \times \dot{x}$, generated by $\{(x, x)\}$, belongs to $\mathcal{J}_X$ for each $x \in X$.
UC$_2$) $\mathcal{G} \in \mathcal{J}_X$ whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{F} \subset \mathcal{G}$.
UC$_3$) $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-1} = \{F^{-1}, F \in \mathcal{F}\} \in \mathcal{J}_X$,
where $F^{-1} = \{(y, x) : (x, y) \in F\}$.
If $(X, \mathcal{J}_X)$ is a semiuniform convergence space, then the elements of $\mathcal{J}_X$ are called *uniform filters.*
2) A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between semiuniform convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{F}) \in \mathcal{J}_Y$ for each $\mathcal{F} \in \mathcal{J}_X$.
3) The construct of semiuniform convergence spaces (and uniformly continuous maps) is denoted by **SUConv**.

0.3 Remarks. 1) **SUConv** is a topological construct, where initial and final structures have an easy description:
If X is a set, $((X_i, \mathcal{J}_{X_i}))_{i\in I}$ a family of semiuniform convergence spaces and $(f_i : X \to X_i)_{i\in I}$ (resp. $(f_i : X_i \to X)_{i\in I}$) a family of maps, then
$\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : (f_i \times f_i)(\mathcal{F}) \in \mathcal{J}_{X_i} \text{ for each } i \in I\}$
(resp. $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) :$ there exist $i \in I$ and $\mathcal{F}_i \in \mathcal{J}_{X_i}$ with $(f_i \times f_i)(\mathcal{F}_i) \subset \mathcal{F}\} \cup \{\dot{x} \times \dot{x} : x \in X\}$) is the *initial* (*resp. final*) **SUConv**-*structure* w.r.t. the given data.
2) **SUConv** is a strong topological universe (=cartesian closed and extensional topological construct such that products of quotients are quotients). This implies that there are natural function spaces and one-point extensions in **SUConv**.
[If $\mathbf{X} = (X, \mathcal{J}_X)$ and $\mathbf{Y} = (Y, \mathcal{J}_Y)$ are semiuniform convergence spaces, then on the set $[\mathbf{X}, \mathbf{Y}]$ of all uniformly continuous maps from $\mathbf{X}$ into $\mathbf{Y}$ there is a coarsest **SUConv**-structure $\mathcal{J}_{X,Y}$ such that the evaluation map $e_{X,Y} : \mathbf{X} \times ([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y}) \to \mathbf{Y}$ is uniformly continuous. $([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is called a *natural function space*. Further, let $X^* = X \cup \{\infty_X\}$ with $\infty_X \notin X$. Then there is a coarsest **SUConv**-structure $\mathcal{J}_{X^*}$ on X^* such that $(X, \mathcal{J}_X)$ is a subspace of $(X^*, \mathcal{J}_{X^*})$. $(X^*, \mathcal{J}_{X^*})$ is called a *one-point extension* of $(X, \mathcal{J}_X)$.]

0.4 Definitions. 1) Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space. A filter $\mathcal{F}$ on X is called
a) a *Cauchy filter* provided that $\mathcal{F} \times \mathcal{F}$ is uniform,
b) *convergent* to $x \in X$ provided that $\mathcal{F} \cap \dot{x}$ is a Cauchy filter, where $\dot{x} = \{A \subset X : x \in A\}$.
2) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *convergence space* provided that it is generated by its convergent filters, i.e.
$\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there exists a convergent filter $\mathcal{G}$ on X with $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$.

0.5 Remarks. 1) The construct **Conv** of convergence spaces (and uniformly continuous maps) is concretely isomorph to $\mathbf{KConv}_\mathrm{S}$, i.e. the construct of symmetric Kent convergence spaces (and continuous maps) [remember: A *symmetric Kent convergence space* is a pair (X, q), where X is a set and q a relation between the set $F(X)$ of all filters on X and the set X, called a convergence relation, such that the following are valid:
C_1) $(\dot{x}, x) \in q$ for each $x \in X$,
C_2) $(\mathcal{G}, x) \in q$ whenever $(\mathcal{F}, x) \in q$ and $\mathcal{F} \subset \mathcal{G}$,

C_3) $(\mathcal{F}, x) \in q$ implies $(\mathcal{F} \cap \dot{x}, x) \in q$,
S) $(\mathcal{F}, x) \in q$ and $y \in \bigcap_{F \in \mathcal{F}} F$ imply $(\mathcal{F}, y) \in q$.
A map $f : (X, q) \to (X', q')$ between symmetric Kent convergence spaces is called *continuous* provided that $(f(\mathcal{F}), f(x)) \in q'$ for each $(\mathcal{F}, x) \in q$.]
If $(X, \mathcal{X})$ is a symmetric topological space (=R_0-space), then $(X, q_{\mathcal{X}})$ is a symmetric Kent convergence space, where $(\mathcal{F}, x) \in q_{\mathcal{X}}$ iff $\mathcal{F}$ contains the neighborhood filter $\mathcal{U}_{\mathcal{X}}(x)$ of x in $(X, \mathcal{X})$.
2) Every semiuniform convergence space $(X, \mathcal{J}_X)$ has an underlying symmetric Kent convergence space $(X, q_{\gamma_{\mathcal{J}_X}})$ defined by $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$ iff $\mathcal{F}$ is convergent to x. It results that **Conv** is bicoreflective in **SUConv**. Furthermore, $\mathbf{KConv}_{\mathrm{S}}$ ($\cong$ **Conv**) contains the construct $\mathbf{Top}_{\mathrm{S}}$ of symmetric topological spaces (=R_0-spaces) [and continuous map] as a bireflective subconstruct. Thus it makes sense to call a semiuniform convergence space $(X, \mathcal{J}_X)$ *topological* iff it is a convergence space and its underlying symmetric Kent convergence space $(X, q_{\gamma_{\mathcal{J}_X}})$ is topological (i.e. there is a symmetric topology $\mathcal{X}$ on X such that $q_{\mathcal{X}} = q_{\gamma_{\mathcal{J}_X}}$).
3) **Conv** is also concretely isomorphic to the construct **CFil** of complete filter spaces (and Cauchy continuous maps) [remember: If X is a set and γ a subset of $F(X)$, then (X, γ) is called a *complete filter space* provided that the following are satisfied:
1^0 filter space axioms: F_1) $\dot{x} \in \gamma$ for each $x \in X$
F_2) $\mathcal{G} \in \gamma$ whenever $\mathcal{F} \in \gamma$ and $\mathcal{F} \subset \mathcal{G}$
2^0 completeness condition: C) $\mathcal{F} \in \gamma$ implies $\mathcal{F} \cap \dot{x} \in \gamma$ for some $x \in X$.
A map $f : (X, \gamma) \to (X', \gamma')$ between filter spaces is called *Cauchy continuous* provided that $f(\mathcal{F}) \in \gamma'$ for each $\mathcal{F} \in \gamma$].
4) It is easily checked that **Conv** is closed under formation of products and closed subspaces in **SUConv**, but it is not closed under formation of arbitrary subspaces, since completeness is not hereditary.

1. Topologically generated spaces

1.1 Proposition. *Coproducts* (*in* **SUConv**) *of topological semiuniform convergence spaces are topological.*

Proof. Let $((X_i, \mathcal{J}_{X_i}))_{i \in I}$ be a family of topological semiuniform convergence spaces, i.e. each $(X_i, q_{\gamma_{\mathcal{J}_{X_i}}})$ is topological. Then, for each $i \in I$, there is a symmetric topology $\mathcal{X}_i$ on X_i such that $q_{\mathcal{X}_i} = q_{\gamma_{\mathcal{J}_{X_i}}}$. Since **Conv** is bicoreflective in **SUConv**, the coproduct $(X, \mathcal{J}_X)$ of $((X_i, \mathcal{J}_{X_i}))_{i \in I}$ is a convergence space. Let $(X, \mathcal{X})$ be the sum space (=coproduct) of $((X_i, \mathcal{X}_i))_{i \in I}$ in **Top** (or $\mathbf{Top}_\mathrm{S}$) [note: symmetry is summable in **Top** and $\mathbf{Top}_\mathrm{S}$ is bireflective in **Top**]. It is easily checked that for each $x_i \in X_i$,

$$(*) \qquad j_i(\mathcal{U}_{\mathcal{X}_i}(x_i)) = \mathcal{U}_{\mathcal{X}}(j_i(x_i)),$$

where $j_i : X_i \to X$ denotes the i-th injection. In order to prove that $(X, \mathcal{J}_X)$ is a topological semiuniform convergence space, it suffices to show that
$q_{\gamma_{\mathcal{J}_X}} = q_{\mathcal{X}}$.
By definition,

$$(**) \quad (\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}} \Leftrightarrow (j_i(\mathcal{U}_{\mathcal{X}_i}(x_i)) \subset \mathcal{F} \cap \dot{x} \text{ for some } i \in I \text{ and some } x_i \in X_i)$$

If $(\mathcal{F}, x) \in q_{\mathcal{X}}$, i.e. $\mathcal{F} \supset \mathcal{U}_{\mathcal{X}}(x)$, then $x = j_i(x_i)$ for some $i \in I$ and some $x_i \in X_i$, and by $(*), \mathcal{F} \supset j_i(\mathcal{U}_{\mathcal{X}_i}(x_i))$, which implies $\mathcal{F} \cap \dot{x} \supset j_i(\mathcal{U}_{\mathcal{X}_i}(x_i)) \cap \dot{j_i(x_i)} = j_i(\mathcal{U}_{\mathcal{X}_i}(x_i))$, i.e. by $(**), (\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$. Conversely, if $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$, then $\mathcal{F} \supset \mathcal{F} \cap \dot{x} \supset j_i(\mathcal{U}_{\mathcal{X}_i}(x_i)) = \mathcal{U}_{\mathcal{X}}(j_i(x_i))$ for some $i \in I$ and some $x_i \in X_i$. Since $(X, \mathcal{X})$ is symmetric, $\mathcal{U}_{\mathcal{X}}(j_i(x_i)) = \mathcal{U}_{\mathcal{X}}(x)$, i.e. $(\mathcal{F}, x) \in q_{\mathcal{X}}$.

1.2 Theorem. *A semiuniform convergence space is a quotient* (*in* **SUConv**) *of a topological semiuniform convergence space iff it is a convergence space.*

Proof. "$\Rightarrow$". Since each topological semiuniform convergence space is a convergence space and **Conv** is closed under formation of quotients in **SUConv** (as a bicoreflective subconstruct of **SUConv**), this implication is obvious.
"$\Leftarrow$". Let $(X, \mathcal{J}_X)$ be a convergence space and $(X, q_{\gamma_{\mathcal{J}_X}})$ its underlying symmetric Kent convergence space. Further, let $(\mathcal{E}, x) \in q_{\gamma_{\mathcal{J}_X}}$ and $\mathcal{F} = \mathcal{E} \cap \dot{x}$. Then $(X, \gamma_{(\mathcal{E},x)})$ is a complete filter space, where
$\gamma_{(\mathcal{E},x)} = \{\mathcal{H} \in F(X) : \mathcal{H} \supset \mathcal{F}\} \cup \{\dot{y} : y \neq x\}$.
Moreover, $(X, \gamma_{(\mathcal{E},x)})$ is subtopological (cf. [7; 7.2.1] for the definition):
Let $\mathcal{G} \in \gamma_{(\mathcal{E},x)}$. Then 1) $\mathcal{G} \supset \mathcal{F}$ or 2) $\mathcal{G} = \dot{y}$ with $y \neq x$. In the first case the proof is finished if each $F \in \mathcal{F}$ is open, i.e. $F = int_{q_{\gamma_{(\mathcal{E},x)}}} F := \{y \in F :$ for each $\mathcal{K} \in F(X)$ with $(\mathcal{K}, y) \in q_{\gamma_{(\mathcal{E},x)}}, F \in \mathcal{K}\}$. Obviously, $int_{q_{\gamma_{(\mathcal{E},x)}}} F \subset F$. Conversely, let $y \in F$ and $(\mathcal{K}, y) \in q_{\gamma_{(\mathcal{E},x)}}$. Hence, a) $\mathcal{K} \cap \dot{y} \supset \mathcal{F}$ or b) $\mathcal{K} \cap \dot{y} = \dot{z}$ with $z \neq x$. In case a), $\mathcal{K} \supset \mathcal{K} \cap \dot{y} \supset \mathcal{F}$ implies $F \in \mathcal{K}$, i.e. $y \in int_{q_{\gamma_{(\mathcal{E},x)}}} F$. In case b), $\mathcal{K} = \dot{z} = \dot{y}$, which implies $F \in \mathcal{K}$, i.e. $y \in int_{q_{\gamma_{(\mathcal{E},x)}}} F$.

In the second case, a) $y \in \bigcap_{F \in \mathcal{F}} F$ or b) there is some $F \in \mathcal{F}$ with $y \notin F$. In case 2)a), $\mathcal{F} \subset \dot{y}$, which implies the assertion (cf. case 1)a)). In case 2)b), there is no $\mathcal{H} \in \gamma_{(\mathcal{E},x)}$ with $\mathcal{H} \subset \dot{y}$ and $\mathcal{H} \neq \dot{y}$. It suffices to prove that $\{y\}$ is open. Let $(\mathcal{K}, y) \in q_{\gamma_{\mathcal{E},x)}}$. Then 1. $\mathcal{K} \cap \dot{y} \supset \mathcal{F}$ or 2. $\mathcal{K} = \dot{y}$. 1. cannot occur, since there is some $F \in \mathcal{F}$ with $y \notin F$, whereas 2. implies $\{y\} \in \mathcal{K}$.
Consequently, for each $(\mathcal{E}, x) \in q_{\gamma_{\mathcal{J}_X}}$, $(X, \gamma_{(\mathcal{E},x)})$ is a topological filter space, i.e. the semiuniform convergence space $(X, \mathcal{J}_{\gamma_{(\mathcal{E},x)}})$ is topological, where $\mathcal{J}_{\gamma_{(\mathcal{E},x)}} = \{\mathcal{F} \in F(X \times X) : \text{there is some } \mathcal{G} \in \gamma_{(\mathcal{E},x)} \text{ with } \mathcal{G} \times \mathcal{G} \subset \mathcal{F}\}$. Further, $1_X : (X, \mathcal{J}_{\gamma_{(\mathcal{E},x)}}) \to (X, \mathcal{J}_X)$ is uniformly continuous for each $(\mathcal{E}, x) \in q_{\gamma_{\mathcal{J}_X}}$, and $(1_X : (X, \mathcal{J}_{\gamma_{(\mathcal{E},X)}}) \to (X, \mathcal{J}_X))_{(\mathcal{E},x) \in q_{\gamma_{\mathcal{J}_X}}}$ is a final epi-sink in **SUConv**. The finality is proved as follows:
Let $\mathcal{F} \in \mathcal{J}_X$. Then $\mathcal{F} \supset \mathcal{G} \times \mathcal{G}$ for some $\mathcal{G} \in F(X)$ such that there is some $x \in X$ with $(\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}}$, since $(X, \mathcal{J}_X)$ is a convergence space. Obviously, $\mathcal{G} \in \gamma_{(\mathcal{G},x)}$ and $(1_X \times 1_X)(\mathcal{G} \times \mathcal{G}) = \mathcal{G} \times \mathcal{G} \in \mathcal{J}_{\gamma_{(\mathcal{G},x)}}$, which implies the assertion. By 1.1, $\coprod_{(\mathcal{E},x) \in q_{\gamma_{\mathcal{J}_X}}} (X, \mathcal{J}_{\gamma_{(\mathcal{E},x)}})$ is topological and the map $f : \coprod_{(\mathcal{E},x) \in q_{\gamma_{\mathcal{J}_X}}} (X, \mathcal{J}_{\gamma_{(\mathcal{E},x)}}) \to (X, \mathcal{J}_X)$ defined by $f \circ j_{(\mathcal{E},x)} = 1_X$ for each $(\mathcal{E}, x) \in q_{\gamma_{\mathcal{J}_X}}$ is a quotient map, where $j_{(\mathcal{E},x)} : (X, \mathcal{J}_{\gamma_{(\mathcal{E},x)}}) \to \coprod_{(\mathcal{E},x) \in q_{\gamma_{\mathcal{J}_X}}} (X, \mathcal{J}_{\gamma_{(\mathcal{E},x)}})$ denotes the $(\mathcal{E}, x)$-th injection.

1.3 Remarks. In the realm of nearness, H. L. Bentley, H. Herrlich and W. A. Robertson [2] have obtained a result corresponding to 1.2.

1.4 Corollary. *A semiuniform convergence space is topologically generated, i.e. it belongs to the (bi-)coreflective hull of* $\mathbf{Top}_{\mathrm{S}}$ *in* **SUConv**, *iff it is a convergence space.*

Proof. "$\Rightarrow$". Since **Conv** is bicoreflective in **SUConv** and contains $\mathbf{Top}_{\mathrm{S}}$ (up to isomorphism), this implication is obvious.
"$\Leftarrow$". Apply 1.2 and see 0.1.

1.5 Remark. The construct **Conv** ($\cong$ $\mathbf{KConv}_{\mathrm{S}}$) is strongly cartesian closed (cf. [7]), where the natural function spaces are obtained as follows: Let $\mathbf{X} = (X, q)$ and $\mathbf{X}' = (X', q')$ be symmetric Kent convergence spaces and $[\mathbf{X}, \mathbf{X}']$ the set of all continuous maps from $\mathbf{X}$ to $\mathbf{X}'$. Then the natural *function space structure* $\tilde{q}$ *in* $\mathbf{KConv}_{\mathrm{S}}$ on $[\mathbf{X}, \mathbf{X}']$ is given by

$$(\psi, f) \in \tilde{q} \Leftrightarrow e_{X,X'}(\mathcal{F} \times (\psi \cap \dot{f})) \text{ converges in } \mathbf{X'} \\ \text{for each convergent filter } \mathcal{F} \text{ in } \mathbf{X} \quad ,$$

where $e_{X,X'} : X \times [\mathbf{X}, \mathbf{X'}] \to X'$ denotes the evaluation map.
If a convergence relation q on a set X fulfills the axioms C_1), C_2) and S) (cf. 0.5.1)), then (X, q) is called a symmetric generalized convergence space and $\mathbf{GKonv}_S$ denotes the corresponding construct. It is easily verified that *$\tilde{q}$ is finer than the* $\mathbf{GKonv}_S$-structure *$\hat{q}$ of continuous convergence* which is defined as follows:

$$(\psi, f) \in \hat{q} \Leftrightarrow (e_{X,X'}(\mathcal{F} \times \psi), f(x)) \in q' \text{ for each } (\mathcal{F}, x) \in q \ ,$$

and $\tilde{q} = \hat{q}$ provided that $\mathbf{X'}$ is a (symmetric) limit space (cf. the proof of 3.1.10 in [7]). It has been shown by F. Schwarz [8] that $\tilde{q} \neq \hat{q}$ in general.
2) **Conv** is not a topological universe, i.e. there don't exist one-point extensions in general (cf. [9] or [7; 3.2.7.⑤ c)]).
3) There are convergence spaces $(X, \mathcal{J}_X)$ such that the intersection of two filters converging to some $x \in X$ does not converge to x (i.e. the corresponding symmetric Kent convergence space $(X, q_{\gamma_{\mathcal{J}_X}})$ is not a limit space and thus not a topological space) as the following *example* shows: Let $\mathbb{R}_t$ be the usual topological space of real numbers regarded as a semiuniform convergence space. Then its quotient space $(\mathbb{R}/\mathbb{N}, \mathcal{J}_{\mathbb{R}/\mathbb{N}})$ in **SUConv** obtained by identifying the set $\mathbb{N}$ of natural numbers to a point is a convergence space, but $(\mathbb{R}/\mathbb{N}, q_{\gamma_{\mathcal{J}_{\mathbb{R}/\mathbb{N}}}})$ is not a limit space. Obviously,

$$(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_{\mathbb{R}/\mathbb{N}}}} \Leftrightarrow \mathcal{F} \cap \dot{x} \supset \omega(\mathcal{U}(y)) \text{ for some } y \in \mathbb{R},$$

where $\mathcal{U}(y)$ denotes the neighborhood filter of y in $\mathbb{R}_t$ and $\omega : \mathbb{R} \to \mathbb{R}/\mathbb{N}$ is the natural map. Then $\omega(\mathcal{U}(1)) \to \omega(1)$ and $\omega(\mathcal{U}(2)) \to \omega(1)$, but $\omega(\mathcal{U}(1)) \cap \omega(\mathcal{U}(2))$ does not converge to $\omega(1)$, since there is no $z \in \mathbb{R}$ such that

$$\omega(\mathcal{U}(z)) \subset \omega(\mathcal{U}(1)) \cap \omega(\mathcal{U}(2)) \cap \dot{\omega(1)} = \omega(\mathcal{U}(1)) \cap \omega(\mathcal{U}(2)).$$

2. Uniformly generated spaces

2.1 Remark. Semiuniformities have been studied extensively by E. Čech [3]. A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a semiuniform space (resp. uniform space) provided that there is a semiuniformity (resp. uniformity) $\mathcal{V}$ on X such that $\mathcal{J}_X = [\mathcal{V}]$, where $[\mathcal{V}] = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \mathcal{V}\}$. The construct of semiuniform spaces (resp. uniform spaces) [and uniformly continuous maps] is denoted by **SUnif** (resp. **Unif**). **SUnif** and **Unif** are bireflective in **SUConv**:
$1_X : (X, \mathcal{J}_X) \to (X, [\mathcal{W}])$ is the bireflection of $(X, \mathcal{J}_X) \in |\mathbf{SUConv}|$ w.r.t. **SUnif** (resp. **Unif**), where $\mathcal{W}$ is the finest semiuniformity (resp. uniformity) on X which is contained in each $\mathcal{F} \in \mathcal{J}_X$.

2.2 Proposition. *Quotients* (*in* **SUConv**) *of semiuniform spaces are semiuniform.*

Proof. This follows immediately from the fact that for a semiuniformity $\mathcal{V}$ on a set X and a surjective map $f : X \to Y$ from X to a set Y, $f \times f(\mathcal{V})$ is a semiuniformity on Y.

2.3 Remark. Every semiuniform space $(X, \mathcal{J}_X)$ is diagonal, i.e. the filter (Δ_X) generated by the diagonal $\Delta_X = \{(x, x) : x \in X\}$ belongs to $\mathcal{J}_X$, if it exists. This implies that coproducts (in **SUConv**) of semiuniform spaces need not be semiuniform, since they are not necessarily diagonal (e.g. consider the disjoint closed intervals [0,1] and [2,3] on the real line $\mathbb{R}$ and endow them with the subspace structure of the usual uniformity on $\mathbb{R}$; then their coproduct in **SUConv** is not diagonal). On the other hand they are uniformly generated as we will see in the following (cf. 2.5 and 2.7).

2.4 Definition. 1) A semiuniform convergence space $(X, \mathcal{J}_X)$ is called a *uniformly generated space* provided that the following is satisfied:
UG) For each $\mathcal{F} \in \mathcal{J}_X$ there are a subset Y of X and a semiuniformity $\mathcal{V}$ on Y such that $\mathcal{V} \subset \mathcal{F}$ and $i \times i(\mathcal{V}) \in \mathcal{J}_X$, where $i : Y \to X$ denotes the inclusion map.
2) **UG-SUConv** denotes the construct of uniformly generated spaces (and uniformly continuous maps).

2.5 Remark. Every semiuniform space is uniformly generated.

2.6 Proposition. **UG-SUConv** *is bicoreflective in* **SUConv**, *where* $1_X : (X, (\mathcal{J}_X)_{UG}) \to (X, \mathcal{J}_X)$ *is the bicoreflection of* $(X, \mathcal{J}_X) \in |\mathbf{SUConv}|$ *w.r.t.* **UG-SUConv** $[(\mathcal{J}_X)_{UG} = \{\mathcal{F} \in F(X \times X)$: *there exist* $Y \subset X$ *and a semiuniformity* $\mathcal{V}$ *on* Y *with* $\mathcal{V} \subset \mathcal{F}$ *and* $i_Y \times i_Y(\mathcal{V}) \in \mathcal{J}_X\}$ *and* $i_Y : Y \to X$ *is the inclusion map*].

Proof. Obviously, $(X, (\mathcal{J}_X)_{UG})$ is a uniformly generated space. Further, let $(Z, \mathcal{J}_Z) \in |\mathbf{UG\text{-}SUConv}|$ and let $f : (Z, \mathcal{J}_Z) \to (X, \mathcal{J}_X)$ be a uniformly continuous map. If $\mathcal{F} \in \mathcal{J}_Z$, there are $Y_\mathcal{F} \subset Z$ and a semiuniformity $\mathcal{V}_\mathcal{F}$ on $Y_\mathcal{F}$ such that $\mathcal{V}_\mathcal{F} \subset \mathcal{F}$ and $i_\mathcal{F} \times i_\mathcal{F}(\mathcal{V}_\mathcal{F}) \in \mathcal{J}_Z$ where $i_\mathcal{F} : Y_\mathcal{F} \to Z$ is the inclusion map. Then $g \times g(\mathcal{V}_\mathcal{F}) \subset f \times f(\mathcal{F})$, where $g = (f \,|\, Y_\mathcal{F})' : Y_\mathcal{F} \to f[Y_\mathcal{F}]$, is a semiuniformity on $f[Y_\mathcal{F}]$ and, if $j_\mathcal{F} : f[Y_\mathcal{F}] \to X$ is the inclusion map,
$j_\mathcal{F} \times j_\mathcal{F}(g \times g(\mathcal{V}_\mathcal{F})) = (j_\mathcal{F} \circ g \times j_\mathcal{F} \circ g)(\mathcal{V}_\mathcal{F}) = (f \,|\, Y_f \times f \,|\, Y_\mathcal{F})(\mathcal{V}_\mathcal{F})$
$= (f \circ i_\mathcal{F} \times f \circ i_\mathcal{F}) = f \times f(i_\mathcal{F} \times i_\mathcal{F})(\mathcal{V}_\mathcal{F})) \in \mathcal{J}_X$.
Consequently, $f \times f(\mathcal{F}) \in (\mathcal{J}_X)_{UG}$, i.e. $f : (Z, \mathcal{J}_Z) \to (X, (\mathcal{J}_X)_{UG})$ is uniformly continuous.

2.7 Corollary. *Coproducts and quotients* (*formed in* **SUConv**) *of uniformly generated spaces are uniformly generated as well as all discrete* **SUConv**-*objects.*

2.8 Theorem. **UG-SUConv** *is the* (*bi*)*coreflective hull of* **Unif** *in* **SUConv**.

Proof. 1) By 2.5, each uniform space is uniformly generated. Using 2.6 the bicoreflection hull of **Unif** is contained in **UG-SUConv**.
2) Let $(X, \mathcal{J}_X) \in |\mathbf{UG\text{-}SUConv}|$. Then for each $\mathcal{F} \in \mathcal{J}_X$, there are a $Y_\mathcal{F} \subset X$ and a semiuniformity $\mathcal{V}_\mathcal{F}$ on $Y_\mathcal{F}$ with $\mathcal{V}_\mathcal{F} \subset \mathcal{F}$ and $i_\mathcal{F} \times i_\mathcal{F}(\mathcal{V}_\mathcal{F}) \in \mathcal{J}_X$, where $i_\mathcal{F} : Y_\mathcal{F} \to X$ is the inclusion map. By [3; 37.A.8] there is a uniform space $(X_\mathcal{F}, [\mathcal{W}_\mathcal{F}]$ and a quotient map $f_\mathcal{F} : (X_\mathcal{F}, [\mathcal{W}_\mathcal{F}]) \to (Y_\mathcal{F}, [\mathcal{V}_\mathcal{F}])$ in **SUnif**. Since **SUnif** is bireflective in **SUConv**, it follows from 2.2 that
$f_\mathcal{F} : (X_\mathcal{F}, [\mathcal{W}_\mathcal{F}]) \to (Y_\mathcal{F}, [\mathcal{V}_\mathcal{F}])$ is also a quotient map in **SUConv**. Then the sink $(i_\mathcal{F} \circ f_\mathcal{F} : (X_\mathcal{F}, [\mathcal{W}_\mathcal{F}]) \to (X, \mathcal{J}_X))_{\mathcal{F} \in \mathcal{J}_X}$ is final in **SUConv**, which implies that $(X, \mathcal{J}_X)$ is uniformly generated (cf. 0.1).

2.9 Remark. Obviously, **UG-SUConv** is also the bicoreflective hull of **SUnif** in **SUConv** (put $f_\mathcal{F} = 1_{Y_\mathcal{F}}$ in the proof of 2.8).

2.10 Proposition. *Subspaces (in* **SUConv***) of uniformly generated spaces are uniformly generated.*

Proof. Let $\mathbf{X} = (X, \mathcal{J}_X)$ be a uniformly generated space, i.e. there is a final sink $(f_i : \mathbf{X}_i \to \mathbf{X})_{i \in I}$ in **SUConv** such that $\mathbf{X}_i$ is uniform for each $i \in I$. If $Y \subset X$ and $\mathbf{Y}$ is the subspace (in **SUConv**) of $\mathbf{X}$ with underlying set Y, then, since **SUConv** is extensional, $(g_i : \mathbf{Y}_i \to \mathbf{Y})_{i \in I}$ is final in **SUConv**, where $\mathbf{Y}_i$ is the subspace von $\mathbf{X}_i$ with underlying set $f_i^{-1}[X]$ and $g_i : f_i^{-1}[X_i] \to Y$ is the corresponding restriction of f_i. Thus, $\mathbf{Y}$ is uniformly generated, because each $\mathbf{Y}_i$ is uniform.

2.11 Proposition. *Finite products (in* **SUConv***) of uniformly generated spaces are uniformly generated.*

Proof. Let $\mathbf{X} = (X, \mathcal{J}_X)$ and $\mathbf{Y} = (Y, \mathcal{J}_Y)$ be uniformly generated spaces. Hence, there are final epi-sinks $(f_i : \mathbf{X}_i \to \mathbf{X})_{i \in I}$ and $(g_j : \mathbf{Y}_j \to \mathbf{Y})_{j \in J}$, where $\mathbf{X}_i$ and $\mathbf{Y}_j$ are uniform spaces. Obviously, $(f_i \times g_j : \mathbf{X}_i \times \mathbf{Y}_j \to \mathbf{X} \times \mathbf{Y})_{(i,j) \in I \times J}$ is an epi-sink, and $f_i \times g_j = (1_{\mathbf{X}} \times g_j) \circ (f_i \times 1_{\mathbf{Y}_i})$ for each $(i,j) \in I \times J$. Since **SUConv** is cartesian closed, $(f_i \times 1_{\mathbf{Y}_j} : \mathbf{X}_i \times \mathbf{Y}_j \to \mathbf{X} \times \mathbf{Y}_j)_{i \in I}$ is a final epi-sink for each $j \in J$, and $(1_{\mathbf{X}} \times g_j : \mathbf{X} \times \mathbf{Y}_j \to \mathbf{X} \times \mathbf{Y})_{j \in J}$ is likewise a final epi-sink. Thus, the epi-sink $(f_i \times g_j)_{(i,j) \in I \times J}$ is final and, since **Unif** is closed under formation of products, $\mathbf{X} \times \mathbf{Y}$ is uniformly generated.

2.12 Theorem. **UG-SUConv** *is a topological universe whose natural function spaces and one-point extensions arise from the corresponding ones in* **SUConv** *by bicoreflective* **UG-SUConv***-modification.*

Proof. Using 2.6, 2.10, 2.11 and the fact that **SUConv** is a topological universe, the result follows from [7; 3.1.7] and [7; 3.2.5]

2.13 Proposition. *A convergence space $(X, \mathcal{J}_X)$ is uniformly generated iff the following is satisfied: For each $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$, there exist some $\mathcal{G} \in F(X)$ with $(\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}}$, a subset Y of X and a semiuniformity $\mathcal{V}$ on Y such that*

$$\mathcal{G} \times \mathcal{G} \subset i_Y \times i_Y(\mathcal{V}) \subset \mathcal{F} \times \mathcal{F}$$

where $i_Y : Y \to X$ is the inclusion map.

2.14 Corollary. *A topological semiuniform convergence space $(X, \mathcal{J}_X)$ is uniformly generated iff for each $x \in X$, there are a subset Y_X of X and a semiuniformity $\mathcal{V}_{Y_X}$ (on Y_X), which is a base for $\mathcal{U}(x) \times \mathcal{U}(x)$, where $\mathcal{U}(x)$ denotes the neighborhood filter of x in the symmetric topological space corresponding to $(X, \mathcal{J}_X)$.*

2.15 Examples. 1) Quotients of coproducts of indiscrete semiuniform convergence spaces are uniformly generated convergence spaces (indiscrete spaces are topological and uniform!).
2) Discrete semiuniform convergence spaces are uniformly generated (as coproducts of their one-point subspaces) and topological.
3) If D_2 denotes the two-point discrete semiuniform convergence space (=two-point discrete topological space), then the product $D_2^{\mathbb{N}}$ of countably many copies of D_2 are not uniformly generated, but topological, i.e. the *Cantor set*. (If $D_2^{\mathbb{N}}$ would be uniformly generated, by 2.14 there were some $x \in D_2^{\mathbb{N}}$, a subset Y_X of X and a semiuniformity $\mathcal{V}_{Y_X}$ with
$(\mathcal{V}_{Y_X}) \in \mathcal{U}(x) \times \mathcal{U}(x)$. Choose $x = \overline{0}$, i.e. $x(n) = 0$ for each $n \in \mathbb{N}$, and
$U_1(\overline{0}) = \prod_{i=1}^{\infty} U_1^i(0)$, where $U_1^1 = \{0\}$ and $U_1^i = \{0,1\}$ for each $i \in \mathbb{N} \setminus \{1\}$
$U_2(\overline{0}) = \prod_{i=1}^{\infty} V_2^i(0)$, where $V_2^2 = \{0\}$ and $V_2^i = \{0,1\}$ for each $i \in \mathbb{N} \setminus \{2\}$
$\vdots$
By assumption, $Y_{\overline{0}}$ would be contained in each $U_i(\overline{0})$, i.e. $Y_{\overline{0}} = \{\overline{0}\}$, since for each $y \in Y_{\overline{0}}, y(i) \in \{0\}$ for each $i \in \mathbb{N}$. Thus, $\mathcal{V}_{Y_{\overline{0}}} = \{Y_{\overline{0}} \times Y_{\overline{0}}\} = \{\{\overline{0}\} \times \{\overline{0}\}\}$. Consequently, $(\mathcal{V}_{Y_{\overline{0}}}) = \dot{\overline{0}} \times \dot{\overline{0}} = \mathcal{U}(\overline{0}) \times \mathcal{U}(\overline{0})$, which would imply $\mathcal{U}(\overline{0}) = \dot{\overline{0}}$, i.e. $\overline{0}$ would be an isolated point. But this is impossible, since the Cantor set does not have isolated points.)
This example shows that *countable products of uniformly generated spaces need not be uniformly generated.*

2.16 Problem. M. Hušek and M. D. Rice [4] have proved that in **Unif** (arbitrary) products of quotients are quotients. It is an open problem, whether the same is true for **UG-SUConv**.

2.17 Diagram
In the following diagram **ULim** [resp. **SULim**] denotes the construct of uniform limit spaces [resp. semiuniform limit spaces] (cf. [6] or [7] for the definition) and r (resp. c) stands for embedding as a bireflective (resp. bicoreflective) subconstruct. It shows how **Unif** is related to **UG-SUConv** and to other topological subconstructs of **SUConv**.

$$
\begin{tikzcd}
 & \mathbf{SUConv} \\
\mathbf{UG\text{-}SUConv} \arrow[ur, "c"] & \\
 & \mathbf{SULim} \arrow[uu, "r"'] \\
\mathbf{SUnif} \arrow[uu, "r"'] \arrow[ur, "r"] & \\
 & \mathbf{ULim} \arrow[uu, "r"'] \\
\mathbf{Unif} \arrow[uu, "r"'] \arrow[ur, "r"] &
\end{tikzcd}
$$

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