A BETTER FRAMEWORK FOR FIRST COUNTABLE SPACES

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Dedicated to Professor Som Naimpally on the occasion of his 70th birthday

Abstract. In the realm of semiuniform convergence spaces first countability is divisible and leads to a well-behaved topological construct with natural function spaces and one-point extensions such that countable products of quotients are quotients. Every semiuniform convergence space (e.g. symmetric topological space, uniform space, filter space etc.) has an underlying first countable space. Several applications of first countability in a broader context than the usual one of topological spaces are studied.

Mathematics Subject Classifications (2000): 54A05, 54C35, 54D55, 54E15, 18A40, 18D15.

Key words: First axiom of countability, second axiom of countability, countably compact, sequentially compact, sequentially complete, continous convergence, sequentially continous, semiuniform convergence spaces, convergence spaces, filter spaces, topological spaces, uniform spaces, bicoreflective subconstruct, cartesian closedness.

0. Introduction

Recently, the author ([4], [5]) has shown that a topologist's life is easier in Convenient Topology, where mainly semiuniform convergence spaces are considered which form a strong topological universe, i.e. a topological construct which has natural function spaces and one-point extensions such that products of quotients are quotients, denoted by **SUConv**. In **SUConv**, topological and uniform concepts are available and fundamental constructions can be easily described. In the realm of topological spaces the first axiom of countability plays an essential role whenever sequences are prefered rather than filters (e.g. in Analysis). Unfortunately, quotients of first countable topological spaces need not be first countable. This situation can be improved in the framework of semiuniform convergence spaces: *First countability* *is divisible*. Since first countability is also summable, this implies that the construct **FC-SUConv** of first countable semiuniform convergence spaces is bicoreflective in **SUConv**, i.e. every semiuniform convergence space has an underlying first countable space. Another consequence is the following: **FC-SUConv** is a topological universe (= cartesian closed and extensional topological construct) such that countable products of quotients are quotients.

Not only first countable (symmetric) topological spaces, but also uniform spaces with a countable base, i.e. pseudometrizable uniform spaces, are first countable (as semiuniform convergence spaces). For uniform spaces, there is no difference between the first and the second axiom of countability, when these axioms are generalized to semiuniform convergence spaces.

The applications of first countability, studied here, cover the following themes:

1. The definition of the closure of a subset by means of sequences.
2. The definition of completeness by means of sequences.
3. The equivalence of countable compactness and sequential compactness.
4. The definition of continuous convergence by means of sequences.
5. The definition of continuity by means of sequences.

1. Preliminaries

1.1 Definitions. 1) A *semiuniform convergence space* is a pair $(X, \mathcal{J}_X)$, where X is a set and $\mathcal{J}_X$ a set of filters on $X \times X$ such that the following are satisfied:
UC$_1$) The filter $\dot{x} \times \dot{x}$, generated by $\{(x,x)\}$, belongs to $\mathcal{J}_X$ for each $x \in X$,
UC$_2$) $\mathcal{G} \in \mathcal{J}_X$, whenever $\mathcal{F} \in \mathcal{J}_X$ and $\mathcal{F} \subset \mathcal{G}$,
UC$_3$) $\mathcal{F} \in \mathcal{J}_X$ implies $\mathcal{F}^{-\perp} = \{F^{-\perp} : F \in \mathcal{F}\} \in \mathcal{J}_X$, where $F^{-\perp} = \{y, x) : (x,y) \in F\}$.
If $(X, \mathcal{J}_X)$ is a semiuniform convergence space, then the elements of $\mathcal{J}_X$ are called *uniform filters*.
2) A map $f : (X, \mathcal{J}_X) \to (Y, \mathcal{J}_Y)$ between semiuniform convergence spaces is called *uniformly continuous* provided that $(f \times f)(\mathcal{F}) \in \mathcal{J}_Y$ for each $\mathcal{F} \in \mathcal{J}_X$.

3) The construct of semiuniform convergence spaces (and uniformly continuous maps) is denoted by **SUConv**.

1.2 Remark. **SUConv** is a topological construct, where initial and final structures have an easy description:
If X is a set, $((X_i, \mathcal{J}_{X_i}))_{i\in I}$ a family of semiuniform convergence spaces and $(f_i : X \to X_i)_{i\in I}$ (resp. $(f_i : X_i \to X)_{i\in I}$) a family of maps, then
$\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : (f_i \times f_i)(\mathcal{F}) \in \mathcal{J}_{X_i}$ for each $i \in I\}$ (resp. $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X)$: there is some $i \in I$ and some $\mathcal{F}_i \in \mathcal{J}_{X_i}$ with $(f_i \times f_i)(\mathcal{F}_i) \subset \mathcal{F}\} \cup \{\dot{x} \times \dot{x} : x \in X\}$)
is the *initial* (resp. *final*) **SUConv**-structure w.r.t. the given data, where $F(X \times X)$ denotes the set of all filters on $X \times X$.

1.3 Examples. 1) Let $(X, \mathcal{X})$ be a (symmetric) topological space. Then its corresponding semiuniform convergence space $(X, \mathcal{J}_{\gamma_{q_\mathcal{X}}})$ is defined by
$\mathcal{F} \in \mathcal{J}_{\gamma_{q_\mathcal{X}}}$ iff there is some $x \in X$ such that $\mathcal{F} \supset \mathcal{U}_\mathcal{X}(x) \times \mathcal{U}_\mathcal{X}(x)$, where $\mathcal{U}_\mathcal{X}(x)$ denotes the neighborhood filter of x in $(X, \mathcal{X})$.
2) Let $(X, \mathcal{W})$ be a uniform space. Then its corresponding semiuniform convergence space $(X, [\mathcal{W}])$ is defined by

$$\mathcal{F} \in [\mathcal{W}] \text{ iff } \mathcal{F} \supset \mathcal{W}.$$

3) Let $\mathbf{X} = (X, \mathcal{J}_X)$ and $\mathbf{Y} = (Y, \mathcal{J}_Y)$ be semiuniform convergence spaces. Then there is a coarsest **SUConv**-structure $\mathcal{J}_{X,Y}$ on the set $[\mathbf{X}, \mathbf{Y}]$ of all uniformly continuous maps from $\mathbf{X}$ into $\mathbf{Y}$ such that the evaluation map $e_{X,Y} : \mathbf{X} \times ([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y}) \to \mathbf{Y}$ (defined by $e_{X,Y}(x, f) = f(x)$) is uniformly continuous. It is defined by

$$\Phi \in \mathcal{J}_{X,Y} \text{ iff } \Phi(\mathcal{F}) \in \mathcal{J}_Y \text{ for each } \mathcal{F} \in \mathcal{J}_X$$

($\Phi(\mathcal{F})$ denotes the filter generated by $\{A(F) : A \in \Phi$ and $F \in \mathcal{F}\}$ with $A(F) = \{(f(x), g(x)) : (f, g) \in A$ and $(x, y) \in F\}$).
$([\mathbf{X}, \mathbf{Y}], \mathcal{J}_{X,Y})$ is called a *natural function space.*
4) Let $\mathbf{X} = (X, \mathcal{J}_X)$ be a semiuniform convergence space. Let $X^* = X \cup \{\infty_X\}$ with $\infty_X \notin X$ and $\mathcal{J}_{X^*} = \{\mathcal{H} \in F(X^* \times X^*)$: the trace of $\mathcal{H}$ on $X \times X$ exists and belongs to $\mathcal{J}_X$ or the trace on $X \times X$ does not exist$\}$.
Then $(X, \mathcal{J}_{X^*})$ is called a *one-point extension* of $(X, \mathcal{J}_X)$. Obviously, $\mathcal{J}_{X^*}$ is the coarsest **SUConv**-structure on X^* such that $(X, \mathcal{J}_X)$ is a subspace of $(X^*, \mathcal{J}_{X^*})$.

1.4 Remarks. 1) The existence of one-point extensions implies that quotients are hereditary, i.e. if $f : X \to Y$ is a quotient map in **SUConv**, then for every $Z \subset Y$, the map $f \mid f^{-\perp}[Z] : f^{-\perp}[Z] \to Z$ is again a quotient map.
2) Concerning the definitions of (symmetric) Kent convergence spaces and filter spaces, and their relations to semiuniform convergence spaces, see [4] (or [5]).

2. First countable spaces

2.1 Definitions. 1) A semiuniform convergence space $(X, \mathcal{J}_X)$ is said to fulfill the *first axiom of countability* or to be *first countable* iff each uniform filter has a uniform subfilter with a countable base.
2) A filter space (X, γ) is called *first countable* provided that each $\mathcal{F} \in \gamma$ has a subfilter $\mathcal{G} \in \gamma$ with a countable base.
3) A Kent convergence space (X, q) is *first countable* iff for each $(\mathcal{F}, x) \in q$, there is some $(\mathcal{G}, x) \in q$ such that $\mathcal{G}$ is a subfilter of $\mathcal{F}$ with a countable base.

2.2 Proposition. Let $(X, \mathcal{J}_X)$ be a semiuniform convergence space fulfilling the first axiom of countability. Then the underlying filter space $(X, \gamma_{\mathcal{J}_X})$ is first countable.

Proof. Let $\mathcal{F} \in \gamma_{\mathcal{J}_X}$. Then $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$ and by assumption there is some $\mathcal{G} \in \mathcal{J}_X$ with a countable base $\mathcal{B} = \{B_1, B_2, \dots, B_n \dots\}$ such that $\mathcal{G} \subset \mathcal{F} \times \mathcal{F}$. Thus, for each $n \in \mathbb{N} = \{1, 2, \dots\}$, there exists some $F_n \in \mathcal{F}$ with $F_n \times F_n \subset B_n$. Obviously, the finite intersections of elements of $\{F_n : n \in \mathbb{N}\}$ form a countable base of some subfilter $\mathcal{H}$ of $\mathcal{F}$. Since $\mathcal{G} \subset \mathcal{H} \times \mathcal{H}$, it follows that $\mathcal{H} \times \mathcal{H}$ belongs to $\mathcal{J}_X$, i.e. $\mathcal{H} \in \gamma_{\mathcal{J}_X}$.

2.3 Proposition. Let (X, γ) be a filter space and $(X, \mathcal{J}_\gamma)$ its corresponding semiuniform convergence space. Then $(X, \mathcal{J}_\gamma)$ is first countable iff (X, γ) is first countable in the sense of 2.1.2).

Proof. "$\Rightarrow$". This follows immediately from 2.2, since $\gamma = \gamma_{\mathcal{J}_\gamma}$.
"$\Leftarrow$". Let $\mathcal{F} \in \mathcal{J}_\gamma$. Then there is some $\mathcal{G} \in \gamma$ with $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}$. Furthermore, there is some $\mathcal{H} \in \gamma$ with a countable base $\mathcal{B}$ such that $\mathcal{H} \subset \mathcal{G}$. Consequently, $\mathcal{H} \times \mathcal{H} \subset \mathcal{G} \times \mathcal{G} \subset \mathcal{F}$, where $\mathcal{H} \times \mathcal{H} \in \mathcal{J}_\gamma$ has a countable base, namely $\{B \times B : B \in \mathcal{B}\}$.
2.4 Proposition. Let (X, γ) be a first countable filter space. Then the un-

derlying (symmetric) Kent convergence space (X, q_γ) is also first countable.
Proof. Let $(\mathcal{F}, x) \in q_\gamma$, i.e. $\mathcal{F} \cap \dot{x} \in \gamma$. By assumption, there is some $\mathcal{G} \in \gamma$ with a countable base such that $\mathcal{G} \subset \mathcal{F} \cap \dot{x}$. Then $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G} \cap \dot{x} = \mathcal{G} \in \gamma$, i.e. $(\mathcal{G}, x) \in q_\gamma$.

2.5 Proposition. Let (X, q) be a symmetric Kent convergence space and $(X, \mathcal{J}_{\gamma_q})$ its corresponding semiuniform convergence space. Then $(X, \mathcal{J}_{\gamma_q})$ is first countable iff (X, q) is first countable in the sense of 2.1 3).

Proof. "$\Rightarrow$". Since $\gamma_{\mathcal{J}_{\gamma_q}} = \mathcal{S}_q$ and $q_{\gamma_q} = q, (X, q)$ is first countable by means of the above propositions.
"$\Leftarrow$". Let $\mathcal{F} \in \mathcal{J}_{\gamma_q}$. Then there is some $(\mathcal{G}, x) \in q$ such that $\mathcal{G} \times \mathcal{G} \subset \mathcal{F}$. By assumption, there exists $(\mathcal{H}, x) \in q$ such that $\mathcal{H}$ is a subfilter of $\mathcal{G}$ with a countable base $\mathcal{B}$. Hence, $\mathcal{H} \times \mathcal{H} \subset \mathcal{F}$, where $\mathcal{H} \times \mathcal{H} \in \mathcal{J}_{\gamma_q}$ has the countable base $\{B \times B : B \in \mathcal{B}\}$.

2.6 Corollary. The underlying (symmetric) Kent convergence $(X, q_{\gamma_{\mathcal{J}_X}})$ of a first countable semiuniform convergence space $(X, \mathcal{J}_X)$ is first countable.

Proof. Apply 2.2 and 2.4.

2.7 Corollary. Let $(X, \mathcal{X})$ be a symmetric topological space (=R_0-space) and define $q_\mathcal{X} \subset F(X) \times X$ by $(\mathcal{F}, x) \in q_\mathcal{X}$ iff $\mathcal{F} \supset \mathcal{U}_\mathcal{X}(x)$. Then the following are equivalent:
(1) $(X, \mathcal{X})$ is first countable in the usual sense, i.e. for each $x \in X$, $\mathcal{U}_\mathcal{X}(x)$ has a countable base.
(2) $(X, \mathcal{X})$ is first countable as a semiuniform convergence space, i.e. $(X, \mathcal{J}_{\gamma_{q_\mathcal{X}}})$ is first countable.
(3) $(X, \mathcal{X})$ is first countable as a Kent convergence space, i.e. $(X, q_\mathcal{X})$ is first countable in the sense of 2.1.3).

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{F} \in \mathcal{J}_{\gamma_{q_\mathcal{X}}}$. Then there is some $x \in X$ such that $\mathcal{U}_\mathcal{X}(x) \times \mathcal{U}_\mathcal{X}(x) \subset \mathcal{F}$. Since, by assumption, $\mathcal{U}_\mathcal{X}(x)$ has a countable base, the subfilter $\mathcal{U}_\mathcal{X}(x) \times \mathcal{U}_\mathcal{X}(x)$ of $\mathcal{F}$ has also a countable base and belongs to $\mathcal{J}_{\gamma_{q_\mathcal{X}}}$.
(2) $\Rightarrow$ (3). Apply 2.6.
(3) $\Rightarrow$ (1). Let $x \in X$ and consider $\mathcal{U}_\mathcal{X}(x)$. Then $(\mathcal{U}_\mathcal{X}(x), x) \in q_\mathcal{X}$ and, by assumption, there is a subfilter $\mathcal{G}$ of $\mathcal{U}_\mathcal{X}(x)$ with a countable base such that $(\mathcal{G}, x) \in q_\mathcal{X}$. Hence, $\mathcal{U}_\mathcal{X}(x) \subset \mathcal{G} \subset \mathcal{U}_\mathcal{X}(x)$, which implies $\mathcal{G} = \mathcal{U}_\mathcal{X}(x)$.

2.8 Proposition. Let $(X, \mathcal{W})$ be a uniform space and $(X, [\mathcal{W}])$ its corresponding semiuniform convergence space.
Then the following are equivalent:
(1) $(X, \mathcal{W})$ is first countable as a semiuniform convergence space, i.e. $(X, [\mathcal{W}])$ is first countable.
(2) The filter $\mathcal{W}$ of entourages has a countable base.

Proof. Since $[\mathcal{W}]$ consists of all filters $\mathcal{F}$ on $X \times X$ containing $\mathcal{W}$, the proof is obvious.

2.9 Remark. It is well-known that a uniform space is pseudometrizable, i.e. its uniformity is induced by a pseudometric, iff its filter of encourages has a countable base. Thus, *a semiuniform convergence space* $(X, \mathcal{J}_X)$ *is pseudometrizable* (i.e. there is a pseudometric d on X such that $\mathcal{J}_X = \{\mathcal{F} \in F(X \times X) : \mathcal{F} \supset \{V_\varepsilon : \varepsilon > 0\}\}$ with $\vee_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}$) *iff it is uniform* (i.e. there is a uniformity $\mathcal{W}$ on X such that $\mathcal{J}_X = [\mathcal{W}]$) *and first countable.*

2.10 Proposition. The construct **FC-SUConv** of first countable semiuniform convergence spaces (and uniformly continous maps) is bicoreflective in **SUConv**, where $1_X : (X, (\mathcal{J}_X)_{FC}) \to (X, \mathcal{J}_X)$ with $(\mathcal{J}_X)_{FC} = \{\mathcal{F} \in \mathcal{J}_X$: there is some $\mathcal{G} \in \mathcal{J}_X$ with $\mathcal{G} \subset \mathcal{F}$ and $\mathcal{G}$ has a countable base$\}$ is the bicoreflection of $(X, \mathcal{J}_X) \in|$ **SUConv** $|$ w.r.t. **FC-SUConv**, and $(X, (\mathcal{J}_X)_{FC})$ is called the underlying first countable semiuniform convergence space of $(X, \mathcal{J}_X)$.

Proof. Obviously, if $f : (Y, \mathcal{J}_Y) \to (X, \mathcal{J}_X)$ is a uniformly continuous map from a first countable semiuniform convergence space $(Y, \mathcal{J}_Y)$ into a semiuniform convergence space $(X, \mathcal{J}_X)$, then $f : (Y, \mathcal{J}_Y) \to (X, (\mathcal{J}_X)_{FC})$ is also uniformly continuous.

2.11 Corollary. First countability is summable and divisible.

2.12 Remarks. 1) In classical General Topology first countability is not divisible, e.g. the usual topological space $\mathbb{R}_t$ of real numbers is first countable, whereas the topological quotient space $\mathbb{R}_t/\mathbb{N}$, obtained by identifying the set $\mathbb{N}$ of positive integers to a point is not first countable.
If $\mathbb{R}_t$ is regarded as a semiuniform convergence space and the quotient $\mathbb{R}_t/\mathbb{N}$

is formed in **SUConv**, then $\mathbb{R}_t/\mathbb{N}$ is first countable, but no longer topological. Indeed, $\mathbb{R}_t/\mathbb{N}$ is a convergence space (=symmetric Kent convergence space), since $\mathbb{R}_t$ is a convergence space and convergence spaces are closed under formation of quotients in **SUConv**.
2) The convergence spaces are exactly the quotients (in **SUConv**) of the topological semiuniform convergence spaces (= symmetric topological spaces). This has been proved in the language of nearness by W. A. Robertson [6] (cf. also [1]).

2.13 Proposition. 1) **FC-SUConv** is closed under formation of subspaces in **SUConv**.
2) **FC-SUConv** is closed under formation of countable products in **SU-Conv**.

Proof. 1) Let $(X, \mathcal{J}_X) \in\mid$ **SUConv** $\mid, A \subset X$ and let $i : A \to X$ be the inclusion map. If the initial **SUConv**-structure on A w.r.t. $i : A \to X$ is denoted by $\mathcal{J}_A$, then let $\mathcal{F} \in \mathcal{J}_A$. By definition, $(i \times i)(\mathcal{F}) \in \mathcal{J}_X$, and by assumption, there is some $\mathcal{G} \in \mathcal{J}_X$ with a countable base $\mathcal{B}$ such that $\mathcal{G} \subset (i \times i)(\mathcal{F})$. Hence, the subfilter $(i \times i)^{-\perp}(\mathcal{G})$ of $\mathcal{F}$ belongs to $\mathcal{J}_A$ and has the countable base $\{B \cap (A \times A) : B \in \mathcal{B}\}$.
2) Let $((X_i, \mathcal{J}_{X_i}))_{i \in I}$ be a countable family of first countable semiuniform convergence spaces and let $(\Pi_{i \in I} X_i, \mathcal{J}_X)$ be their product. Further, let $\mathcal{F} \in \mathcal{J}_X$. Then $(p_i \times p_i)(\mathcal{F}) \in \mathcal{J}_{X_i}$ for each $i \in I$, where $p_i : \Pi_{i \in I} X_i \to X_i$ denotes the $i - th$ projection. By assumption, there is some subfilter $\mathcal{G}_i \in \mathcal{J}_{X_i}$ of $(p_i \times p_i)(\mathcal{F})$ with a countable base $\mathcal{B}_i$. Hence, $\Pi_{i \in I} \mathcal{G}_i \subset \Pi_{i \in I}(p_i \times p_i)(\mathcal{F}) \subset \mathcal{F}$ where $\Pi_{i \in I}(X_i \times X_i)$ and $(\Pi X_i) \times (\Pi_{i \in I} X_i)$ are not distinguished. Obviously, $\Pi_{i \in I} \mathcal{G}_i \in \mathcal{J}_X$. The set $\mathcal{B}$ consisting of all finite intersections of elements of the countable set $\mathcal{B}' = \{p_i \times p_i)^{-1}[B_i] : i \in I, B_i \in \mathcal{B}_i\}$ is a countable base of $\Pi_{i \in I} \mathcal{G}_i$.

2.14 Remark. Since the construct $\mathbf{Top}_S$ of symmetric topological spaces (and continuous maps) is closed under formation of products in **SUConv**, arbitrary products of first countable semiuniform convergence spaces need not be first countable, e.g. it is well-known that the topological product space $\mathbb{R}_t^{\mathbb{R}}$ is not first countable.
2.15 Theorem. **FC-SUConv** is a topological universe, where the natural function spaces (resp. one-point extensions) arise from the natural function spaces (resp. one-point extensions) in **SUConv** by forming the underlying

first countable spaces. Furthermore, in **FC-SUConv** countable products of quotients are quotients.

Proof. 1) Since **FC-SUConv** is bicoreflective in the topological universe **SUConv** and closed under formation of finite products and subspaces, it is a topological universe.
2) Since countable products and quotients are formed in **FC-SUConv** as in **SUConv**, where products of quotients are quotients, in **FC-SUConv** countable products of quotients are quotients.

2.16. A much stronger condition than the first axiom of countability is the second axiom of countability which can be defined in the realm of semiuniform convergence spaces as follows:
Definition. A semiuniform convergence space $(X, \mathcal{J}_X)$ is said to fulfill the *second axiom of countability* or to be *second countable* if there is a countable set $\mathcal{A} \subset \mathcal{P}(X \times X)$ such that each $\mathcal{F} \in \mathcal{J}_X$ has a subfilter $\mathcal{G} \in \mathcal{J}_X$ with a base of elements of $\mathcal{A}$.

2.17 Proposition. Every second countable semiuniform convergence space is first countable.

2.18 Proposition. Let $(X, \mathcal{W})$ be a uniform space and $(X, [\mathcal{W}])$ its corresponding semiuniform convergence space. Then the following are equivalent:
(1) $(X, [\mathcal{W}])$ is first countable,
(2) $(X, [\mathcal{W}])$ is second countable,
(3) $\mathcal{W}$ has a countable base.

Proof. The equivalence of (1) and (3) has been proved under 2.8. Since 2.17 is valid, it suffices to prove:
(1) $\Rightarrow$ (2). Let $\mathcal{F} \in \mathcal{J}_X$. Then $\mathcal{F} \supset \mathcal{W}$ and by (3), $\mathcal{W}$ has a countable base $\mathcal{A}$.

2.19 Proposition. Let $(X, \mathcal{X})$ be a symmetric topological space and $(X,\mathcal{J}_{\gamma_{q\mathcal{X}}})$ its corresponding semiuniform convergence space. Then $(X, \mathcal{J}_{\gamma_{q\mathcal{X}}})$ is second countable iff $(X, \mathcal{X})$ is second countable in the usual sense.

Proof. "$\Rightarrow$". Let $\mathcal{A} \subset \mathcal{P}(X \times X)$ be countable and let each $\mathcal{F} \in \mathcal{J}_{\gamma_{q\mathcal{X}}}$ have a subfilter $\mathcal{G} \in \mathcal{J}_{\gamma_{q\mathcal{X}}}$ with a base of elements of $\mathcal{A}$.
For each $x \in X, \mathcal{U}_{\mathcal{X}}(x) \times \mathcal{U}_{\mathcal{X}}(x) \in \mathcal{J}_{\gamma_{q\mathcal{X}}}$, and by assumption there is some

filter $\mathcal{G} \in \mathcal{J}_{\gamma_{q_{\mathcal{X}}}}$, i.e. $\mathcal{G} \supset \mathcal{U}_{\mathcal{X}}(y) \times \mathcal{U}_{\mathcal{X}}(y)$ for some $y \in X$, such that

$$(*) \;\; \mathcal{U}_{\mathcal{X}}(y) \times \mathcal{U}_{\mathcal{X}}(y) \subset \mathcal{G} \subset \mathcal{U}_{\mathcal{X}}(x) \times \mathcal{U}_{\mathcal{X}}(x)$$

where $\mathcal{G}$ has a base of elements of $\mathcal{A}$. If $p_1 : X \times X \to X$ denotes the first projection, it follows from $(*)$

$$\mathcal{U}_{\mathcal{X}}(y) \subset p_1(\mathcal{G}) \subset \mathcal{U}_{\mathcal{X}}(x)$$

and, since $(X, \mathcal{X})$ is symmetric,

$$p_1(\mathcal{G}) = \mathcal{U}_{\mathcal{X}}(x) = \mathcal{U}_{\mathcal{X}}(y).$$

Thus, $\mathcal{U}_{\mathcal{X}}(x)$ has a base of elements of the countable set $\mathcal{A}' = \{p_1[A] : A \in \mathcal{A}\}$. Let $\mathcal{A}'' = \{(p_1[A])^0 : A \in \mathcal{A}\}$. Then $\mathcal{A}''$ is countable and $\mathcal{U}_{\mathcal{X}}(x)$ has also a base of elements of $\mathcal{A}''$. Each $O \in \mathcal{X}$ is a union of elements of $\mathcal{A}''$, i.e. $(X, \mathcal{X})$ is second countable.
"$\Leftarrow$". Let $\mathcal{X}$ have a countable base $\mathcal{B}$ and let $\mathcal{F} \in \mathcal{J}_{\gamma_{q_{\mathcal{X}}}}$, i.e. there is some $x \in X$ such that $\mathcal{F} \supset \mathcal{U}_{\mathcal{X}}(x) \times \mathcal{U}_{\mathcal{X}}(x)$.
Obviously, $\mathcal{U}_{\mathcal{X}}(x)$ has a base of elements of $\mathcal{B}$, and $\mathcal{U}_{\mathcal{X}}(x) \times \mathcal{U}_{\mathcal{X}}(x)$ has a base of elements of the countable set $\mathcal{A} = \{B \times B : B \in \mathcal{B}\}$. Consequently, $(X, \mathcal{J}_{\gamma_{q_{\mathcal{X}}}})$ is second countable.

2.20 Remark. The second axiom of countable does not have the nice structural behaviour as the first axiom of countability. It is easily checked that second countable semiuniform convergence spaces are hereditary, countably productive, countable summable and divisible, but they don't form a bicoreflective subconstruct of **SUConv**, since arbitrary sums of second countable spaces are not second countable, which is known from second countable topological spaces (note: coproducts [=sums] of topological semiuniform convergence spaces are topological). Nevertheless, in the realm of semiuniform convergence spaces, *second countability is divisible*, a result which is not true in the framework of topological spaces.

3. Applications of first countability

3.1 Remark. If $(X, \mathcal{J}_X)$ is a semiuniform convergence space, then a filter $\mathcal{F}$ on X is called a *Cauchy filter* iff $\mathcal{F} \times \mathcal{F} \in \mathcal{J}_X$ (i.e. $\mathcal{F} \in \gamma_{\mathcal{J}_X}$), and it is said to *converge* to x (i.e. $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$) iff $\mathcal{F} \cap \dot{x}$ is a Cauchy filter.

A sequence (x_n) in X is a *Cauchy sequence* iff the elementary filter $\mathcal{F}_e$ of (x_n) is a Cauchy filter, and it converges to x iff the elementary filter of (x_n) converges to x.

3.2 Definition. For each subset A in a semiuniform convergence space $(X, \mathcal{J}_X)$, the closere $\overline{A}$ is defined as follows:

$$\overline{A} = \{x \in X : \text{there is some filter } \mathcal{F} \text{ on } X \text{ converging to } x \text{ such that } A \in \mathcal{F}\}.$$

3.3 Proposition. Let $(X, \mathcal{J}_X)$ be a first countable semiuniform convergence space and A a subset in $(X, \mathcal{J}_X)$. Then

$$\overline{A} = \{x \in X : \text{there is a sequence in } A \text{ converging to } x \text{ in } (X, \mathcal{J}_X)\}.$$

Proof. Let $x \in \overline{A}$. Then there is some filter $\mathcal{F}$ on X such that $A \in \mathcal{F}$ and $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$, i.e. $\mathcal{F}$ converges to x in $(X, \mathcal{J}_X)$. By 2.6, there is some $(\mathcal{G}, x) \in q_{\gamma_{\mathcal{J}_X}}$ such that $\mathcal{G} \subset \mathcal{F}$ has a countable base $\mathcal{B} = \{B_1, B_2, \dots\}$, where we may assume without loss of generality that $B_1 \supset B_2 \supset \dots$. For each $n \in \mathbb{N}$, choose some $x_n \in B_n \cap A$. Then $\mathcal{G} \subset \mathcal{F}_e((x_n))$, which implies that (x_n) converges to x in $(X, \mathcal{J}_X)$.
The inverse implication is evident.

3.4. Uniform spaces have been generalized to uniform limit spaces, which are semiuniform convergence spaces fulfilling the following two additional axioms:
$\mathrm{UC}_{4)}$ If $\mathcal{F}$ and $\mathcal{G}$ are uniform filters, then $\mathcal{F} \cap \mathcal{G}$ is also a uniform filter;
$\mathrm{UC}_{5)}$ If the composition $\mathcal{F} \circ \mathcal{G}$ of two uniform filters $\mathcal{F}$ and $\mathcal{G}$ exists, it is a uniform filter, which has $\{F \circ G : F \in \mathcal{F}, G \in \mathcal{G}\}$ as a base, where $\circ$ stands for the composition of relations.
The underlying filter spaces of uniform limit spaces are called Cauchy spaces.

3.5 Proposition. A first countable Cauchy space (X, γ) is complete iff it is sequentially complete (i.e. each Cauchy sequence in (X, γ) converges).

Proof. "$\Rightarrow$". Obvious.
"$\Leftarrow$". Let $\mathcal{F} \in \gamma$. By assumption, there is some $\mathcal{G} \in \gamma$ with a countable base $\mathcal{B} = \{B_1, B_2, \dots\}$ such that $\mathcal{G} \subset \mathcal{F}$. Without loss of generality, let $B_1 \supset B_2 \supset \dots$. For each $n \in \mathbb{N}$, choose some $x_n \in B_n$. Since the elementary filter $\mathcal{F}_e$ of (x_n) is finer than $\mathcal{G}$, it is a Cauchy filter. Furthermore,

$\gamma = \gamma_{\mathcal{J}^*_\gamma}$, where $\mathcal{J}^*_\gamma = \{\mathcal{G} \in F(X \times X) :$ there are $\mathcal{K}_1, \ldots \mathcal{K}_n \in \gamma$ with $\mathcal{G} \supset \cap_{i=1}^n \mathcal{K}_i \times \mathcal{K}_i\}$ is a uniform limit space structure for X. It follows from

$$\mathcal{G} \times \mathcal{F}_e \supset (\mathcal{G} \cap \mathcal{F}_e) \times (\mathcal{G} \cap \mathcal{F}_e) \in \mathcal{J}^*_\gamma,$$

$\mathcal{G} \times \mathcal{F}_e \in \mathcal{J}^*_\gamma$, and since by assumption $\mathcal{F}_e$ converges to some $x \in X, \mathcal{F}_e \times \dot{x} \in \mathcal{J}^*_\gamma$. Thus, $\mathcal{G} \times \dot{x} = (\mathcal{F}_e \times \dot{x}) \circ (\mathcal{G} \circ \mathcal{F}_e) \in \mathcal{J}^*_\gamma$, i.e. $\mathcal{G}$ converges to x, which implies that $\mathcal{F}$ converges to x.

3.6 Corollary. A first countable uniform limit space $(X, \mathcal{J}_X)$ is complete iff each Cauchy sequence in $(X, \mathcal{J}_X)$ converges.

Proof. By 2.2, the underlying filter space (=Cauchy space) of the first countable uniform limit space $(X, \mathcal{J}_X)$ is first countable. Applying 3.5, 3.6 is proved.

3.7 Definition. A semiuniform convergence space $(X, \mathcal{J}_X)$ is called
1) *countably compact* provided that each filter $\mathcal{F}$ on X with a countable base has an adherence point $x \in X$, i.e. there is some filter $\mathcal{G}$ on X converging to x such that $\mathcal{G} \supset \mathcal{F}$,
2) *sequentially compact* iff each sequence in X has a subsequence converging in $(X, \mathcal{J}_X)$.

3.8 Proposition. A semiuniform convergence space $(X, \mathcal{J}_X)$ is countably compact iff each sequence in X has an accumulation point (in $(X, q_{\gamma_{\mathcal{J}_X}})$).

Proof. "$\Rightarrow$". Let (x_n) be a sequence in X and $\mathcal{F}_e$ its elementary filter. Then $\mathcal{F}_e$ has a countable base and by assumption there is an adherence point $x \in X$ of $\mathcal{F}_e$, i.e. x is an accumulation point of (x_n).
"$\Leftarrow$". Let $\mathcal{F}$ be a filter on X with a countable base $\mathcal{B} = \{B_1, B_2, \ldots\}$ such that, without loss of generality, $B_1 \supset B_2 \supset \ldots$. For each $n \in \mathbb{N}$, choose some $x_n \in B_n$. The elementary filter $\mathcal{F}_e$ of (x_n) is finer that $\mathcal{F}$ and has an adherence point by assumption. Hence, $\mathcal{F}$ has an adherence point.

3.9 Proposition. 1) Every sequentially compact semiuniform convergence space is countably compact.
2) Every countably compact semiuniform convergence space, which is first countable, is sequentially compact.

Proof. 1) is obvious.
2) Let (x_n) be a sequence in X and $\mathcal{F}_e$ its elementary filter. Since $(X, \mathcal{J}_X)$ is countably compact, $\mathcal{F}_e$ has an adherence point $x \in X$, i.e. there is some filter $\mathcal{F} \supset \mathcal{F}_e$ such that $(\mathcal{F}, x) \in q_{\gamma_{\mathcal{J}_X}}$. By 2.6, there is a subfilter $\mathcal{G}$ of $\mathcal{F}$ with a countable base $\mathcal{B} = \{B_1, B_2, \dots\}$ converging to x. Without loss of generality, let $B_1 \supset B_2 \supset \dots$. For each $m \in \mathbb{N}$, $E_m \cap B_m \neq \emptyset$, where $E_m = \{x_n : n \geq m\}$. Choose $y_m \in B_m \cap E_m$ for each $m \in \mathbb{N}$. Then $(y_m)_{m\in\mathbb{N}}$ converges to x, since the elementary filter $\mathcal{F}_e'$ of (y_m) is finer than $\mathcal{G}$. Furthermore, there exists a subsequence (x_{n_j}) of (x_n) such that the elementary filter of (x_{n_j}) coincides with the elementary filter of (y_m), i.e. (x_{n_j}) converges to x.

3.10. Continuous convergence, introduced by H. Hahn [3] in Analysis, is nowadays mainly considered in the realm of limit spaces, where a Kent convergence space (X, q) is called a *limit space* provided that for any $x \in X$ and any two filter $\mathcal{F}$ and $\mathcal{G}$ on X converging to x, the intersection $\mathcal{F} \cap \mathcal{G}$ converges to x. Let $\mathbf{X} = (X, q)$ and $\mathbf{Y} = (Y, r)$ be limit spaces and let $[\mathbf{X}, \mathbf{Y}]$ be the set of all continuous maps between $\mathbf{X}$ and $\mathbf{Y}$, where a map $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff $(f(\mathcal{F}), f(x)) \in r$ for each $(\mathcal{F}, x) \in q$. A filter $\mathcal{F}$ on $[\mathbf{X}, \mathbf{Y}]$ *converges continously* to $f \in [\mathbf{X}, \mathbf{Y}]$ provided that for each $x \in X$ and each filter $\mathcal{G}$ on X converging to x in $\mathbf{X}$, the filter $e_{X,Y}(\mathcal{G} \times \mathcal{F})$ converges to $f(x)$ in $\mathbf{Y}$, where $\mathcal{G} \times \mathcal{F}$ denotes the product filter and $e_{X,Y}(x, g) = g(x)$ for each $(x, g) \in X \times [\mathbf{X}, \mathbf{Y}]$). A sequence (f_n) in $[\mathbf{X}, \mathbf{Y}]$ is said to converge continuously to $f \in [\mathbf{X}, \mathbf{Y}]$ iff the elementary filter of (f_n) converges continuously to f.
If a Kent convergence space $\mathbf{X} = (X, q)$ has the property that for each $x \in X$, the neighborhood filter $\mathcal{U}(x) = \bigcap\{\mathcal{F} \in F(X) : (\mathcal{F}, x) \in q\}$ converges to x, then it is called a pretopological space.

3.11 Proposition. Let $\mathbf{X} = (X, q)$ be a first countable limit space and $\mathbf{Y} = (Y, r)$ a pretopological space. Then a sequence (f_n) in $[\mathbf{X}, \mathbf{Y}]$ converges continuously to $f \in [\mathbf{X}, \mathbf{Y}]$ iff for each $x \in X$ and each sequence (x_n) in $\mathbf{X}$ converging to x, the sequence $(f_n(x_n))$ converges to $f(x)$ in $\mathbf{Y}$.
Proof. "$\Leftarrow$" (indirectly). If the elementary filter $\mathcal{F}_e$ of (f_n) does not converge continuously to $f \in [\mathbf{X}, \mathbf{Y}]$, there is some $(\mathcal{G}, x) \in q$ such that $e_{X,Y}(\mathcal{G} \times \mathcal{F})$ does not converge to $f(x)$, i.e. there is some $U \in \mathcal{U}(f(x))$ in $\mathbf{Y}$ such that for each $G \in \mathcal{G}$ and each $F \in \mathcal{F}$, there are $y \in G$ and $g \in F$ with $g(y) \notin U$. By assumption, there exists some $(\mathcal{H}, x) \in q$ such that $\mathcal{H}$ has a countable base

$\mathcal{B} = \{B_1, B_2, \dots\}$ and $\mathcal{H} \subset \mathcal{G}$. Without loss of generality, let $B_1 \supset B_2 \supset \dots$. Since for each $n \in \mathbb{N}, B_n \in \mathcal{G}$ and $\{f_m : m \geq n\} \in \mathcal{F}$, one can construct sequences (x_n) and $m_0 < m_1 < \dots$ such that $x_n \in B_n$ and $f_{m_n}(x_n) \notin U$. Define a sequence (y_n) in X by $y_0 = \dots = y_{m_0} = x_0, y_{m_0+1} = \dots = y_{m_1} = x_1, \dots$ and so on. Obviously, (y_n) converges to x in $\mathbf{X}$, but $(f_n(y_n))$ does not converge to $f(x)$, since $f_{m_n}(y_{m_n}) = f_{m_n}(x_n) \notin U$.
"$\Rightarrow$". This implication is straightforward.

3.12 Proposition. Let $\mathbf{X} = (X, q)$ be a first countable Kent convergence space and $\mathbf{Y} = (Y, r)$ a pretopological space. A map $f : \mathbf{X} \to \mathbf{Y}$ is continuous, where continuity is defined as under 3.10., iff for each $x \in X$ and each sequence (x_n) in X converging to x in $\mathbf{X}$, the sequence $(f(x_n))$ converges to $f(x)$ in $\mathbf{Y}$.
Proof. "$\Rightarrow$". This implication is evident.
"$\Leftarrow$". Since each filter $\mathcal{F}$ on X with a countable base is the intersection of all elementary filters on X containing $\mathcal{F}$ (cf. [2; 2.8.6.]) and each map $f : X \to Y$ preserves intersection of filters, the proof is obvious.

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