RECENT PROGRESS ON THE DAUGAVET PROPERTY

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1. Introduction

It is a remarkable result due to I.K. Daugavet [7] that the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|, \tag{1.1}$$

which has become known as the Daugavet equation, holds for compact operators on $C[0,1]$; shortly afterwards the same result for compact operators on $L_1[0,1]$ was discovered by G.Ya. Lozanovskii [19]. Over the years, (1.1) has been extended to larger classes of operators on various spaces; in particular, the Daugavet equation is known to hold for operators not fixing a copy of $C[0,1]$ defined on certain "large" subspaces of $C(K)$, where K is a compact space without isolated points, and for operators not fixing a copy of $L_1[0,1]$ defined on certain "large" subspaces of $L_1[0,1]$ ([14], [21], [27]). See also the examples in the next section.

Methods encountered in the investigation of (1.1) include Banach lattice techniques ([1], [2]), stochastic kernels and random measures ([12], [26], [28]) and other arguments from the geometry of Banach spaces ([13], [29]).

The Daugavet equation has proved useful in approximation theory, where it was used to find the best constants in certain inequalities [25], and in the geometry of Banach spaces. In [12], G. Godefroy, N. Kalton and D. Li observed for an operator V with $\|V\| < 2$ and $T := V - \mathrm{Id}$ satisfying (1.1) that V must be an isomorphism; indeed, (1.1) implies that $\|V - \mathrm{Id}\| < 1$, and hence the result follows from the Neumann series. They go on to apply this consequence to quotients by nicely placed subspaces of L_1. V. Kadets [13] used the Daugavet equation to give a very simple argument that neither $C[0,1]$ nor $L_1[0,1]$ have unconditional bases; see Proposition 3.1 below.

This survey describes some results along these lines that were recently obtained in [16], [17] and [24], where full details can be found.

2. The Daugavet property

Our starting point is the observation that those Banach spaces on which (1.1) holds for rank-1 operators can be characterised geometrically. So we first give a convenient definition.

Definition 2.1. A Banach space X has the *Daugavet property* if

$$\|\mathrm{Id} + T\| = 1 + \|T\|$$

for every rank-1 operator $T\colon X \to X$.

Clearly, it is enough to check this definition for operators of norm 1. Indeed, if two elements in a normed space satisfy $\|v+w\| = \|v\| + \|w\|$, then the function $\varphi(\lambda) = \|\lambda v + (1-\lambda)w\| - (\lambda\|v\| + (1-\lambda)\|w\|)$ is convex on $[0,1]$, takes values ≤ 0 and satisfies $\varphi(1/2) = 0$; therefore $\varphi = 0$ which implies that $\|rv + sw\| = r\|v\| + s\|w\|$ for all $r, s > 0$.

We remark that the Daugavet property is an isometric property of a Banach space that is liable to be spoilt by passing to an equivalent norm.

Before giving examples, we mention that it will turn out that the Daugavet equation holds for much wider classes of operators if it merely holds for one-dimensional ones, see Theorem 2.7 and Theorem 2.8 below and Section 4.

Examples. (a) If K is a compact Hausdorff space without isolated points, then $C(K)$ has the Daugavet property; this is essentially I.K. Daugavet's original result [7]. On the other hand, if K has an isolated point k_0, then the one-dimensional operator $Tf = -f(k_0)\chi_{\{k_0\}}$ does not satisfy (1.1); instead we have $\|\mathrm{Id} + T\| = 1$, $1 + \|T\| = 2$.

(b) If μ is a nonatomic measure, then $L_1(\mu)$ and $L_\infty(\mu)$ have the Daugavet property; this is originally due to G.Ya. Lozanovskii [19] and A. Pełczyński [10]. Again it is easy to see that (1.1) fails in general if μ has atoms. Note that the Daugavet property passes from X^* to X, but not necessarily from X to X^*; indeed $C[0,1]$ has the Daugavet property, but its dual fails it. We mention that (a) and (b) extend to the Banach space valued function spaces $C(K,E)$ and $L_1(\mu,E)$ irrespective of the range space; see below.

(c) In general, the Daugavet property does not pass to subspaces, not even to 1-complemented ones; we shall have more to say on this in Section 5. However, it does pass to M-ideals and L-summands [16]. On the other hand, if X and Y have the Daugavet property, then so do $X \oplus_1 Y$ and $X \oplus_\infty Y$ ([16], [29] and, for the special case $C(K) \oplus_1 C(K)$, [1]). Very recently, D. Bilik has shown that the ℓ_1- and ℓ_∞-sums are the only unconditional sums that preserve the Daugavet property.

(d) If $A \subset C(K)$ is a uniform algebra, i.e., a closed subalgebra containing the constant functions and separating the points of K, and if the Choquet boundary of A does not contain any isolated points, then A has the Daugavet property ([28], [29]). In particular, the Daugavet property holds for the disk algebra $A(\mathbb{D})$ and the algebra of bounded analytic functions H^∞.

(e) A real L_1-predual space X, i.e., a Banach space whose dual is isometric to some space $L_1(\mu)$, has the Daugavet property if the set of extreme points $\operatorname{ex} B_{X^*}$ has no isolated points; in the complex case one has to consider the quotient space $\operatorname{ex} B_{X^*}/{\sim}$ instead, where $p^* \sim q^*$ if they are multiples of each other [28].

(f) The noncommutative analogues of (a) and (b) hold as well [20]: A C^*-algebra has the Daugavet property if and only if it is nonatomic, and then (1.1) is automatically valid for the completely bounded norm. Consequently, the predual of a nonatomic von Neumann algebra has the Daugavet property.

In order to describe the Daugavet property in geometric terms, we recall that a slice of the unit ball of X is a set given by

$$S(x^*, \alpha) = \{x \in B_X \colon \operatorname{Re} x^*(x) \geq 1 - \alpha\}$$

for some functional $x^* \in X^*$ of norm 1 and some $\alpha > 0$. Here $B_X = \{x \in X \colon \|x\| \leq 1\}$ denotes the closed unit ball of X, and we let $S_X = \{x \in X \colon \|x\| = 1\}$ stand for its unit sphere.

In the following we shall assume for simplicity that we are dealing with real scalars.

Lemma 2.2. *The following assertions about a Banach space X are equivalent:*

(i) *X has the Daugavet property.*

(ii) *For every slice $S = S(x_0^*, \varepsilon_0)$ of B_X, every $x_0 \in S_X$ and every $\varepsilon > 0$ there exists a point $x \in S$ such that $\|x + x_0\| \geq 2 - \varepsilon$.*

(iii) *For every slice $S = S(x_0^*, \varepsilon_0)$ of B_X, every $x_0 \in S_X$ and every $\varepsilon > 0$ there exists a slice S' of B_X contained in S such that $\|x+x_0\| \geq 2-\varepsilon$ for all $x \in S'$.*

(iv) *For every relatively weakly open subset U of B_X, every $x_0 \in S_X$ and every $\varepsilon > 0$ there exists a relatively weakly open subset U' of B_X contained in U such that $\|x + x_0\| \geq 2 - \varepsilon$ for all $x \in U'$.*

Proof. The equivalence of (i) and (ii) is quickly established by looking at the rank-1 operator defined by $Tx = x_0^*(x)x_0$ when a slice and a point are given; conversely, every such operator determines a slice and a point. For the implication (i) $\Rightarrow$ (iii) note that $\|\mathrm{Id} + T^*\| = \|\mathrm{Id} + T\| = 2$ for the above T. Hence there is a functional $x^* \in S_{X^*}$ such that $\|x^* + T^*x^*\| \geq 2 - \varepsilon_0$ and $x^*(x_0) \geq 0$. Put

$$x_1^* = \frac{x^* + T^*x^*}{\|x^* + T^*x^*\|}, \quad \varepsilon_1 = 1 - \frac{2 - \varepsilon_0}{\|x^* + T^*x^*\|}.$$

Then we have, given $x \in S' := S(x_1^*, \varepsilon_1)$,

$$\langle(\mathrm{Id} + T^*)x^*, x\rangle \geq (1 - \varepsilon_1)\|x^* + T^*x^*\| = 2 - \varepsilon_0,$$

therefore

$$x^*(x) + x^*(x_0)x_0^*(x) \geq 2 - \varepsilon_0, \tag{2.1}$$

which implies that $x_0^*(x) \geq 1 - \varepsilon_0$, i.e., $x \in S(x_0^*, \varepsilon_0)$. Moreover, by (2.1) we have $x^*(x) + x^*(x_0) \geq 2 - \varepsilon_0$ and hence $\|x + x_0\| \geq 2 - \varepsilon_0$; note that there is no loss of generality in assuming that $\varepsilon = \varepsilon_0$.

The inclusion (i) $\Rightarrow$ (iv) is more difficult and originally due to R. Shvidkoy [24]. He starts off by rediscovering a lemma that has originated in Bourgain's Paris lecture notes on the Radon-Nikodým property (cf. [11, Lemma II.1]) to the effect that a relatively weakly open subset of B_X contains a convex combination of slices, and then one uses (iii). Finally, the implications (iv) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) are clear. $\square$

By means of the Hahn-Banach theorem the equivalence of (i) and (ii) above can be rephrased as follows. We use the notation

$$\Delta_\varepsilon(x) = \{y \in B_X \colon \|x - y\| \geq 2 - \varepsilon\}$$

for $x \in S_X$.

Corollary 2.3. *A Banach space X has the Daugavet property if and only if $B_X = \overline{\mathrm{co}}\, \Delta_\varepsilon(x)$ for all $x \in S_X$ and $\varepsilon > 0$.*

Example. Let us use Lemma 2.2 to show that $L_1[0,1]$ has the Daugavet property. In fact, let $f_0 \in S_{L_1}$ and $g_0 \in S_{L_\infty}$. For $\varepsilon > 0$, find a measurable subset B of $[0,1]$ such that $\|\chi_B f_0\|_{L_1} \leq \varepsilon/2$ and $\|\chi_B g_0\|_{L_\infty} \geq 1 - \varepsilon/2$, and pick $f \in S_{L_1}$ so that $\chi_B f = f$ and $\langle g_0, f\rangle \geq 1 - \varepsilon$. Since clearly $\|f + f_0\| \geq 2 - \varepsilon$, condition (ii) of Lemma 2.2 is fulfilled.

The same arguments work for $L_1(\mu)$ if (Ω, Σ, μ) is a nonatomic measure space, and they easily extend to the Bochner space $L_1(\mu, E)$ irrespective of the range space.

Similarly one can show that $C(K)$ and indeed the space of vector-valued functions $C(K,E)$ has the Daugavet property if K has no isolated points. It is somewhat more convenient to work with Corollary 2.3 now. Thus, let $f \in S_{C(K,E)}$ and $\varepsilon > 0$ be given. Let U be the open set $\{t \in K\colon \|f(t)\|_E > 1 - \varepsilon/2\}$ and pick, given $n \in \mathbb{N}$, open pairwise disjoint nonvoid subsets $U_1, \dots, U_n \subset U$ and points $t_j \in U_j$. Now let $h \in B_{C(K,E)}$. Choose continuous E-valued functions g_j such that $g_j = h$ off U_j, $g_j(t_j) = -f(t_j)$ and $\|g_j\|_\infty \leq 1$; this can be achieved by multiplying f and h by suitable Urysohn functions. Then $g_j \in \Delta_\varepsilon(f)$, and for $t \in U_i$

$$\begin{aligned}\Bigl\| h(t) - \frac{1}{n}\sum_{j=1}^n g_j(t) \Bigr\|_\infty &= \Bigl\| h(t) - \frac{n-1}{n} h(t) - \frac{1}{n} g_i(t) \Bigr\|_\infty \\ &= \frac{1}{n} \|h(t) - g_i(t)\|_\infty \leq \frac{2}{n},\end{aligned}$$

whereas for $t \notin \bigcup_j U_j$

$$h(t) - \frac{1}{n}\sum_{j=1}^n g_j(t) = 0.$$

This proves that $h \in \overline{\mathrm{co}}\, \Delta_\varepsilon(f)$ and, consequently, that $B_{C(K,E)} = \overline{\mathrm{co}}\, \Delta_\varepsilon(f)$.

The same argument implies that the spaces of weakly resp. weak* continuous functions $C_w(K,E)$ resp. $C_{w^*}(K,E^*)$ have the Daugavet property if K fails to have isolated points.

There is also a weak* version of Lemma 2.2 that can be verified by working with the adjoint of the above operator T.

Lemma 2.4. *The following assertions about a Banach space X are equivalent:*

(i) *X has the Daugavet property.*

(ii) *For every weak* closed slice S^* of B_{X^*}, every $x_0^* \in S_{X^*}$ and every $\varepsilon > 0$ there exists a point $x^* \in S^*$ such that $\|x^* + x_0^*\| \geq 2 - \varepsilon$.*

(iii) *For every weak* closed slice S^* of B_{X^*}, every $x_0^* \in S_{X^*}$ and every $\varepsilon > 0$ there exists a slice $S^{*\prime}$ of B_{X^*} contained in S^* such that $\|x^* + x_0^*\| \geq 2 - \varepsilon$ for all $x^* \in S^{*\prime}$.*

Lemmas 2.2 and 2.4 allow us to deduce an immediate necessary condition for the Daugavet property.

Corollary 2.5. *If X has the Daugavet property, then every slice of the unit ball has diameter 2. Therefore, X fails to have the Radon-Nikodým property, and in particular X is nonreflexive. Likewise X^* fails to have the Radon-Nikodým property.*

This follows from (ii) of Lemma 2.2 if we apply it to an x_0 with $-x_0 \in S$, respectively from (ii) of Lemma 2.4; note that closed convex bounded sets in spaces with the Radon-Nikodým property admit slices of arbitrarily small diameter (cf. [3, Th. 5.8]).

The next theorem gives another reason why a space with the Daugavet property must be nonreflexive.

Theorem 2.6. *If X has the Daugavet property, then X contains a subspace isomorphic to ℓ_1.*

Proof. (Sketch.) Start with any vector $x_1 \in S_X$ and any slice S_1 of B_X. Lemma 2.2 provides us with a subslice $S_2 \subset S_1$ such that $\|x + x_1\|$ is close to 2 for all $x \in S_2$. Now apply Lemma 2.2 with $-x_1$ and S_2 to obtain a slice $S_3 \subset S_2$ such that $\|x - x_1\|$ is close to 2 for all $x \in S_3$; that is

$$\|x \pm x_1\| \approx 2 \qquad \forall x \in S_3.$$

That means that for any such x the linear span of x and x_1 is approximately a 2-dimensional ℓ_1-space. Denote a subspace thus obtained by E_2. If we repeat the above procedure for sufficiently many points $y_1, \dots, y_N$ in the unit sphere of E_2 and for the resulting slices $S_3 \supset S_4 \supset S_5 \supset \dots$, we eventually obtain a slice S such that

$$\|x + y_j\| \approx 2 \qquad \forall x \in S,\ j = 1, \dots, N,$$

and if the points $y_1, \dots, y_N$ form a δ-net for a sufficiently small δ, then

$$\|x + y\| \approx 2 \qquad \forall x \in S,\ y \in S_{E_2}.$$

This implies that any such x and E_2 generate a 3-dimensional subspace $E_3 \supset E_2$ that is approximately a 3-dimensional ℓ_1-space, etc. With a judicious choice of the epsilons and deltas involved that determine the accuracy of the approximations one obtains a nested sequence $E_2 \subset E_3 \subset \dots$, with each E_k approximately a finite-dimensional ℓ_1-space. Therefore the closed linear span of the E_k is isomorphic to ℓ_1. □

An inspection of the proof reveals that a space X with the Daugavet property actually contains asymptotically isometric copies of ℓ_1, meaning copies of ℓ_1 spanned by a basis satisfying, for some null sequence (ε_n),

$$\sum_{n=1}^{\infty}(1-\varepsilon_n)|a_n| \le \left\|\sum_{n=1}^{\infty} a_n e_n\right\| \le \sum_{n=1}^{\infty}|a_n| \qquad \forall (a_n)\in \ell_1.$$

By [9] this implies that X^* contains an isometric copy of $L_1[0,1]$. However, in [16] we have shown that a certain space constructed by Talagrand in his study of the three-space problem for L_1 has the Daugavet property; this yields an example of a space with the Daugavet property that fails to contain an isomorphic copy of $L_1[0,1]$.

We now discuss some transfer theorems for operators that satisfy the Daugavet equation. The definition of the Daugavet property modestly requires (1.1) to hold only for rank-1 operators, but we shall see that then automatically much more is true.

Theorem 2.7. *If X has the Daugavet property and $T\colon X\to X$ is a weakly compact operator, then*

$$\|\mathrm{Id}+T\| = 1+\|T\|.$$

Let us remark that, due to the nonlinearity of the norm, it is not obvious that the Daugavet equation passes form rank-1 to rank-2 operators; yet Theorem 2.7 shows this and much more. Instead of indicating the proof of this theorem now we only give the argument for the special case of a compact operator; a generalisation of Theorem 2.7 will be discussed later (Theorem 4.5 and Theorem 4.7).

So let $T\colon X\to X$ be a compact operator on a space with the Daugavet property; assume without loss of generality that $\|T\|=1$. We shall use the fact that the restriction of T^* to the dual unit ball is weak*-to-norm continuous. Therefore, by the Krein-Milman theorem, there exists an extreme point p^* of B_{X^*} such that $\|T^*p^*\| = \|T^*\| = 1$. The converse of the Krein-Milman theorem implies that p^* has a neighbourhood base for the weak* topology of B_{X^*} consisting of weak* closed slices, say $(S^*_\alpha)_\alpha$; cf. [6, p. 107] for details. Now Lemma 2.4 shows that each S^*_α contains some x^*_α such that $\|x^*_\alpha + T^*p^*\| \ge 2-\varepsilon_\alpha$, where (ε_α) is a net converging to 0. This means that $x^*_\alpha \to p^*$ weak* and consequently $T^*x^*_\alpha \to T^*p^*$ in norm so that $\|x^*_\alpha + T^*x^*_\alpha\| \to 2$ and thus $\|\mathrm{Id}+T\| = \|\mathrm{Id}+T^*\| = 2$.

Incidentally, Theorem 2.7 shows that our concept of the Daugavet property coincides with the one that made a brief appearance in [2].

Comparing Theorem 2.7 with Corollary 2.5 we see that the Banach space properties of a space with the Daugavet property and the properties of the operators on that space that satisfy the Daugavet equation seem to complement each other. Therefore, in view of Theorem 2.6, it is natural to suspect that an even larger class of operators might satisfy the Daugavet equation, namely the class of operators not fixing a copy of ℓ_1. This was verified in a number of special cases ([14], [21] and [27]) before R. Shvidkoy obtained the general statement in [24].

Recall that an operator $T\colon X \to Y$ between Banach spaces is said to fix a copy of ℓ_1 if there is a closed subspace E of X isomorphic to ℓ_1 such that $T|_E\colon E \to T(E)$ is an isomorphism; otherwise one says that T does not fix a copy of ℓ_1 (or that T is ℓ_1-singular).

Theorem 2.8. *If X has the Daugavet property and $T\colon X \to X$ is an operator not fixing a copy of ℓ_1, then*

$$\|\mathrm{Id} + T\| = 1 + \|T\|.$$

Again, we shall discuss an extension of this theorem later (Theorem 4.6) and therefore bypass its proof.

For example, let Q be a quotient map from $C[0,1]$ onto c_0 and let J: $c_0 \to C[0,1]$ be an embedding; then JQ does not fix a copy of ℓ_1 and hence satisfies the Daugavet equation. This cannot be deduced from Theorem 2.7 since JQ is not weakly compact.

3. Unconditional bases

Recall that a Schauder basis (e_n) with coefficient functionals (e_n^*) on a (separable) Banach space X is called an unconditional basis if for each $x \in X$ the expansion

$$x = \sum_{n=1}^{\infty} e_n^*(x) e_n$$

converges unconditionally. This can be equivalently rephrased by saying that for each x the net of finite-rank projections defined by

$$P_F(x) = \sum_{n\in F} e_n^*(x) e_n, \qquad F \subset \mathbb{N} \text{ finite},$$

converges to x along the index set FIN of finite subsets of $\mathbb{N}$.

It is a well-known fact that neither $C[0,1]$ nor $L_1[0,1]$ have unconditional bases. Indeed, this follows easily from the Daugavet property as shown by V. Kadets [13].

Proposition 3.1. *Any separable Banach space with the Daugavet property fails to have an unconditional basis.*

Proof. Suppose X has an unconditional basis, and define the operators P_F as above. Then $\sup\{\|P_F x\|\colon F \subset \mathbb{N} \text{ finite}\} < \infty$ for each x so that the uniform boundedness principle implies

$$\alpha := \sup\{\|P_F\|\colon F \subset \mathbb{N} \text{ finite}\} < \infty.$$

If X has the Daugavet property, this leads to a contradiction as follows.

Pick F_0 such that $\|P_{F_0}\| > \alpha - 1/2$. Then the Daugavet equation implies that

$$\|\mathrm{Id} - P_{F_0}\| = 1 + \|P_{F_0}\| > \alpha + 1/2; \tag{3.1}$$

note that it is applicable by Theorem 2.7 (in fact by the special case we have proved above). But

$$\|(\mathrm{Id} - P_{F_0})x\| = \Big\| \sum_{n \notin F_0} e_n^*(x) e_n \Big\| \le \sup\{\|P_G x\| : G \subset \mathbb{N} \setminus F_0 \text{ finite}\} \le \alpha \|x\|$$

so that $\|\mathrm{Id} - P_{F_0}\| \le \alpha$, contradicting (3.1). ☐

The same argument applies to show that a Banach space with the Daugavet property fails to have an unconditional finite-dimensional Schauder decomposition or indeed an unconditional Schauder decomposition into reflexive spaces or even into spaces not containing ℓ_1; we shall have more to say on this in a moment.

At this point it might be worth mentioning that there are Banach spaces with the Daugavet property failing the approximation property, since every Banach space X_0 can be embedded as a complemented subspace of some uniform algebra X having the Daugavet property (see [29]); if X_0 fails the approximation property, then so does X.

The arguments presented in this paper so far seem to provide the most easily accessible proof to show that $C[0,1]$ and $L_1[0,1]$ don't have unconditional bases. But for these spaces more is true: they do not even embed into spaces with unconditional bases. In fact, this also results from the Daugavet property.

Theorem 3.2. *A separable Banach space with the Daugavet property does not embed into a space having an unconditional basis.*

Let us sketch the main ideas of the proof ([15], [16]).

Suppose X has the Daugavet property and suppose X embeds isomorphically into the separable space Y. Let us pretend for the moment that X is isometric to a subspace of Y (with $J\colon X \to Y$ denoting the inclusion operator) and that the pair (X, Y) has the Daugavet property in the sense that

$$\|J + T\| = 1 + \|T\| \tag{3.2}$$

for every rank-1 operator $T\colon X \to Y$. Then one could work out the analogous statements of Section 2 to arrive at the conclusion that (3.2) extends to weakly compact operators or ℓ_1-singular operators from X to Y. In particular, (3.2) holds for finite-rank operators, and the method of Proposition 3.1 applies to show that J cannot be expanded into a pointwise unconditionally converging sum of finite-rank projections; therefore Y cannot have an unconditional basis.

To get rid of the extra assumption, the main idea is to prove that one can renorm Y to obtain $\tilde{Y}$, say, in such a way that the pair $(X, \tilde{Y})$ satisfies the above extra condition. So, starting from X we wish to find a renorming $\tilde{Y}$ of Y such that X is isometric to a subspace of $\tilde{Y}$ and the pair $(X, \tilde{Y})$ has the Daugavet property. This is achieved by means of a universal embedding procedure.

Let U be the space $\ell_\infty(B_{X^*})$ of all bounded functions on B_{X^*} with the supremum norm; then X embeds into U in a canonical fashion. If X embeds isomorphically into a separable space Y, then it is possible to show that Y in turn embeds isomorphically into U so that the embedded copy $\tilde{Y}$ extends X. If we equip $\tilde{Y}$ with the supremum norm inherited from U, we obtain isometric embeddings $X \subset \tilde{Y} \subset U$. In order to show that the pair $(X, \tilde{Y})$ has the Daugavet property it is enough to establish this for the pair (X, U). But now the proof breaks down since U is based on isolated points; in fact, $U = C(K)$ for K the Stone-Čech compactification of the discrete space B_{X^*} which has a dense set of isolated points.

In order to remedy this situation we reconsider our choice of U; the point is to get rid of the isolated points. The following idea works. Let Ω denote B_{X^*} with its weak* topology, and let $\mathrm{fc}(\Omega)$ denote the subspace of all bounded functions $f\colon \Omega \to \mathbb{R}$ for which $\{\omega\colon f(\omega) \neq 0\}$ is of first category. Let $m_0(\Omega) = \ell_\infty(\Omega)/\mathrm{fc}(\Omega)$; note that by the Baire category theorem the restriction of the quotient map $Q\colon \ell_\infty(\Omega) \to m_0(\Omega)$ to $C(\Omega)$ is an isometry. Therefore X embeds isometrically into $m_0(\Omega)$ by means of $Q|_X$. Now the following statements turn out to be true.

Theorem 3.3. *Let X and Y be separable Banach spaces, with X isomorphic to a subspace of Y.*

(a) *There is an isomorphic embedding $S\colon Y \to m_0(\Omega)$ with $S|_X = Q|_X$.*
(b) *If X has the Daugavet property, then the pair $(X, m_0(\Omega))$ has the Daugavet property.*

With this theorem in hand the above outline works with $m_0(\Omega)$ in place of U, which yields the proof of Theorem 3.2. We remark that in [16] we used $\Omega = \overline{\mathrm{ex}}\, B_{X^*}$ instead of B_{X^*} in order to be able to work with slices; given part (iv) of Lemma 2.2 one can now proceed as above (see [24]).

4. Narrow operators

In [17] we have launched another approach to the Daugavet property building on previous attempts in [14] and [21]. In [14] V. Kadets and M. Popov defined an operator $T\colon C[0,1] \to E$ to be C-narrow if for every proper closed subset F of $[0,1]$ and every $\varepsilon > 0$ there is a continuous function of norm 1 vanishing on F such that $\|Tf\| \le \varepsilon$. In other words, a C-narrow operator is not bounded from below on any M-ideal of $C[0,1]$. (Actually, they speak of narrow operators – but see below.) They went on to show that a C-narrow operator $T\colon C[0,1] \to C[0,1]$ satisfies the Daugavet equation (1.1), thus proving (1.1) for operators on $C[0,1]$ not fixing a copy of $C[0,1]$ itself or – modulo a theorem due to H.P. Rosenthal [22] – for ℓ_1-singular operators on $C[0,1]$.

An L_1-version of narrowness appears in [21]. In [17] the following definition is proposed.

Definition 4.1. Let $T\colon X \to E$ be an operator between Banach spaces.

(a) T is called *almost narrow* (or *strong Daugavet operator*) if for every two elements $x, y \in S_X$ and every $\varepsilon > 0$ there is some $z \in B_X$ such that $\|T(y-z)\| \le \varepsilon$ and $\|x+z\| \ge 2-\varepsilon$.

(b) T is called *narrow* if for every functional $x^* \in X^*$ the operator $T \oplus x^*\colon X \to E \oplus_1 \mathbb{R}$ defined by

$$(T \oplus x^*)(x) = \bigl(T(x), x^*(x)\bigr)$$

is almost narrow.

The necessity to distinguish between narrow and almost narrow operators will become apparent only later (cf. Theorem 5.2 and Theorem 5.4).

It follows from Lemma 2.2 that a finite-rank operator on a space with the Daugavet property is almost narrow, and conversely, if every rank-1 operator is almost narrow, then X has the Daugavet property.

Also, if X admits at least one narrow operator (with values in some Banach space E), then by definition all rank-1 operators are almost narrow; hence X has the Daugavet property if it admits at least one narrow operator.

The following lemma geometrically describes the difference between narrow and almost narrow operators.

Lemma 4.2. *An operator $T\colon X \to E$ is narrow if and only if for every two elements $x, y \in S_X$, $\varepsilon > 0$ and every slice S of the unit ball of X containing y there is some $z \in S$ such that $\|T(y-z)\| < \varepsilon$ and $\|x+z\| > 2-\varepsilon$.*

Proof. We only prove the forward direction. So let T be a narrow operator on X, and let $x, y \in S_X$, $\varepsilon > 0$ and a slice $S = S(x^*, \alpha)$ be given such that $y \in S$, that is, $x^*(y) \ge 1-\alpha$. Pick a point $y' \in S_X$ with $\|y-y'\| \le \varepsilon/(2\|T\|)$ and $x^*(y') > 1-\alpha$. Let $\delta = \min\{x^*(y') - (1-\alpha), \varepsilon/2\}$. Since $T \oplus x^*$ is almost narrow, we can find a point $z \in B_X$ such that $\|z+x\| \ge 2-\delta \ge 2-\varepsilon$ and $\|T(y'-z)\| + |x^*(y'-z)| \le \delta$. This implies that $z \in S(x^*, \alpha)$ and $\|T(y-z)\| \le \varepsilon$. □

There is an obvious connection between almost narrow operators and the Daugavet equation.

Lemma 4.3. *If $T\colon X \to X$ is an almost narrow operator, then T satisfies the Daugavet equation* (1.1).

Proof. We assume without loss of generality that $\|T\| = 1$. Given $\varepsilon > 0$ pick $y \in S_X$ such that $\|Ty\| \ge 1-\varepsilon$. If $x = Ty/\|Ty\|$ and z is chosen according to Definition 4.1, then

$$2-\varepsilon \le \|z+x\| \le \|z+Ty\| + \varepsilon \le \|z+Tz\| + 2\varepsilon,$$

hence

$$\|z+Tz\| \ge 2-3\varepsilon,$$

which proves the lemma. □

Therefore it is an important task to find sufficient conditions for an operator to be narrow or almost narrow. First of all, for operators on $C(K)$ there is essentially no difference between C-narrow and narrow operators.

Theorem 4.4. *On the space $C(K)$ an operator is almost narrow if and only if it is C-narrow. If K has no isolated points, then an operator on $C(K)$ is narrow if and only if it is C-narrow.*

For the proof see [14] and [17].

The following two results extend, via Lemma 4.3, Theorem 2.7 and Theorem 2.8.

Theorem 4.5. *If T is a weakly compact operator on a space with the Daugavet property, then T is narrow.*

Proof. Since $T \oplus x^*$ is again weakly compact, it is enough to show that T is almost narrow. So let $x, y \in S_X$ and $\varepsilon > 0$ be given. Denote $K = \overline{TB_X}$; this is a weakly compact set. Therefore K is the closed convex hull of its strongly exposed points; see e.g. Theorem 5.11 and Theorem 5.17 of [3]. Hence there exists a convex combination of strongly exposed points of K such that

$$\left\| Ty - \sum_{j=1}^{n} \lambda_j u_j \right\| < \varepsilon.$$

By the definition of a strongly exposed point, each u_j has a neighbourhood base for the relative norm topology on K consisting of slices of K. Thus, there exist slices $S_1', \dots, S_n'$ of K such that

$$\sum_{j=1}^{n} \lambda_j S_j' \subset \{u \in K \colon \|Ty - u\| < \varepsilon\}.$$

Now $S_j := T^{-1}(S_j') \cap B_X$ is a slice of B_X, and

$$U := \sum_{j=1}^{n} \lambda_j S_j \subset \{z \in B_X \colon \|Ty - Tz\| < \varepsilon\}.$$

By Lemma 2.2(iv) there is some $z \in U$ such that $\|z + x\| \geq 2 - \varepsilon$, which shows that T is almost narrow. $\square$

Theorem 4.6. *If T is an ℓ_1-singular operator on a space with the Daugavet property, then T is narrow.*

Proof. (Sketch.) Again it is enough to show that T is almost narrow. We shall only consider the case of separable spaces; in fact, it is possible to reduce the general case to the separable one. By [8, Lemma 1(xii)] an operator that does not fix a copy of ℓ_1 can be factored through a space not containing ℓ_1. Therefore, it is left to prove that an operator $T \colon X \to E$ between separable spaces, with X having the Daugavet property and E not containing a copy of ℓ_1, is almost narrow.

So let $x, y \in S_X$ and $\varepsilon > 0$ be given. We introduce the following directed set $(\Gamma, \preceq)$: the elements of Γ are finite sequences in S_X of the form $\gamma = (x_1, \dots, x_n)$, $n \in \mathbb{N}$, with $x_1 = x$. The (strict) ordering is defined by

$$(x_1, \dots, x_n) \prec (y_1, \dots, y_m) \iff n < m \ \&\ \{x_1, \dots, x_n\} \subset \{y_1, \dots, y_{m-1}\}$$

and of course $\gamma_1 \preceq \gamma_2$ iff $\gamma_1 \prec \gamma_2$ or $\gamma_1 = \gamma_2$. Now define a bounded function $F\colon \Gamma \to E \times \mathbb{R}$ by

$$F(\gamma) = (Tx_n, \alpha(\gamma)),$$

where

$$\alpha(\gamma) = \sup\bigl\{a > 0\colon \|z + x_n\| > a(\|z\| + \|x_n\|)\ \forall z \in \operatorname{lin}\{x_1, x_2, \dots, x_{n-1}\}\bigr\}.$$

We claim that $(Ty, 1)$ is a weak limit point of this net. Indeed, given $\gamma_0 = (x_1, \dots, x_{n-1})$, a weak neighbourhood V of Ty and $\delta > 0$, an argument similar to the one of Theorem 2.6, this time based on Lemma 2.2(iv), shows that there is some x_n in the weak neighbourhood $U := T^{-1}(V)$ of y such that x_n is "up to δ" ℓ_1-orthogonal over $\operatorname{lin}\{x_1, \dots, x_{n-1}\}$; i.e., for $\gamma = (x_1, \dots, x_n)$ we have $F(\gamma) \in V \times (1 - \delta, 1 + \delta)$.

Now we use a modification of a theorem due to H.P. Rosenthal [23] which tells us that there is a strictly increasing sequence (γ_j) such that $F(\gamma_j) \to (Ty, 1)$ weakly. (It is here that the assumption that E is separable and fails to contain ℓ_1 enters.) If we write $\gamma_j = (x_1, \dots, x_{n(j)})$, then there is a subsequence (x'_n) of (x_n) such that $Tx'_n \to Ty$ weakly and (x'_n) is "up to ε" an ℓ_1-basis. Therefore there exists a sequence of convex combinations $z_n \in \operatorname{co}\{x'_n, x'_{n+1}, \dots\}$ such that $\|Tz_n - Ty\| \to 0$ and $\|x_1 + z_n\| \ge 2 - \varepsilon$. Hence for large enough n, $z = z_n$ will satisfy the requirements of Definition 4.1(a). $\square$

These results can be extended further. First we note that the essential property of a weakly compact set that we have used in the proof of Theorem 4.5 is that a convex weakly compact set is the norm-closed convex hull of its strongly exposed points. This is known to hold in a larger class of sets called Radon-Nikodým sets [3, Th. 5.17]. An operator $T\colon X \to E$ is called a strong Radon-Nikodým operator if the closure of $T(B_X)$ is a Radon-Nikodým set; this is certainly the case if E has the Radon-Nikodým property. Thus, the above proof actually shows:

Theorem 4.7. *If T is a strong Radon-Nikodým operator on a space with the Daugavet property, then T is narrow.*

In fact, the following more general statements hold. For two operators $T\colon X \to E$ and $S\colon X \to F$ we define $T \oplus S\colon X \to E \oplus_1 F$ by

$$(T \oplus S)(x) = \bigl(T(x), S(x)\bigr).$$

Theorem 4.8. *Suppose X has the Daugavet property and let T and S be two operators on X.*

(a) *If T is a strong Radon-Nikodým operator (or merely weakly compact) and S is narrow, then $T \oplus S$ is narrow.*

(b) *If T does not fix a copy of ℓ_1 and S is a narrow operator, then $T \oplus S$ is narrow.*

The previous theorems correspond to the choice $S = 0$ in this result. The proof of Theorem 4.8 can be found in [17]. Theorem 4.8 raises the following problem that has remained open:

- *If T and S are narrow operators, does it follow that $T \oplus S$ is narrow?*

5. Rich subspaces

We have seen that a space with the Daugavet property is sort of large: it is not reflexive, in fact neither the space nor its dual have the Radon-Nikodým property, and it is very rich in ℓ_1-subspaces. Thus it is natural to suspect that the Daugavet property is inherited by "large" subspaces. It turns out that the following concept describes the appropriate notion of largeness. Again, this definition draws on precursors from [14] and [21].

Definition 5.1. A subspace Y of a Banach space X is called *rich* (resp. *almost rich*) if the quotient map from X onto X/Y is narrow (resp. almost narrow).

Example. Let $A \subset C(K)$ be a uniform algebra whose Silov boundary is K. Then A is an almost rich subspace. Indeed, by Theorem 4.4 it is enough to show that the quotient map $Q\colon C(K) \to C(K)/A$ is C-narrow; that is, we have to show for a closed subset $F \subsetneq K$ and for $\varepsilon > 0$ that there is a continuous function f of norm 1 vanishing on F whose distance to A is $< \varepsilon$. But a fundamental theorem in the theory of uniform algebras ensures that there is some $g \in A$ with $\|g\| = 1$ and $|g| < \varepsilon$ on F ([18, p. 49 and p. 78]), hence an obvious modification of g yields a function f as requested.

If K has no isolated points, then A is even rich.

Another example worth mentioning (from [14]) is that a subspace of $C[0,1]$ (resp. $L_1[0,1]$) containing a complemented copy of $C[0,1]$ (resp. $L_1[0,1]$) is rich.

We now have:

Theorem 5.2. *A rich subspace Y of a Daugavet space X is a Daugavet space itself. Moreover, the pair (Y, X) has the Daugavet property.*

Proof. Consider elements $x \in S_X$, $y \in S_Y$, and a slice $S = S(x^*, \varepsilon)$ with $y \in S$. According to our assumption the quotient map $Q\colon X \to X/Y$ is a narrow operator. So by Lemma 4.2 there is an element $z \in S$ such that $\|z + x\| > 2 - \varepsilon$ and $\|Q(y - z)\| = \|Q(z)\| < \varepsilon$. The last condition means that the distance from z to Y is smaller than ε, so there is an element $v \in Y$ with $\|v - z\| < \varepsilon$. The norm of v is close to 1, viz. $1 - 2\varepsilon < \|v\| < 1 + \varepsilon$. Put $w = v/\|v\|$. For this w we have $\|w - z\| < 3\varepsilon$, so $w \in S(x^*, 4\varepsilon)$ and $\|w + x\| > 2 - 4\varepsilon$. □

Corollary 5.3. *Suppose X has the Daugavet property and let Y be a closed subspace of X.*

(a) *If the quotient space X/Y has the Radon-Nikodým property, then Y is rich.*
(b) *If the quotient space X/Y contains no copy of ℓ_1, then Y is rich.*
(c) *If $(X/Y)^*$ has the Radon-Nikodým property, then Y is rich.*
(d) *In particular, every finite-codimensional subspace of X is rich.*

In either case Y is a Daugavet space itself.

Proof. (a) follows from Theorem 4.7 and (b) from Theorem 4.6; (c) and (d) follow from (b). □

Example. Consider the unit circle $\mathbb{T}$ and the Fourier transform $\mathscr{F}\colon f \mapsto \widehat{f}$ from $L_1(\mathbb{T})$ to $c_0(\mathbb{Z})$. A subset Λ of $\mathbb{Z}$ is a Sidon set if the mapping $\mathscr{F}_\Lambda$: $f \mapsto \widehat{f}|_\Lambda$ maps $L_1(\mathbb{T})$ onto $c_0(\Lambda)$; an example is the set $\{1, 4, 16, 64, \dots\}$. The kernel of $\mathscr{F}_\Lambda$ is $L_{1,\Lambda'}$, the space of L_1-functions whose Fourier transforms are supported on $\Lambda' = \mathbb{Z} \setminus \Lambda$. Thus, if Λ' is the complement of a Sidon set Λ, then $L_1/L_{1,\Lambda'}$ is isomorphic to $c_0(\Lambda)$, a space whose dual $\ell_1(\Lambda)$ is separable and hence has the Radon-Nikodým property. Consequently $L_{1,\Lambda'}$ is a rich subspace of L_1 if Λ' is the complement of a Sidon set.

We remark that Theorem 5.2 would not hold if we had replaced the assumption of richness by almost richness, for there is the following example.

Theorem 5.4. *There is a subspace Y of $L_1 := L_1[0,1]$ such that Y fails the Daugavet property, but the quotient map $Q\colon L_1 \to L_1/Y$ is almost narrow. Hence Q is an almost narrow operator that fails to be narrow and Y is an almost rich subspace that fails to be rich.*

Outline of the construction. The space Y is the subspace of $L_1[0,1]$ spanned by the constant functions and certain step functions with integral 0.

We first partition $[0,1]$ into 2^4 subintervals $I_{1,1}, \dots, I_{1,2^4}$ of length $1/2^4$ and consider the function $g_{0,1}$ defined as follows: On $I_{1,2} \cup \dots \cup I_{1,2^4}$ it takes the value -1, and on $I_{1,1}$, the leftmost of these intervals, it takes a value a_0 such that $\int g_{0,1} = 0$. Next $[0,1]$ is partitioned into 2^{16} subintervals $I_{2,1}, \dots, I_{2,2^{16}}$ of length $1/2^{16}$. We let $g_{1,k}$ $(k = 1, \dots, 2^4)$ be the function with support $I_{1,k}$ defined as follows: On the leftmost of the small subintervals that make up $I_{1,k}$ it takes a value a_1 and on the remainder of $I_{1,k}$ it takes the value -1, where a_1 is chosen so as to guarantee that $\int g_{1,k} = 0$. Then this construction is repeated with 2^{64}, 2^{256}, etc. subintervals giving step functions $g_{2,k}$ $(k = 1, \dots, 2^{16})$, $g_{3,k}$ $(k = 1, \dots, 2^{64})$, etc.

We define Y to be the closed linear span of the $g_{n,k}$ and the constant functions. That the quotient map $Q\colon L_1 \to L_1/Y$ is almost narrow can be shown by approximating the function y appearing in Definition 4.1 by step functions supported on the intervals $I_{n,k}$ $(k = 1, \dots, 2^{4^n})$ for sufficiently large n and then "shifting" the mass onto the leftmost subinterval of the next generation. However Y fails the Daugavet property, since it can be shown that the operator $Tf = \int_S f(x)\,dx\,\mathbf{1}$, with $S = [0,1] \setminus \bigcup_{n,k}\{g_{n,k} > 0\}$, fails the Daugavet equation (1.1). $\square$

We finally mention another result from [17]. If $Y \subset Z \subset X$ and Y is a rich subspace of X, then so is Z by the definition of a narrow operator. Consequently, all superspaces of a rich subspace have the Daugavet property by Theorem 5.2. The converse is valid as well:

Theorem 5.5. *For a subspace Y of a Banach space X the following are equivalent:*

(i) *Y is a rich subspace of X.*
(ii) *Every finite-codimensional subspace of Y is a rich subspace of X.*
(iii) *Every superspace $Y \subset Z \subset X$ has the Daugavet property ("Y is wealthy").*

6. Open problems

Here we list some problems which have remained open.

(1) Does the Banach space of Lipschitz functions on the unit square have the Daugavet property with respect to one of its natural norms? This is true for the space of Lipschitz functions on the unit interval, which is isomorphic to $L_\infty[0,1]$.
(2) Is there a Banach space X such that X^{**} has the Daugavet property?
(3) If X and/or Y have the Daugavet property, what about their tensor products $X\widehat{\otimes}_\varepsilon Y$ and $X\widehat{\otimes}_\pi Y$? Note that $C(K)\widehat{\otimes}_\varepsilon Y = C(K,Y)$ and $L_1(\mu)\widehat{\otimes}_\pi Y = L_1(\mu, Y)$.
(4) Is the sum $S \oplus T$ of two narrow operators narrow?
(5) If X has the Daugavet property, does X have a subspace isomorphic to ℓ_2?
(6) If T is an operator on a space X with the Daugavet property which does not fix a copy of ℓ_2, is T then narrow? We remark that the answer is affirmative in the case $X = C[0,1]$ by Theorem 4.6 and a result due to J. Bourgain [4].
(7) Is there a rich subspace of L_1 with the Schur property? Is there a Schur space with the Daugavet property? (Obviously, not both of Problems 5 and 7 can have a positive answer.)
(8) Specifically, let $X \subset L_1$ be the subspace constructed by J. Bourgain and H.P. Rosenthal in [5]. It has the Schur property and fails the Radon-Nikodým property. Is it rich? Does it have the Daugavet property? This space has the following property:

    For $\varepsilon > 0$ and $\|x\| = 1$, let $\Delta_\varepsilon(x) = \{y \in B_X\colon \|x - y\| \geq 2 - \varepsilon\}$; then $x \in \overline{\mathrm{co}}\,\Delta_\varepsilon(x)$.

    For the Daugavet property, we must have $B_X = \overline{\mathrm{co}}\,\Delta_\varepsilon(x)$ for all ε and x; cf. Corollary 2.3. By contrast, the above property means that $\|\mathrm{Id} - P\| = 2$ for every norm-1 rank-1 projection P.

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