On boundary value problems for elliptic equations with boundary operators of fractional order

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Abstract

We deal with boundary value problems for pseudo-differential equations of elliptic type with boundary operators depending on a positive real parameter α. In particular, boundary conditions can be given through the one-sided Marchaud, Grünwald-Letnikov or Liouville-Weyl fractional derivatives of order α. We prove solvability theorems in spaces of different type including some subspaces of the Sobolev space $H^s(I\!R^n)$, the Lizorkin type spaces and some modifications of the space $\Psi_{G,p}(I\!R^n)$, $1 < p < \infty$, of functions in L_p whose Fourier transforms are compactly supported in a domain $G \subset I\!R^n$. The distributions defined on the last space may have singularities of infinite order and can be considered as hyper-functions. We show that they can be treated also as boundary limits in a special weak topology of the one-sided Marchaud, Grünwald-Letnikov or Liouville-Weyl fractional derivatives of harmonic functions.

1 Introduction

This paper is devoted to solvability conditions of boundary value problems with boundary operators given through fractional derivatives in Marchaud, Grünwald-Letnikov or Liouville-Weyl sense. Conceptually it can be considered as a continuation of [ULG00], where, in particular, the processes intermediate between heat conduction and wave propagation have been studied. In contrast to [ULG00], problems considered here interpolate between corresponding stationary states.

Consider the boundary value problem

$$\frac{\partial^2 u}{\partial y^2} + A(D)u(y,x) = 0,\ y > 0,\ x \in \mathbb{R}^n, \tag{1}$$

$$B^\alpha u(+0,x) = \varphi(x),\ x \in \mathbb{R}^n, \tag{2}$$

$$|u(y,x)| \to 0 \ \text{ as } y \to \infty, \ \text{ uniformly for } \ x \in \mathbb{R}^n, \tag{3}$$

where $D = -i(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, $A(D)$ is an elliptic pseudo-differential operator of order $2m,\ m \in \mathbb{N}$, acting on $u(y,x)$ with respect to the variable x. Its symbol $A(\xi)$ is a smooth function satisfying the condition

$$C_1|\xi|^{2m} \le -A(\xi) \ \le C_2|\xi|^{2m},\ C_1 > 0,\ C_2 > 0, \tag{4}$$

and B^α is a boundary operator depending on a non-negative real parameter α acting on $u(y,x)$ with respect to the variable y. For what follows we take as B^α the one-sided Marchaud fractional derivative $\mathbf{D}_-^\alpha$ defined for $y > 0$ as

$$\mathbf{D}_-^\alpha f(y) = \begin{cases} f(y), & \alpha = 0, \\ \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{f(y)-f(y+h)}{h^{1+\alpha}} dh, & 0 < \alpha < 1, \\ \frac{d^k}{dy^k} \mathbf{D}_-^{\alpha-k} f(y), & k \le \alpha < k+1,\ k = 1, 2, \ldots. \end{cases}$$

As usual, $\Gamma(z)$ denotes the Euler gamma-function.

We note that in the spaces $L_p,\ 1 \le p < \infty$, the Marchaud derivative exists simultaneously and coincides with the one-sided Grünwald-Letnikov fractional derivative $\mathbb{D}_-^\alpha$ (see [SKM93]) which is defined for $y > 0$ as

$$\mathbb{D}_-^\alpha f(y) = \lim_{h \to 0} \frac{\Delta_h^\alpha f(y)}{h^\alpha},\ h > 0,\ \alpha \ge 0.$$

Here Δ_h^α means the finite difference of fractional order α with step h:

$$\Delta_h^\alpha f(y) = \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} f(y+kh),\ \binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-k+1)k!},\ k = 0, 1, \ldots.$$

If $\alpha = 0$, we put $\Delta_h^\alpha = I$ with I as the identity operator. For $\alpha = 1$ we get

$$\mathbb{D}_-^1 f(y) = \lim_{h \to 0} \frac{f(y+h) - f(y)}{h} = \frac{d}{dy} f(y).$$

For sufficiently well-behaved functions both the Marchaud and Grünwald-Letnikov derivatives coincide (see [SKM93]) with the one-sided Liouville-Weyl fractional derivative $\mathcal{D}_-^\alpha = (-1)^k \frac{d^k}{dy^k}\mathcal{I}_-^{k-\alpha}$, $k-1 < \alpha \le k \in I\!N$, where

$$\mathcal{I}_-^\beta f(y) = \frac{1}{\Gamma(\beta)}\int_y^\infty (\xi - y)^{\beta-1} f(\xi)d\xi,\ y > 0,\ \beta > 0,$$

$$\mathcal{D}_-^0 f(y) \ = \ \mathcal{I}_-^0 f(y) \ = \ If(y) \ = \ f(y).$$

If B^α is the one-sided Marchaud, Grünwald-Letnikov or Liouville-Weyl fractional derivative of order α we get

$$B^\alpha e^{-ay} \ = \ (-a)^\alpha e^{-ay},\ a > 0. \tag{5}$$

This formula plays an essential role in our constructions.

We denote the problem (1)–(3) by P_α. As particular cases we get the Dirichlet problem, P_0, if $\alpha = 0$ and the Neumann problem, P_1, if $\alpha = 1$. When $0 < \alpha < 1$ the problem P_α interpolates between these two problems whose solutions are often used in applications to describe stationary states. It is well-known that the Dirichlet problem is unconditionally solvable, while for solvability of the Neumann problem an additional condition on orthogonality is necessary. We will show that for φ in the Sobolev space $H^s(I\!R)$ (one-dimensional case) the problem P_α remains unconditionally solvable for all $\alpha \in [0, \frac{1}{2})$; for $\alpha \in (\frac{1}{2}, 1]$ it is solvable only under additional orthogonality conditions on φ. We will also show that if α increases then the number of orthogonality conditions required for solvability increases as well. The new orthogonality conditions appear exactly at the values $\alpha = j + \frac{1}{2}, j = 1, 2, \ldots$.

Let $N(\alpha)$ be the number of conditions necessary for solvability of P_α and with $2m$ as the order of the pseudo-differential operator $A(D)$ (see (1), (4)) let $l(\alpha) = [m\alpha - \frac{n}{2} + 1]_+$ where $[x]$ stands for the integer part of x and $[x]_+ = \max\{[x],\ 0\}$. Then the formula

$$N(\alpha) = \frac{(l(\alpha) + n - 1)!}{(l(\alpha) - 1)!n!}$$

holds. It is easy to see that, if $n = 1$ (one-dimensional case) and $m = 1$, then the last formula is reduced to $N(\alpha) = [\alpha + \frac{1}{2}]$ which justifies our arguments in the one-dimensional case.

For mixed equations boundary value problems with boundary operators of fractional order have been considered in [Nak75] in connection with the problem of flow of underground water. Similar boundary value problems with fractional order boundary operators defined in another sense have been studied in [Uma93], [TU94], [Naz97].

We note that the order α of the boundary operator (2) is not connected directly with the order $2m$ of the equation (1). In particular, α can be greater than $2m$. Such problems with the boundary operators whose symbols are polynomials of order higher than that of the corresponding equations have been considered in [Kar96], where also the number of orthogonality conditions necessary for solvability has been found.

The structure of our paper is as follows. In Section 2 we will introduce some function spaces to be used to describe solvability results for the problem P_α given in the succeeding sections. We consider function spaces of three different types. The spaces of the first type are subspaces of the Sobolev space $H^s(I\!R^n)$ satisfying some orthogonality conditions. The number of these conditions depends on the order α of the operator B_α in (2) and on the order $2m$ of the pseudo-differential operator $A(D)$ in (1). These spaces are most suitable in view of the exact number of conditions necessary for solvability of the problem P_α.

The spaces of second type are the well-known Lizorkin type spaces. On the one hand these spaces are natural from the point of view of describing the solvability of P_α for any $\alpha \geq 0$. On the other hand, unfortunately, it is not possible to provide the exact number of solvability conditions of P_α in these spaces for a given α.

The spaces of the third type are some modifications of the duality construction given in [Uma97], [Uma98] (see also [ULG00]). The distributions arising in the framework of this duality may have singularities of infinite order. So they can be considered as hyper-functions (cf. [Van89]). As a simple corollary we obtain in Section 3 that such distributions may be treated as boundary limits of the Marchaud (or Grünwald-Letnikov or Liouville-Weyl) fractional derivatives of harmonic functions in a special weak topology.

In Section 3 we will prove solvability theorems for the problem P_α in the spaces introduced in Section 2.

2 Function spaces

I. *Sobolev spaces.*

For a function $u \in S$, the space of rapidly decreasing test functions, we define its Fourier transform $\hat{u}$ as

$$\hat{u}(\xi) \;=\; Fu(\xi) = \int_{I\!R^n} u(x) e^{-ix\xi} dx, \tag{6}$$

and as usual we extend the operator F to the space S' of tempered distributions (for definition of S' see [GS68]).

Denote by $H^s(I\!R^n)$, $s \in I\!R$, the Sobolev space of tempered distributions $u \in S'$ for which the norm

$$\|u\|_s = \|(1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi)\|_{L_2} = (\int_{R^n}(1+|\xi|^2)^s|\hat{u}(\xi)|^2 d\xi)^{\frac{1}{2}}$$

is finite. For convenience, we denote by $|\xi|$ the Euclidean norm $\|\cdot\|_2$ of a vector $\xi \in I\!R^n$, whereas later for a multi-index $\beta \in I\!N_0^n$ we denote by $|\beta|$ its one-norm $\|\beta\|_1 = \beta_1 + \cdots + \beta_n$. Let $H^s_{comp}(I\!R^n)$ be the space of functions $u \in H^s(I\!R^n)$ with compact support in $I\!R^n$, i.e., $\operatorname{supp} u \subset\subset I\!R^n$. We endow the space $H^s_{comp}(I\!R^n)$ with the following convergence: a sequence of functions $u_m \in H^s_{comp}(I\!R^n), m = 1, 2, \ldots$, is said to converge to a function $u_0 \in H^s_{comp}(I\!R^n)$ iff:

(i) there exists a compact set $K \subset I\!R^n$ such that $\operatorname{supp} u_m \subset K$ for each $m = 1, 2, \ldots$;

(ii)$\|u_m - u_0\|_s \to \infty$ as $m \to \infty$.

The space $H^s_{comp}(I\!R^n)$ is a locally convex topological vector space and can be represented as an inductive limit of a sequence of Banach spaces. As members of such sequence we can take the spaces

$$X^s_N \equiv \{u \in H^s : \operatorname{supp} u \subset \{|x| \leq N\}\}$$

with the norms $\|u\|_N = \inf_{\{U\}} \|U\|_s$, inf taken over all $U \in H^s(I\!R^n)$ which, restricted to $\{x \in I\!R^n : |x| \leq N\}$, coincide with u for $|x| \leq N$.

Further, denote by $H^s_{comp,l}(I\!R^n)$, $l \in I\!N$, the space of functions $\varphi \in H^s_{comp}(I\!R^n)$ with $(\varphi, x^\beta) = 0$ if $|\beta| \leq l-1$. Here $(\cdot,\cdot)$ means the L_2-inner product. For $l = 0$ we let $H^s_{comp,0}(R^n) = H^s_{comp}(R^n)$.

Lemma 1. *Let $\varphi \in H^s_{comp}(I\!R^n)$, $s \geq 0$, and $\operatorname{supp}\varphi \subset Q_N = \{|x_j| \leq N,\ j = 1, \ldots, n\}$. Then there exists a positive constant C independent of φ and N such that*

$$\sup_{|\xi|\leq 1} |\hat{\varphi}(\xi)| \leq CN^{\frac{n}{2}}\|\varphi\|_s.$$

P r o o f. Let, first, $s = 0$. Then it is readily seen that the estimate

$$|\hat{\varphi}(\xi)| \leq \int_{Q_N} |\varphi(x)|dx \leq \sqrt{m(Q_N)}\|\varphi\|_0$$

holds, where $m(Q_N)$ is the volume of Q_N. Further, taking into account the embedding $H^s \hookrightarrow L_2$, we arrive at the desired estimate. ∎

Remark 1. *Analogously, one can show that, for every multi-index $\beta = (\beta_1, \ldots, \beta_n)$ and $\varphi \in H^s_{comp}(I\!R^n)$, $s \geq 0$, with $\operatorname{supp}\varphi \subset Q_N$ there exists a* *positive constant C independent of φ and N such that*

$$\sup_{|\xi|\leq 1} |D^\beta \hat{\varphi}(\xi)| \leq C\frac{N^{\frac{n}{2}+|\beta|}}{\sqrt{B(\beta)}}\|\varphi\|_s. \tag{7}$$

Here $|\beta| = \sum_{i=1}^n \beta_i,\ B(\beta) = \prod_{i=1}^n(1+\beta_i)$.

Lemma 2. *Let $\varphi \in H^s_{comp,l}(I\!R^n)$ and $supp\,\varphi \subset Q_N$. Then there exists a function $v \in H^{s+l}(I\!R^n)$ and a positive constant C_N independent of φ such that $\hat{\varphi}(\xi) = |\xi|^l\hat{v}(\xi)$ and $\|v\|_{s+l} \leq C_N\|\varphi\|_s$.*

P r o o f. For any $\varphi \in H^s_{comp,l}(I\!R^n)$ the orthogonality conditions $(\varphi, x^\beta) = 0,\ |\beta| \leq l-1$ are satisfied. Then $D^\beta\hat{\varphi}(0) = 0$ for $|\beta| \leq l-1$. Hence

$$\hat{\varphi}(\xi) = |\xi|^l(\sum_{|\beta|\geq l} |\xi|^{|\beta|-l}\frac{1}{\beta!}D^\beta\hat{\varphi}(0)\theta^\beta)$$

with $\theta = \frac{\xi}{|\xi|}$. Consider the function

$$v(x) = F^{-1}(\sum_{|\beta|\geq l} |\xi|^{|\beta|-l}\frac{1}{\beta!}D^\beta\hat{\varphi}(0)\theta^\beta),$$

where F^{-1} stands for the inverse Fourier transform. We have

$$\|v\|^2_{s+l} = \int_{R^n}(1+|\xi|^2)^{s+l}|\sum_{|\beta|\geq l}|\xi|^{|\beta|-l}\frac{1}{\beta!}D^\beta\hat{\varphi}(0)\theta^\beta|^2 d\xi = I_1 + I_2,$$

I_1 and I_2 being integrals over the unit ball and its exterior, respectively. For I_1, taking into account (7), we have

$$I_1 \leq C(\sum_{|\beta|\geq l}\frac{|D^\beta\hat{\varphi}(0)|}{\beta!})^2 \leq CN^n \cdot \|\varphi\|^2_s \cdot (\sum_{|\beta|\geq l}\frac{N^\beta}{\beta!\sqrt{B(2\beta)}})^2$$

$$\leq CN^n(e^{nN}-1)^2\|\varphi\|^2_s.$$

Here $B(2\beta) = \prod_{i=1}^n(1+2\beta_i)$. Obviously $|\xi|^{-2} \leq 2(1+|\xi|^2)^{-1}$ provided $|\xi| \geq 1$. Hence

$$I_2 \leq \int_{|\xi|\geq 1}(1+|\xi|^2)^{s+l}\frac{1}{|\xi|^{2l}}|\hat{\varphi}(\xi)|^2 dx$$

$$\leq 2^l\int_{|\xi|\geq 1}(1+|\xi|^2)^s|\hat{\varphi}(\xi)|^2 dx \leq 2^l\|\varphi\|^2_s.$$

The obtained estimates for I_1 and I_2 yield the desired result. ∎

Lemma 3. *Let $\varphi \in H^s_{comp,l}(I\!R^n)$ and $supp\,\varphi \subset Q_N$. Then for a function $v \in H^{s+l}(I\!R^n)$ satisfying the conditions of Lemma 2 there exists a positive constant C_N independent of φ such that*

$$\sup_{|\xi|\leq 1} |\hat{v}(\xi)| \leq C_N \|\varphi\|_s.$$

P r o o f. Let $|\xi| \leq 1$. (7) and $D^\beta \hat{\varphi}(0) = 0,\ |\beta| \leq l-1$, imply

$$|\hat{v}(\xi)| \leq \sum_{|\beta|\geq l}^{\infty} \frac{|D^\beta \hat{\varphi}(0)|}{\beta!} \leq C\|\varphi\|_s \sum_{|\beta|\geq l}^{\infty} \frac{N^{n+|\beta|}}{\beta!} \leq C_N \|\varphi\|_s.$$

■

Remark 2. *Lemmas 1-3 have been announced in [Uma93], [TU94] without proofs.*

II. *Lizorkin spaces.*

Let S be the space of rapidly decreasing test functions. Denote by $V(I\!R^n)$ the set of functions $v \in S$ satisfying the conditions:

$$D^\gamma v(0) = 0,\ |\gamma| = 0, 1, \ldots.$$

We introduce the Lizorkin space $\Phi(I\!R^n)$ as the Fourier pre-image of the space $V(I\!R^n)$ in the space S, i.e.,

$$\Phi(I\!R^n) = \{\varphi \in S:\ \hat{\varphi} \in V(I\!R^n)\}.$$

According to the definition of the Lizorkin space, any function $\varphi \in \Phi(I\!R^n)$ satisfies the orthogonality conditions

$$\int_{I\!R^n} x^\gamma \varphi(x)dx = 0,\ |\gamma| = 0, 1, \ldots.$$

We need also the spaces $\Phi'(I\!R^n)$ and $V'(I\!R^n)$ which are topologically dual to $\Phi(I\!R^n)$ and $V(I\!R^n)$, respectively. For $V'(I\!R^n)$ the factorization $V'(I\!R^n) = S'/V_0'$ holds, where V_0 is the set of all functionals of the form

$$v(x) = \sum_{|\gamma|=0}^{M_v} a_\gamma D^\gamma \delta(x),$$

δ is the Dirac distribution, M_v is a finite number depending on v. Analogously, $\Phi'(I\!R^n)$ can be represented as the quotient space S'/P, where P is the set of all polynomials on $I\!R^n$. We note that the Lizorkin spaces have been studied by several authors including [Liz63], [Liz69], [Sam82], [Rub96].

III. $\Psi_G(I\!R^n)$, *its dual* $\Psi'_{-G}(I\!R^n)$ *and their modifications.*

Let G be an open domain of $I\!R^n$ for which there exists a locally finite covering. A function $f \in L_p(I\!R^n),\ 1 < p < \infty$, is said to belong to the space $\Psi_{G,p}(I\!R^n)$ if its Fourier transform $\hat{f}$ has a compact support in G. We will say that a sequence of functions $f_m \in \Psi_{G,p}(I\!R^n),\ m = 1, 2, \ldots$, converges to an element $f_0 \in \Psi_{G,p}(I\!R^n)$ iff:

(i) there exists a compact set $K \subset G$ such that supp $\hat{f}_m(\xi) \subset K$ for every $m = 1, 2, \ldots$;

(ii) $\|f_m - f_0\|_p = (\int_{I\!R^n} |f_m - f_0|^p dx)^{\frac{1}{p}} \to 0$ as $m \to \infty$.

We will denote by $\Psi'_{-G,q}(I\!R^n),\ q = p/(p-1)$, the topologically dual space to the space $\Psi_{G,p}(I\!R^n)$. These spaces have been introduced and studied in [Uma97], [Uma98] (see also [ULG00]). In the case $p = 2$ they have been introduced in [Dub82]. For the sake of simplicity we will deal with the case $p = 2$, writing $\Psi_G(I\!R^n)$ instead of $\Psi_{G,2}(I\!R^n)$.

Now we introduce some modifications of the spaces $\Psi_G(I\!R^n)$ and $\Psi'_{-G}(I\!R^n)$. Let $\alpha \geq 0$ be a real number. Denote by $\Psi_G^{\beta,m}(I\!R^n),\ \beta > \alpha m - \frac{n}{2}$, the set of functions $\varphi \in \Psi_G(I\!R^n)$ for which

$$|\xi|^{-\beta}|\hat{\varphi}(\xi)| \leq C_\phi < \infty$$

holds in a neighborhood (depending on ϕ) of the origin. We furnish $\Psi_G^{\beta,m}(I\!R^n)$ with the topology induced from $\Psi_G(I\!R^n)$. It is obvious that if $\beta_1 > \beta_2 > \alpha m - \frac{n}{2}$, then $\Psi_G^{\beta_1,m}(I\!R^n) \subset \Psi_G^{\beta_2,m}(I\!R^n)$. Let $\beta_l,\ l = 1, 2, \ldots$, be a sequence of real numbers converging monotonically from above to $\alpha m - \frac{n}{2}$. Denote by $\Psi_G^{+,m}(R^n)$ the inductive limit of the sequence of spaces $\Psi_G^{\beta_l,m}(I\!R^n)$, namely

$$\Psi_G^{+,m}(I\!R^n) = \operatorname{ind} \lim_{\beta_l \to (\alpha m - \frac{n}{2})} \Psi_G^{\beta_l,m}(I\!R^n).$$

This definition implies

$$\Psi_G^{+,m}(I\!R^n) \hookrightarrow \Psi_G(I\!R^n).$$

Let $\Psi_G^{-,m}(I\!R^n)$ be the space topologically dual to $\Psi_G^{+,m}(I\!R^n)$. From the general theory of topological vector spaces it follows that

$$\Psi'_G(I\!R^n) \hookrightarrow \Psi_G^{-,m}(I\!R^n),$$

and $\Psi_G^{-,m}(I\!R^n)$ can be represented as the projective limit of the sequence of spaces

$$\Psi_G^{-,m}(I\!R^n) = \operatorname{pr} \lim_{\beta_l \to (\alpha m - \frac{n}{2})} \Psi_{-G}^{\beta_l,m}(I\!R^n)$$

where the $\Psi_{-G}^{\beta_l,m}(I\!R^n),\ l = 1, 2, \ldots$, are the spaces conjugate to the spaces $\Psi_G^{\beta_l,m}(I\!R^n)$, respectively.

3 Main results

We will now study the solvability of the boundary value problem P_α.

The solution of P_α will be understood in the following sense. Let φ be an element of a topological space X. A function $u(y,x) \in L^\infty(y \geq 0; X)$ is said to be a solution of the problem P_α if it satisfies the equation (1) in the distributional sense and the boundary conditions (2), (3) in the sense of the topology of X. Before formulating solvability results we introduce some notations. Let $C^\ell(y \geq 0; X)$ be the space of functions $u(y,x)$ lying in the space X for any fixed $y \in [0,\infty)$ and having continuous derivatives up to the order ℓ with respect to y up to the bound $y = 0$ for fixed x. The corresponding meaning is given to $C^\ell(y > 0; X)$. Analogously, we write $C^\infty(y > 0; X)$ for the space of functions which are infinitely often differentiable with respect to y. As the space X we will take the spaces considered in Section 2. We recall $l(\alpha) = [\alpha m - \frac{n}{2} + 1]_+$.

Theorem 1. *Let $\alpha \geq 0$, $\varphi \in H^s_{comp,l(\alpha)}(I\!R^n)$, $s \geq 0$. Then P_α has a unique solution $u(y,x)$ and for this solution the following inclusions hold true:*

(i) $u(y,x) \in C(y \geq 0; H^{s+\alpha m})$;

(ii) $\mathbf{D}^\alpha_- u(y,x) \in C(y \geq 0; H^s)$;

(iii) $u(y,x) \in C^\infty(y > 0; H^{s+\alpha m})$.

P r o o f. Let $0 \leq \alpha < \frac{n}{2m}$, $\varphi \in H^s_{comp}(I\!R^n)$ and $\operatorname{supp} \varphi \subset Q_N$. By using (5) it can be straightforward verified that the solution of the problem P_α has a representation in the form

$$u(y,x) = \Pi_\alpha(y,D)\varphi(x),$$

where $\Pi_\alpha(y,D)$ is the pseudo-differential operator with the symbol

$$\Pi_\alpha(y,\xi) = [-A(\xi)]^{-\alpha} \exp(-i\alpha\pi - y\sqrt{-A(\xi)}), \tag{8}$$

Further, by $|\exp(-i\pi\alpha - y\sqrt{-A(\xi)})| \leq 1$ and (4) we have

$$\|u(y,\cdot)\|^2_{s+\alpha m} = \|\Pi_\alpha(y,D)u(y,\cdot)\|^2_{s+\alpha m} \leq C \int_{I\!R^n} (1+|\xi|^2)^{s+\alpha m} \frac{|\hat{\varphi}|^2}{|\xi|^{2\alpha m}} d\xi$$

$$= C(\int_{|\xi|\leq 1} \{\ldots\} d\xi + \int_{|\xi|\geq 1} \{\ldots\} d\xi). \tag{9}$$

Since $2\alpha m < n$, Lemma 1 implies

$$\int_{|\xi|\leq 1} \{\ldots\} d\xi \leq C_N \|\varphi\|^2_s \int_{|\xi|\leq 1} \frac{d\xi}{|\xi|^{2\alpha m}} \leq C_N \|\varphi\|^2_s. \tag{10}$$

For the second term in the right-hand side of (9) we have

$$\int_{|\xi|\leq 1}\{...\}d\xi \leq C\int_{|\xi|\leq 1}(1+|\xi|^2)^s|\varphi|^2 d\xi \leq C\|\varphi\|_s^2. \tag{11}$$

Further, let $\alpha > \frac{n}{2m}$ be an arbitrary number, $\varphi \in H^s_{comp,l(\alpha)}(I\!R^n)$ and $\operatorname{supp}\varphi \subset Q_N$. Then, according to Lemma 2, there exists a function $v \in H^{s+l(\alpha)}$ such that $\hat{\varphi}(\xi) = |\xi|^{l(\alpha)}\hat{v}(\xi)$ and $\|v\|_{s+l(\alpha)} \leq C_N\|\varphi\|_s$. We note that in this case the solution of P_α is represented in the form $u(y,x) = \Pi'_\alpha(y,D)v(x)$, where $\Pi'_\alpha(y,D)$ is the pseudo-differential operator with the symbol

$$\Pi_\alpha(y,\xi) = |\xi|^{-\alpha m+l(\alpha)}\exp(-i\alpha\pi - y\sqrt{-A(\xi)})$$

and $\alpha m - l(\alpha) < \frac{n}{2}$. So, we can use the same technique as in the case $0 \leq \alpha < \frac{n}{2m}$. Then, we have the estimate

$$\|u(t,x)\|_{s+\alpha m} \leq C'_N\|v\|_{s+l(\alpha)} \leq C''_N\|\varphi\|_s,$$

which, together with (10) and (11), gives the inclusion (i) in the theorem.

Now we will show that the boundary conditions are fulfilled. In fact,

$$\|\mathbf{D}^\alpha_- u(y,\cdot) - \varphi\|_s^2 = \|(\exp(-i\alpha\pi - y\sqrt{A(-\xi)}) - 1)|\hat{\varphi}|^2(1+|\xi|^2)^s\|_{L_2} \to 0$$

as $y \to +0$ according to the Lebesgue theorem on dominated convergence.

Further, let $Y > 0$ and $y \geq Y$. Using again the representation

$$\hat{\varphi}(\xi) = |\xi|^{l(\alpha)}\hat{v}(\xi),\ v \in H^{s+l(\alpha)}(I\!R^n),$$

we obtain

$$|u(y,x)| = |\int_{I\!R^n} e^{-Y\sqrt{-A(\xi)}}\exp(-i(\alpha\pi - x\xi) - (y-Y)\sqrt{-A(\xi)})\frac{\hat{v}(\xi)}{(-A(\xi))^{m\alpha - l(\alpha)}}d\xi|$$

$$\leq C\|v\|_{s+l(\alpha)}\left(\int_{I\!R^n}\frac{e^{-2Y\sqrt{-A(\xi)}}}{|\xi|^{2m\alpha-2l(\alpha)}(1+|\xi|^2)l(\alpha)}d\xi\right)^{\frac{1}{2}}$$

$$\leq C\|\varphi\|_s \cdot Y^{2m\alpha-2l(\alpha)-n},\ y \geq Y,\ x \in I\!R^n.$$

Since $2m\alpha - 2l(\alpha) < n$, we see, letting $Y \to \infty$, that the boundary condition (3) is also fulfilled.

Moreover, it is easy to see that $\mathbf{D}^\alpha_- u(y,x) = \Pi_0(y,D)\varphi(x)$, that is, it can be represented as a Poisson type integral. So the assertion (ii) of the theorem follows from the well-known properties of this integral.

Finally, if $y > 0$ then

$$\frac{\partial^k}{\partial y^k}\Pi_\alpha(y,\xi) = (-1)^k[A(\xi)]^{\frac{k-\alpha}{2}}\exp(-i\alpha\pi - y\sqrt{A(-\xi)}),\ k = 1, 2, \ldots.$$

By taking (4) into account we arrive at

$$|\frac{\partial^k}{\partial y^k}\Pi_\alpha(y,\xi)| \ = \ O(e^{-\epsilon|\xi|^m}),\ |\xi| \to \infty,$$

with some $\epsilon > 0$. This yields the assertion (iii). ■

The next two assertions follow immediately from Theorem 1.

Corollary 1. *For the dimensions of the kernel and co-kernel of the operator $P_\alpha,\ \alpha \geq 0$, corresponding to the problem P_α we have*

$$dim\, Ker\, P_\alpha \ = \ 0, \quad dim\, Co\, Ker\, P_\alpha \ = \ \frac{(l(\alpha)+n-1)!}{(l(\alpha)-1)!n!}.$$

Corollary 2. *Let $\varphi \in C_0^\infty(I\!R^n)$. Then there exists a unique solution $u(y,x)$ of P_α and for this solution the following inclusions hold true:*

(a) $u(y,x) \in C(y \geq 0; C^\infty(I\!R^n))$;

(b) $\mathbf{D}_-^\alpha u(y,x) \in C(y \geq 0; C^\infty(I\!R^n))$;

(c) $u(y,x) \in C^\infty(y > 0; C^\infty(I\!R^n))$.

Now we consider the problem P_α in the Lizorkin space $\Phi(I\!R^n)$.

Theorem 2. *Let $\varphi \in \Phi(I\!R^n)$. Then for arbitrary $\alpha \geq 0$ there exists a unique solution $u(y,x)$ of P_α and for this solution the following inclusions hold true:*

(i) $u(y,x) \in C(y \geq 0; \Phi(I\!R^n))$;

(ii) $\mathbf{D}_-^\alpha u(y,x) \in C(y \geq 0; \Phi(I\!R^n))$;

(iii) $u(y,x) \in C^\infty(y > 0; \Phi(I\!R^n))$.

First we prove an auxiliary result.

Lemma 4. *For every fixed $y \geq 0$ the pseudo-differential operator $\Pi_\alpha(y,D)$ acts continuously from $\Phi(I\!R^n)$ into $\Phi(I\!R^n)$.*

P r o o f. Let $\varphi \in \Phi(I\!R^n)$. We will first show that $\Pi_\alpha(y,D)\varphi \in \Phi(I\!R^n)$ as well. Indeed, using the well-known Leibnitz formula, we have

$$D_\xi^\gamma[F(\Pi_\alpha(y,D)\varphi(x))] = D_\xi^\gamma(\Pi_\alpha(y,\xi)\hat{\varphi}(\xi))$$

$$= \sum_\beta \binom{\gamma}{\beta} D_\xi^{\gamma-\beta}\Pi_\alpha(y,\xi)D_\xi^\beta\hat{\varphi}(\xi).$$

The function $D_\xi^\theta \Pi_\alpha(y,\xi)$ has a singularity of the order $\alpha m + |\theta|$ exactly at the origin. Since $\varphi \in \Phi(I\!R^n)$, we get

$$D_\xi^\gamma(\Pi_\alpha(y,D)\varphi(x))(0) = 0$$

for all multi-indices γ.

Furthermore, for arbitrary real number l and multi-index γ, there exists a real number l_1 such that

$$(1+|\xi|^2)^l |D_\xi^\gamma F(\Pi_\alpha(y,D)\varphi(x))| \leq C(1+|\xi|^2)^{l_1}|D_\xi^\gamma \hat{\varphi}(\xi)|.$$

This implies the continuity of $\Pi_\alpha(y,D)$. ■

We return to the proof of Theorem 2. (i) is a simple consequence of Lemma 4. (ii) and (iii) can be proved analogously to Theorem 1.

Theorem 3. *Let $\varphi \in \Phi^{'}(I\!R^n)$. Then the problem P_α has a solution $u(y,x)$ unique to within an arbitrary additive polynomial, and for this solution the following inclusions hold true:*

(a) $u(y,x) \in C(y \geq 0; \Phi^{'}(I\!R^n))$;

(b) $\mathbf{D}_-^\alpha u(y,x) \in C(y \geq 0; \Phi'(I\!R^n))$;

(c) $u(y,x) \in C^\infty(y > 0; \Phi'(I\!R^n))$.

The proof of Theorem 3 is based on the following lemma.

Lemma 5. *For every fixed $y \geq 0$ the pseudo-differential operator $\Pi_\alpha(y,-D)$ with the symbol defined by (8) acts continuously from $\Phi^{'}(I\!R^n)$ into $\Phi^{'}(I\!R^n)$.*

P r o o f. Let $\varphi \in \Phi^{'}(I\!R^n)$. Then for arbitrary $v \in \Phi(I\!R^n)$ the equality

$$< \Pi_\alpha(y,-D)\varphi, v > = < \varphi, \Pi_\alpha(y,D)v >$$

holds. According to Lemma 4, $\Pi_\alpha(y,D)v \in \Phi(I\!R^n)$. The continuity of $\Pi_\alpha(y,-D)$ in $\Phi^{'}(I\!R^n)$ follows from the continuity of $\Pi_\alpha(y,D)$ in $\Phi(I\!R^n)$. ■

Finally, we give solvability results for the space $\Psi_G^{+,m}(I\!R^n)$.

Theorem 4. *Let $\varphi \in \Psi_G^{+,m}(I\!R^n)$. Then there exists a unique solution $u(y,x)$ of P_α and this solution has the properties:*

(i) $u(y,x) \in C(y \geq 0; \Psi_G(I\!R^n))$;

(ii) $\mathbf{D}_-^\alpha u(y,x) \in C(y \geq 0; \Psi_G(I\!R^n))$;

(iii) $u(y,x) \in C^\infty(y > 0; \Psi_G(I\!R^n))$.

The proof of the theorem is based on the following auxiliary result:

Lemma 6. *For every fixed $y \geq 0$ and $\alpha \geq 0$ the pseudo-differential operator*

$$\Pi_\alpha(y,D) : \Psi_G^{+,m}(I\!R^n) \to \Psi_G(I\!R^n) \tag{12}$$

is continuous.

P r o o f (of Lemma 6). Let $\varphi \in \Psi_G^{+,m}(I\!R^n)$. We show that $\Pi_\alpha(y,D)\varphi$ belongs to $\Psi_G(I\!R^n)$. In fact, for $\varphi \in \Psi_G^{+,m}(I\!R^n)$ there exists a number $\beta > m\alpha - \frac{n}{2}$ and a constant $C_\phi > 0$ such that

$$|\xi|^{-\beta}|\hat{\varphi}(\xi)| \le C_\phi < \infty$$

in some neighborhood of the origin. Hence,

$$|\Pi_\alpha(y,\xi)\hat{\varphi}(\xi)| = O(|\xi|^{\beta-\alpha m}),\ |\xi| \to \infty.$$

Since $m\alpha - \beta < \frac{n}{2}$, then $\Pi_\alpha(y,\xi)\hat{\varphi} \in L_2(I\!R^n)$. By the Plancherel theorem $\Pi_\alpha(y,D)\varphi \in L_2(I\!R^n)$. Moreover, $\operatorname{supp} F[\Pi_\alpha(y,D)\varphi] \subset \operatorname{supp}\hat{\varphi} \subset G$. Thus we have $\Pi_\alpha(y,D)\varphi \in \Psi_G(R^n)$.

Now we show the continuity of (12). Let a sequence of functions $\varphi_k \in \Psi_G^{+,m}(I\!R^n)$ tend to zero in the topology of $\Psi_G^{+,m}(I\!R^n)$. This means that there exists a compact set $K \subset G$ and numbers $\beta > m\alpha - \frac{n}{2}$ and $C > 0$ such that:

(a) $\operatorname{supp}\hat{\varphi}_k \subset K,\ k = 1,2,\ldots;$

(b) $|\xi|^{-\beta}|\hat{\varphi}_k(\xi)| \le C < \infty,\ k = 1,2,\ldots;$

(c) $\|\varphi_k\|_0 \to 0,\ k \to \infty.$

Obviously $\operatorname{supp} F[\Pi_\alpha(y,D)\varphi_k] \subset K$ for all $k = 1,2,\ldots$. Moreover,

$$\|\Pi_\alpha(y,D)\varphi_k\|_0^2 = (2\pi)^n \int_K |\Pi_\alpha(y,\xi)|^2|\hat{\varphi}(\xi)|^2 d\xi$$

$$\le C\int_{|\xi|\le\epsilon} \frac{d\xi}{|\xi|^{2(m\alpha-\beta)}} + C_\epsilon \int_{R^n} |\hat{\varphi_k}|^2 d\xi,$$

with some positive real numbers ϵ and C_ϵ. Let $\delta > 0$ be an arbitrarily small number. Then we can choose ϵ small enough to ensure that

$$C\int_{|\xi|\le\epsilon} \frac{d\xi}{|\xi|^{2(m\alpha-\beta)}} < \frac{\delta}{2}.$$

Further we can take k large enough to ensure that

$$C_\epsilon \int_{I\!R^n} |\hat{\varphi}_k|^2 d\xi < \frac{\delta}{2}.$$

These inequalities imply

$$\|\Pi_\alpha(y,D)\varphi_k\|_0 \to 0,\ k \to \infty.$$

■

Corollary 3. *Let $m = 1$, $A(D) = \Delta$ with the Laplace operator Δ. Let $\varphi \in \Psi_G^{+,1}(I\!R^n)$. Then there exists a function $u(y,x)$, harmonic on the upper half-space $y > 0$, $x \in I\!R^n$, such that*

$$\lim_{y \to +0} \mathbf{D}_-^\alpha u(y,x) = \varphi(x)$$

in the topology of the space $\Psi_G(I\!R^n)$.

Theorem 5. *Let $\varphi \in \Psi'_{-G}(I\!R^n)$. Then there exists a unique solution $u(t,x)$ of P_α, and this solution has the properties:*
(i) $u(y,x) \in C(y \geq 0; \Psi_G^{-,m}(I\!R^n))$;
(ii) $\mathbf{D}_-^\alpha u(y,x) \in C(y \geq 0; \Psi_G^{-,m}(I\!R^n))$;
(iii) $u(y,x) \in C^\infty(y > 0; \Psi_G^{-,m}(I\!R^n))$.

The proof of this theorem is based on the following lemma:

Lemma 7. *For every fixed $y \geq 0$ and $\alpha \geq 0$ the pseudo-differential operator*

$$\Pi_\alpha(y, -D) : \Psi'_{-G}(I\!R^n) \to \Psi_{-G}^{-;m}(I\!R^n)$$

is continuous.

Lemma 7 follows from Lemma 6, its proof is analogous to that of Lemma 5.

Corollary 4. *Let $m = 1$, $A(D) = \Delta$ and $\varphi \in \Psi'_{-G}(I\!R^n)$. Then there exists a function $u(y,x)$, harmonic on the upper half-space $y > 0$, $x \in I\!R^n$, such that*

$$\lim_{y \to +0} \mathbf{D}_-^\alpha u(y,x) = \varphi(x)$$

in the topology of the space $\Psi_G^{-,1}(I\!R^n)$.

P r o o f. Let $\varphi \in \Psi'_{-G}(I\!R^n)$. Then the functional $u(y,\cdot) = \Pi_\alpha(y,-D)\varphi$ is a harmonic function in the distributional sense. Here $\Pi_\alpha(y,-D)$ is the pseudo-differential operator with the symbol

$$\Pi_\alpha(y,\xi) = |\xi|^{-\alpha} \exp(-\alpha\pi - y|\xi|).$$

According to the Weyl lemma $u(y,x)$ is an ordinary harmonic function. By Lemma 7 $u(y,x)$ is in $\Psi_G^{-,1}(I\!R^n)$ for each fixed y. By Theorem 5 $\mathbf{D}_-^\alpha u(y,x) \to \varphi(x)$ as $y \to +0$. ■

Remark 3. *The obtained results can be easily extended to the more general Sobolev spaces H_p^s with arbitrary $p \in (1,\infty)$, Besov or Lizorkin-Triebel type spaces. See [Uma98], where the case $m = 1$ has been considered in these spaces.*

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