On Monotone Multigrid Methods for the Signorini Problem

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SUMMARY

Using well–known concepts from the scalar case (cf. [Kor94]) we derive globally convergent multigrid methods for the discretized Signorini problem in linear elasticity. Special care has to be taken in case of varying normal directions. The algorithm can be implemented as a multigrid V–cycle with optimal complexity. In numerical experiments we observed similar convergence rates as for linear elasticity.

INTRODUCTION

Contact problems in computational mechanics have significant importance for a variety of practical applications. Examples are metal forming processes, crash analysis, the design of gear boxes, bearings, car tires or implants in biomechanics. As a starting point for future developments, we consider the Signorini problem in linear elasticity [Sig33] describing the linearized frictionless contact of an elastic body with a rigid foundation. Even in this simple case, the construction of fast reliable solvers such as multigrid methods is far from trivial, due to the intrinsic non–differentiable nonlinearity of the problem.

Multigrid methods for scalar obstacle problems have been studied by several authors such as [HM83, Man84, BC83, HK94, Kor94], only to mention a few. Monotone multigrid methods as introduced in [Kor94] are based on inexact successive subspace corrections (cf. Xu [Xu92]). See e.g. [HD95, TX] for smooth nonlinear problems. Truncated monotone multigrid methods turned out to combine reliability and efficiency of previous approaches as proposed in [BC83, Man84].

For contact problems in elasticity, regularization techniques have become very popular, especially in the engineering community. This approach requires careful handling of regularization parameters in order to find a reasonable compromise between efficiency and accuracy. Straightforward penalization may be dangerous. Combinations with dual techniques like augmented Lagrangian methods [Ber82] are more robust but may significantly increase the number of unknowns. Monotone multigrid methods to be developed in this paper do not involve any regularization or additional unknowns.

Multigrid methods for the Signorini problem have been considered for example by Belsky and Schöberl. Belsky [Bel93] applies global damping together with some sort of active set strategie in order to ensure feasibility of the iterates. This approach may provide little progress in comparison with *implicit local damping* (see [Kor94] Lemma 2.1). Moreover, only the case of equidirectional normals is considered. Schöberl [Sch98b, Sch98a] uses an approximate projection method (cf. e.g. [Glo84, p. 5] with DD–preconditioner and a fast solver on the interior nodes. Multigrid efficiency is shown and confirmed by numerical experiments provided that the number of interior nodes grows sufficiently faster than the number of nodes on the (possible) contact boundary.

The basic idea of monotone multigrid methods is to minimize the potential energy in the direction of suitably selected search directions. Well–known Gauß–Seidel relaxation is resulting from choosing the fine grid nodal basis. In order to increase convergence speed by better representation of the low–frequency components of the error, we extend this set of search directions by carefully selected coarse grid functions with large support. Optimal complexity is preserved by suitable restriction of the fine grid constraints to coarser grids. A general convergence result will be given in Section 3. In case of equidirectional normals, we can use a straightforward block version of the scalar case. The situation is more complicated, if different normals occur. In this case, different directions on the fine grid have to be represented by the same coarse grid function. A recursive definition of such functions is given in Section 4. The resulting truncated monotone multigrid method is globally convergent and can be implemented as a V–cycle with projected block Gauß–Seidel smoothing and Galerkin restriction and prolongation. In our numerical experiments we observed similar efficiency as in the linear selfadjoint case. More detailed convergence results and numerical experiments in three space dimensions will be contained in a forthcoming paper.

The Signorini Problem

We consider the deformation of an elastic body as described by the unknown displacement vector $u = (u_1, \dots , u_d)$ defined on the polygonal (polyhedral) reference configuration $\Omega \subset R^d$, $d = 2, 3$. Deformation is caused by volume forces f and traction forces P. We assume that the deformed body is in equilibrium state such that

$$-\sigma_{ij}(u)_{,j} = f_i \qquad \text{in } \Omega. \tag{1}$$

Here and in the following summation is implicitly taken over indices i, j, s occuring twice and $\sigma_{,j} = \frac{\partial \sigma}{\partial x_j}$ is the partial derivative. We further assume that the stress tensor σ is related to the strain tensor ε,

$$\varepsilon_{kl}(u) = \frac{1}{2}(u_{k,l} + u_{l,k}), \quad 1 \le k, l \le d, \tag{2}$$

by Hooke's Law

$$\sigma_{ij}(u) = E_{ijkl}\varepsilon_{kl}(u), \quad 1 \le i,j,k,l \le d. \tag{3}$$

Hooke's tensor E has the symmetry properties

$$E_{ijkl} = E_{klij} = E_{klji}. \tag{4}$$

The body's surface $\partial\Omega$ is decomposed into three disjoint parts

$$\partial\Omega = \Gamma_D \,\cup\, \Gamma_F \,\cup\, \Gamma_C$$

with Γ_D having positive measure. We assume that the body is clamped at Γ_D so that

$$u_i = 0 \qquad \text{on } \Gamma_D \tag{5}$$

and traction forces P are applied at Γ_F, giving

$$\sigma_{ij}(u) \cdot \boldsymbol{n}_j = P_i \qquad \text{on } \Gamma_F. \tag{6}$$

Here $\boldsymbol{n} = (n_1, \dots, n_d)$ denotes the outer normal on $\partial\Omega$. The remaining part Γ_C of the body's surface may (or may not) be in contact with a given rigid foundation. Identifying normals on Ω and on the deformed configuration (cf. e.g. [KO88, p. 19]), we obtain linearized contact conditions

$$\left.\begin{array}{rcl} \sigma_T(u) & = & 0 \\ \sigma_n(u)(u \cdot \boldsymbol{n} - g) & = & 0 \\ u \cdot \boldsymbol{n} - g & \le & 0 \\ \sigma_n(u) & \le & 0 \end{array}\right\} \text{on } \Gamma_C. \tag{7}$$

The function g denotes the initial gap between Ω and the rigid foundation. $\sigma_n = \sigma_{ij}n_in_j$ and σ_T denote the normal and the tangential component of the stress vector, respectively. Observe that friction is not taken into account. We emphasize that it is part of the problem to find out which portion of Γ_C actually is in contact with the rigid foundation.

A solution u of the *Signorini problem* (cf. [Sig33]) satisfies the equilibrium conditions (1) subject to the boundary conditions (5), (6) and (7). Multiplying with a test function v and integrating by parts the Signorini problem can be reformulated in a weak sense. In this way, we obtain the variational inequality

$$u \in \mathcal{K}\colon \qquad a(u, v-u) \ge f(v-u) \qquad \forall v \in \mathcal{K}. \tag{8}$$

The bilinear form

$$a(u,v) = \int_\Omega E_{ijkl}\, u_{k,l}\, v_{i,j}\; dx = \int_\Omega \sigma_{ij}(u)\, v_{i,j}\; dx$$

is symmetric and elliptic on $H = \{v \in \left(H^1(\Omega)\right)^d \mid v|_{\Gamma_D} = 0\}$. For sufficiently smooth data, the functional

$$f(v) = \int_\Omega f_i\, v_i\; dx + \int_{\Gamma_F} P_i\, v_i\; ds.$$

is linear and bounded on H. Finally,

$$\mathcal{K} = \{v \in \left(H^1(\Omega)\right)^d \mid v|_{\Gamma_D} = 0,\ v \cdot \boldsymbol{n} \leq g \text{ on } \Gamma_C\}$$

denotes the set of admissible displacements. Note that $\mathcal{K}$ is a closed convex subset of H. The variational formulation (8) can be equivalently rewritten as the convex minimization problem

$$u \in \mathcal{K}: \qquad J(u) \leq J(v) \qquad \forall v \in \mathcal{K}, \tag{9}$$

for the quadratic *potential energy functional* $J\colon H \times H \longrightarrow \mathbb{R}$,

$$J(v) = \frac{1}{2}a(v,v) - f(v). \tag{10}$$

Existence and uniqueness of a weak solution u can now be established by direct methods of variational calculus (see e.g. [KO88, p. 113]). A discrete version of the convex minimization problem (9) will also serve as a basis for our iterative solvers to be developed later on.

Let $\mathcal{T}_\ell$ be a given partition of Ω into triangles (tetrahedra) with minimal diameter $h_\ell = \mathcal{O}(2^{-l})$. $\mathcal{N}_\ell$ denotes the set of all vertices of $\mathcal{T}_\ell$ that are contained in $\Omega \cup \Gamma_F \cup \Gamma_C$. Discretizing (9) by continuous, piecewise linear finite elements $\mathcal{S}_\ell$,

$$\mathcal{S}_\ell = \{v \in H^1(\Omega) \mid v|_t \text{ is linear } \forall t \in \mathcal{T}_\ell \text{ and } v(p) = 0 \ \ \forall p \in \Gamma_D \cap \mathcal{N}_\ell\},$$

we obtain the discrete minimization problem

$$u_\ell \in \mathcal{K}_\ell: \quad J(u_\ell) \leq J(v) \qquad \forall v \in \mathcal{K}_\ell. \tag{11}$$

Here the set $\mathcal{K} \subset H$ is replaced by its discrete analogue $\mathcal{K}_\ell \subset {\mathcal{S}_\ell}^d$,

$$\mathcal{K}_\ell = \{v \in {\mathcal{S}_\ell}^d \mid \boldsymbol{v}(p) \cdot \boldsymbol{n}_\ell(p) \leq g_\ell(p) \quad \forall p \in \mathcal{N}_\ell \cap \Gamma_\ell\}$$

induced by suitable approximations $\boldsymbol{n}_\ell$ and g_ℓ of $\boldsymbol{n}$ and g. The subscript ℓ will be skipped in the sequel. Note that in general $\mathcal{K} \not\subset \mathcal{K}_\ell$. Other discretizations, for example by piecewise bilinear functions on quadrilaterals, can be constructed in a similar way.

Of course, (11) also admits a unique solution. For shape regular partitions $\mathcal{T}_\ell$ the approximate solutions u_ℓ are converging to u as $\ell \to \infty$. Optimal error estimates in H^1 and L^2 are obtained for H^2–regular problems. We refer to [KO88, p. 125] for details.

Extended Relaxations

The finite element space ${\mathcal{S}_\ell}^d$ is spanned by the nodal basis

$$\lambda_p^{(\ell)} \boldsymbol{E}^i, \quad i = 1, \dots, d, \ \ p, q \in \mathcal{N}_\ell$$

with cartesian unit vectors $\boldsymbol{E}^i \in \mathbb{R}^d$ and the standard nodal basis $\lambda_p^{(\ell)}$, $p \in \mathcal{N}_\ell$, of $\mathcal{S}_\ell$. The successive subspace correction in the sense of Xu [Xu92] based on the splitting $\mathcal{S}_\ell = V_1 + \ldots + V_{n_\ell}$ in the d–dimensional subspaces

$$V_l = \operatorname{span}\{\lambda_{p_l}^{(\ell)} \boldsymbol{E}^1, \dots, \lambda_{p_l}^{(\ell)} \boldsymbol{E}^d\} \qquad l = 1, \dots, n_\ell = \#\mathcal{N}_\ell$$

gives rise to the well–known projected block Gauß-Seidel relaxation $\mathcal{M}_\ell$ (cf. e.g. [Glo84, p. 151]). As the convergence speed of this method deteriorates rapidly with increasing refinement, we consider the *extended splitting*

$$\mathcal{S}_\ell{}^d = V_1 + \ldots + V_{n_\ell} + V^\nu_{n_\ell+1} + \ldots + V^\nu_m. \tag{12}$$

The additional d–dimensional subspaces $V_l^\nu \subset \mathcal{S}_\ell{}^d$, $l > n_\ell$, are intended to improve the low–frequency contributions of the error. Hence, we assume that each of these spaces is spanned by functions $\mu_l^1, \ldots \mu_l^d \in \mathcal{S}_\ell{}^d$ with large support. The resulting *extended relaxtion* now reads as follows.

Starting with the given ν–th iterate $u_\ell^\nu = w_0^\nu \in \mathcal{K}_\ell$, we compute a sequence of intermediate iterates $w_l^\nu = w_{l-1}^\nu + v_l^*$, $l = 1, \ldots, m$. The corrections v_l^* are the unique solutions of the local subproblems

$$v_l^* \in \mathcal{D}_l{}^* : \quad J(w_{l-1} + v_l^*) \le J(w_{l-1} + v) \quad \forall v \in \mathcal{D}_l{}^*, \tag{13}$$

where the closed, convex sets $\mathcal{D}_l{}^* = \mathcal{D}_l{}^*(w_{l-1}^\nu)$ are defined by

$$\mathcal{D}_l{}^*(w_{l-1}^\nu) = \{v \in V_l \mid w_{l-1}^\nu + v \in \mathcal{K}_\ell\}. \tag{14}$$

We emphasize that the additional spaces V_l^ν may change in each iteration step. Hence, the splitting (12) can be adapted to the exact contact set in course of the iteration. The index ν will be frequently suppressed in the sequel.

Adapting multigrid terminology, the leading projected block Gauß-Seidel relaxation $\mathcal{M}_\ell$ plays the role of a (fine grid) *smoother*, $\bar{u}_\ell^\nu = w_\ell^\nu = \mathcal{M}_\ell(u_\ell^\nu)$ is the *smoothed iterate* and all v_l^*, $l > n_\ell$ are called *coarse grid corrections.* Note that $\bar{u}_\ell^\nu \in \mathcal{K}_\ell$ for all $u_\ell^0 \in \mathcal{S}_\ell{}^d$.

The efficient evaluation of v_l^* from (13) suffers from the fact that the definition of $\mathcal{D}_l{}^*$ involves the values of a given $v \in V_l$ at all $p \in \mathcal{N}_\ell \cap \operatorname{int} \operatorname{supp} \mu_l^i$. As the evaluation of coarse grid functions μ_l^i usually requires additional interpolations the optimal complexity of classical multigrid methods may be lost. Hence, the exact coarse grid corrections $v_l^* \in V_l$, $l = n_\ell + 1, \ldots, m$, are replaced by approximations $v_l \in V_l$ provided by the approximate local problems

$$v_l \in \mathcal{D}_l : \quad J(w_{l-1} + v_l) \le J(w_{l-1} + v) \quad \forall v \in \mathcal{D}_l, \tag{15}$$

with closed, convex subsets $\mathcal{D}_l \subset V_l$, defined by

$$\mathcal{D}_l = \{v \in V_l \mid v(p_l) \cdot \boldsymbol{e}^i(p_l) \in [\underline{\psi}_l^i, \overline{\psi}_l^i] \; \forall i = 1, \ldots, d\}.$$

The local unit vectors $\boldsymbol{e}^i(p_l)$ are obtained by extending $\boldsymbol{e}^1(p_l) = \boldsymbol{n}(p_l)$ to an orthonormal set in $\mathbb{R}^d$, $p_l \in \mathcal{N}_\ell$ is suitably chosen and the *local obstacles* $\underline{\psi}_l^i, \overline{\psi}_l^i \in \mathbb{R}$ are understood to depend on ν and on the preceding intermediate iterates $w_0^\nu, \ldots, w_{l-1}^\nu$. We assume that $\underline{\psi}_l^i, \overline{\psi}_l^i$ and μ_l^i are selected in such a way that

$$0 \in \mathcal{D}_l \subset \mathcal{D}_l{}^*. \tag{16}$$

Of course we need no constraints (i.e. $\underline{\psi}_l^i = -\infty, \overline{\psi}_l^i = +\infty$) if $\mu_l^i(p) = 0$, $\forall p \in \mathcal{N}_\ell \cap \Gamma_C$.

Theorem 1 *The approximate extended relaxation*

$$u_\ell^{\nu+1} = u_\ell^\nu + \sum_{l=1}^{n_\ell} v_l^* + \sum_{l=n_\ell+1}^{m} v_l$$

with v_l^* and v_l computed from (13) and (15), respectively, is globally convergent.

Proof. The sequence of iterates u_ℓ^ν, $\nu = 0, 1, \dots$, is bounded because our scheme (13) is *monotone* in the sense that

$$J(u_\ell^{\nu+1}) \le J(w_{l+1}^\nu) \le J(w_l^\nu) \le J(\bar{u}_\ell^0) < \infty\ , \quad \nu = 1, 2, \dots\ ,$$

and we have $J(v^\nu) \to \infty$ for any unbounded sequence $v^\nu \in \mathcal{S}_\ell{}^d$.

As u_ℓ^ν is bounded and $\mathcal{S}_\ell{}^d$ has finite dimension, it is sufficient to show that each convergent subsequence of u_ℓ^ν converges to u_ℓ. Let $u_\ell^{\nu_k}$ be an arbitrary, convergent subsequence of u_ℓ^ν, with some limit $u^* \in \mathcal{S}_\ell{}^d$,

$$u_\ell^{\nu_k} \to u^*, \quad k \to \infty. \tag{17}$$

It is easily checked that $\mathcal{M}_\ell$ is continuous so that

$$\mathcal{M}_\ell(u_\ell^{\nu_k}) \to \mathcal{M}_\ell(u^*), \quad k \to \infty. \tag{18}$$

Again, monotonicity of the iteration implies

$$J(u_\ell^{\nu_{k+1}}) \le J(u_\ell^{\nu_k+1}) \le J(\mathcal{M}_\ell(u_\ell^{\nu_k})) \le J(u_\ell^{\nu_k}).$$

In virtue of the convergence (17), (18) and the continuity of J on $\mathcal{K}_\ell$, this leads to

$$J(\mathcal{M}_\ell(u^*)) = J(u^*). \tag{19}$$

It is easily seen that (19) holds, if and only if all local corrections of the projected block Gauß-Seidel relaxation applied to u^* are zero, i.e. $\mathcal{M}_\ell(u^*) = u^*$. As the finite element solution u_ℓ is the only fixed point of the projected block Gauß-Seidel relaxation (cf. e.g. [Glo84, p. 152]), we have shown $u^* = u_\ell$. This completes the proof.

A truncated Monotone Multigrid Method

Assume that $\mathcal{T}_\ell$ is resulting from ℓ refinements of an intentionally coarse triangulation $\mathcal{T}_0$. Let $\mathcal{N}_\ell$ denote the set of vertices contained in $\Omega \cup \Gamma_F \cup \Gamma_C$. Though the algorithms and convergence results to be presented can be easily generalized to the non–uniform case, let us assume for the moment that the triangulations are uniformly refined. More precisely, each triangle $t \in \mathcal{T}_k$ is subdivided into four congruent subtriangles in order to produce the next triangulation $\mathcal{T}_{k+1}$.

Using the hierarchy of grids and the corresponding hierarchy of finite element spaces, we now choose suitable spaces $V_{n_\ell+1}, \dots, V_m$. Each space $V_l = V_{l(p,k)}$ is associated with a node $p \in \mathcal{N}_k$ on some refinement level k. We frequently use the notation $V_p^{(k)} = V_{l(p,k)}$, $\mu_p^{(k)} = \mu_{l(p,k)}$, etc. The ordering $l = l(p,k)$ is taken from fine to coarse.

The local basis functions $(\mu_p^{(k)})^i$ will depend on the actual guess of the contact set $\mathcal{N}_\ell{}^\bullet(u_\ell^\nu)$,

$$\mathcal{N}_\ell{}^\bullet(\bar{u}_\ell^\nu) = \{p \in \mathcal{N}_\ell \cap \Gamma_C \mid \bar{u}_\ell^\nu(p) \cdot \boldsymbol{n}(p) = g(p)\}.$$

Recall that $\boldsymbol{e}^i(p) \in \mathbb{R}^d,\ i=1,\dots d$, denotes a *local orthonormal system* with $\boldsymbol{e}^1(p)=\boldsymbol{n}(p)$.

Starting with

$$\left(\mu_p^{(\ell)}\right)^i = \begin{cases} \lambda_p^{(\ell)}\boldsymbol{e}^i(p) & \text{if } i \neq 1 \text{ or } p \notin \mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu) \\ 0 & \text{else} \end{cases}, \tag{20}$$

we recursively define truncated basis functions

$$\left(\mu_p^{(k-1)}\right)^i = \sum_{q\in\mathcal{N}_k} \lambda_p^{(k-1)}(q)\ \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^j(q)\ (\mu_q^{(k)})^j \tag{21}$$

(summation over j) and we set

$$V_p^{(k)} = \operatorname{span}\{(\mu_p^{(k)})^1,\dots,(\mu_p^{(k)})^d\}. \tag{22}$$

The definition of $(\mu_p^{(k-1)})^i$ is motivated by the following lemma.

Lemma 1 *Let $0\le k\le \ell$, $p\in\mathcal{N}_k$ and $1\le i,j\le d$. Then*

$$\left(\mu_p^{(k)}\right)^i(q)\cdot\boldsymbol{e}^j(q) = \begin{cases} \lambda_p^{(k)}(q)\ \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^j(q) & \textit{if } j\neq 1 \textit{ or } q\notin\mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu) \\ 0 & \textit{else} \end{cases} \tag{23}$$

holds for all $q\in\mathcal{N}_\ell$.

Proof. The assertion is clear for $k=\ell$. Assuming that (23) holds for some $k\le\ell$, we obtain (summation on s) for $j\neq 1$ or $q\notin\mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu)$

$$\begin{aligned}(\mu_p^{(k-1)})^i(q)\cdot\boldsymbol{e}^j(q) &= \textstyle\sum_{r\in\mathcal{N}_k}\lambda_p^{(k-1)}(r)\ \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^s(r)\ (\mu_r^{(k)})^s(q)\cdot\boldsymbol{e}^j(q)\\ &= \textstyle\sum_{r\in\mathcal{N}_k}\lambda_p^{(k-1)}(r)\ \lambda_r^{(k)}(q)\ \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^s(r)\ \boldsymbol{e}^s(r)\cdot\boldsymbol{e}^j(q)\\ &= \textstyle\sum_{r\in\mathcal{N}_k}\lambda_p^{(k-1)}(r)\ \lambda_r^{(k)}(q)\ \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^j(q)\\ &= \lambda_p^{(k-1)}(q)\ \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^j(q)\end{aligned}$$

exploiting the identity

$$\boldsymbol{e}^i(p)\cdot\boldsymbol{e}^s(r)\ \boldsymbol{e}^s(r)\cdot\boldsymbol{e}^j(q) = \boldsymbol{e}^i(p)\cdot\boldsymbol{e}^j(q) \qquad p,q,r\in\mathcal{N}_\ell.$$

The remaining case $j=1$, $q\in\mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu)$ is obvious.

□

As a consequence of Lemma 1, we have $V_p^{(k)} = \operatorname{span}\{\lambda_p^{(k)}\boldsymbol{E}^1,\dots,\lambda_p^{(k)}\boldsymbol{E}^d\}$, if there is no contact in the neighborhood of p, i.e. if int supp $\lambda_p^{(k)}\cap\mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu)=\emptyset$. Hence, our algorithm *locally* reduces to a classical multigrid method with Galerkin restriction and block Gauß–Seidel smoother. If contact occurs in some $q\in$ int supp $\lambda_p^{(k)}$, then, due to initial truncation (20), there is no contribution in the direction of $\boldsymbol{n}(q)$ from $(\mu_p^{(k)})^i$. In particular, the contact set is not changed by the coarse grid correction.

In order to complete the construction of our multigrid method, we now describe the recursive construction of local obstacles $\underline{\psi}_{l(p,k)}^i = (\underline{\psi}_p^{(k)})^i$, $\overline{\psi}_{l(p,k)}^i = (\overline{\psi}_p^{(k)})^i$ occurring in (14). Starting with

$$\begin{array}{llllll}\left(\underline{\psi}_p^{(\ell)}\right)^1 = -\infty, & \left(\overline{\psi}_p^{(\ell)}\right)^1 = g(p)-\bar{u}_\ell^\nu(p)\cdot n(p) & & \multirow{2}{*}{} \\ \left(\underline{\psi}_p^{(\ell)}\right)^i = -\infty & \left(\overline{\psi}_p^{(\ell)}\right)^i = +\infty & i=2,\dots,d & p\in\mathcal{N}_\ell,\end{array}$$

we assume that local obstacles $(\underline{\psi}_p^{(k)})^i$, $(\overline{\psi}_p^{(k)})^i$ have been constructed for some $k \le \ell$. Once all local corrections $v_p^{(k)} = v_{l(p,k)}$ on level k have been computed from (15), we update the obstacles according to

$$\begin{aligned} \left(\underline{\psi}_p^{(k)}\right)^i &\to \left(\underline{\psi}^{(k)_p}\right)^i - v_p^{(k)}(p) \cdot \boldsymbol{e}^i(p) \\ \left(\overline{\psi}_p^{(k)}\right)^i &\to \left(\overline{\psi}_p^{(k)}\right)^i - v_p^{(k)}(p) \cdot \boldsymbol{e}^i(p) \end{aligned} \qquad p \in \mathcal{N}_k. \tag{24}$$

For fixed $p \in \mathcal{N}_{k-1}$ and $i = 1, \dots d$ local obstacles on the next coarser level are now obtained by *monotone restriction*

$$(\underline{\psi}_p^{(k-1)})^i = d^{-1} \max \Psi_- , \qquad (\overline{\psi}_p^{(k-1)})^i = d^{-1} \min \Psi_+ \tag{25}$$

$$\Psi = \left\{ \frac{(\underline{\psi}_q^{(k)})^j}{\boldsymbol{e}^i(p)\cdot \boldsymbol{e}^j(q)}, \frac{(\overline{\psi}_q^{(k)})^j}{\boldsymbol{e}^i(p)\cdot \boldsymbol{e}^j(q)} \mid q \in \text{int supp } \lambda_p^{(k-1)} \cap \mathcal{N}_k,\ j = 1, \dots, d, \right.$$
$$\left. j \ne 1 \text{ or } q \notin \mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu),\ \boldsymbol{e}^i(p)\cdot \boldsymbol{e}^j(q) \ne 0 \right\},$$

where Ψ_- and Ψ_+ contain the nonpositive and nonnegative elements of Ψ, respectively. Note that in general the weights $\boldsymbol{e}^i(p) \cdot \boldsymbol{e}^j(q)$ may be positive, negative or zero.

Lemma 2 *Let $\bar{u}_\ell^\nu \in \mathcal{K}_\ell$. Then the subsets $\mathcal{D}_l = \mathcal{D}_p^{(k)}$,*

$$\mathcal{D}_p^{(k)} = \{ v \in V_p^{(k)} \mid v(p) \cdot \boldsymbol{e}^i(p) \in [(\underline{\psi}_p^{(k)})^i, (\overline{\psi}_p^{(k)})^i]\ \forall i = 1, \dots, d \}$$

satisfy condition (16).

Proof. From corrections on levels ℓ to k we obtain the intermediate iterate $w^{(k)}$,

$$w^{(k)} = \bar{u}_\ell^\nu + \sum_{j=k}^{\ell} \sum_{p \in \mathcal{N}_j} v_p^{(k)}.$$

We show by induction that

$$w^{(k)} + \sum_{p \in \mathcal{N}_k} z_p^{(k)} \in \mathcal{K}_\ell \qquad \forall z_p^{(k)} \in \mathcal{D}_p^{(k)},\ p \in \mathcal{N}_k \tag{26}$$

holds after update (24) of $\mathcal{D}_p^{(k)}$. The assertion is clear for $k = \ell$. Assume that (26) holds for some $k \le \ell$. Let $z_p^{(k-1)} = (z_p^{(k-1)})_i\ (\mu_p^{(k-1)})^i \in \mathcal{D}_p^{(k-1)}$, $p \in \mathcal{N}_{k-1}$. Then

$$\sum_{p \in \mathcal{N}_{k-1}} z_p^{(k-1)} = \sum_{p \in \mathcal{N}_k} z_p^{(k)}$$

holds with

$$z_p^{(k)} = \sum_{r \in \mathcal{N}_{k-1}} \lambda_r^{(k-1)}(p)\ (z_r^{(k-1)})_i\ \boldsymbol{e}^i(r) \cdot \boldsymbol{e}^j(p)\ (\mu_p^{(k)})^j.$$

As a consequence of monotone restriction (25), we get for all $j = 1, \dots, d$

$$z_p^{(k)}(p) \cdot \boldsymbol{e}^j(p) = \sum_{r \in \mathcal{N}_{k-1}} \lambda_r^{(k-1)}(p)\ (z_p^{(k-1)})_i\ \boldsymbol{e}^i(r) \cdot \boldsymbol{e}^j(p) \in [(\underline{\psi}_p^{(k)})^j, (\overline{\psi}_p^{(k)})^j]$$

if $p \in \mathcal{N}_k \backslash \mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu)$. We clearly have $z_p^{(k)}(p) \cdot \boldsymbol{e}^j(p) = 0 \in \left[(\underline{\psi}_p^{(k)})^j, (\overline{\psi}_p^{(k)})^j\right]$ in the remaining case $p \in \mathcal{N}_\ell^\bullet(\bar{u}_\ell^\nu)$.

□

The proof shows that any correction $v^{(k)} = \sum_{p \in \mathcal{N}_k} z_p^{(k)}$ with $z_p^{(k)} \in \mathcal{D}_p^{(k)}$ is feasible. This allows variants of our method including multigrid W–cycle or post–smoothing.

We emphasize that our approach immediately generalizes to Signorini-type problems involving tangential constraints. On the other hand, monotone restriction (25) may cause tangential constraints on the coarse levels which are not present on the finest grid.

Coupling of different directions leads to more pessimistic constraints in comparison with the scalar case, where similar techniques were used by Mandel [Man84]. Improved monotone restrictions may be obtained by generalizing ideas from Kornhuber [Kor94]. Moreover, the factor d^{-1} in (25) can be replaced by the number of non–zero entries in the i–th row of $\left(\boldsymbol{e}^i(p) \cdot \boldsymbol{e}^j(q)\right)_{1 \le i,j \le d}$.

Now we are ready to state the main result of this section.

Theorem 2 *The truncated monotone multigrid method based on local spaces V_l from (22) and on local obstacles $\underline{\psi}_l$, $\overline{\psi}_l$ from (25) is globally convergent.*

Proof. The assertion is an immediate consequence of Theorem 1 together with Lemma 1 and Lemma 2.

□

More detailed convergence results will be contained in a forthcoming paper.

In order to further improve coarse grid transport, truncation is performed recursively on all refinement levels. Such a *fully truncated* monotone multigrid method requires d different sets of critical nodes $\mathcal{N}_\ell^{\bullet,i}$. A node $p \in \mathcal{N}_k$ on some level k is added to $\mathcal{N}_\ell^{\bullet,i}$ if some local correction $v_p^{(k)}$ hits (in direction of $\boldsymbol{e}^i(p)$) one of the local obstacles $(\underline{\psi}_p^{(k)})^i$, $(\overline{\psi}_p^{(k)})^i$. Truncated basis functions and local obstacles are obtain from straightforward extensions of (20),(21) and (25), respectively. Again, we can apply Theorem 1 to establish global convergence.

Other discretizations like piecewise bilinear functions on quadrilaterals can be treated in a similar way.

Numerical Examples

We consider the reference configuration $\Omega = (0,1)^d$ for $d = 2$. Dirichlet conditions $u = (0, -0.01)$ are prescribed at the upper boundary $\Gamma_D = \{(x,y) \in \partial\Omega \mid y = 1\}$. The lower boundary of the square is the possible contact boundary $\Gamma_C = \{(x,y) \in \partial\Omega | y = 0\}$ and we impose Neumann conditions on the remaining part of $\partial\Omega$. No volume forces f are applied. Hooke's tensor is resulting from plane strain assumption and the Lamé constants $\lambda = 121154$ and $\mu = 161538$. The rigid foundation consists of the straight line $y = -0.4$ and the circle with radius 0.4 centered at $(0.5, -0.4)$ (see Figure 3).

The initial triangulation $\mathcal{T}_0$ is depicted in Figure 1. We enforce successive refinement of all quadrilaterals with vertex on Γ_C. As a consequence, nonlinearity at the possible contact boundary Γ_C dominates linearity in the interior of Ω. The discrete problems are

Figure 1: Coarse grid and deformed grid on level 5

solved with the fully truncated monotone multigrid method as introduced in the preceding section. The algorithm was implemented in the framework of the toolbox UG [BBJ+97]. We always use three pre– and three post–smoothing steps. In our first computation, the approximate admissible set $\mathcal{K}_\ell$ is defined by choosing $g_\ell = g$ and $n_\ell = n$. In this case, even at the contact boundary, the local coarse grid spaces are spanned by (possibly truncated) cartesian nodal basis functions. The resulting deformed grid after 5 refinement steps is depicted in Figure 1. The computed displacements are scaled by the factor of 10 so that the shape of the obstacle is not shown. On the final level 13, we obtained 245, 828 unknowns nodes and 5743 contact nodes. Figure 2 shows the (approximate) averaged convergence rates

$$\rho_k = \sqrt[\nu_0]{\|u_k^{\nu_0} - u_k^{\nu_0-1}\| / \|u_k^1 - u_k^0\|}$$

over the number of unknowns n_k. Here, $\|\cdot\| = a(\cdot,\cdot)^{\frac{1}{2}}$ denotes the energy norm and ν_0 is chosen such that $\|u_k^{\nu_0} - u_k^{\nu_0-1}\| \le 10^{-12}$. Iteration starts with the initial iterates

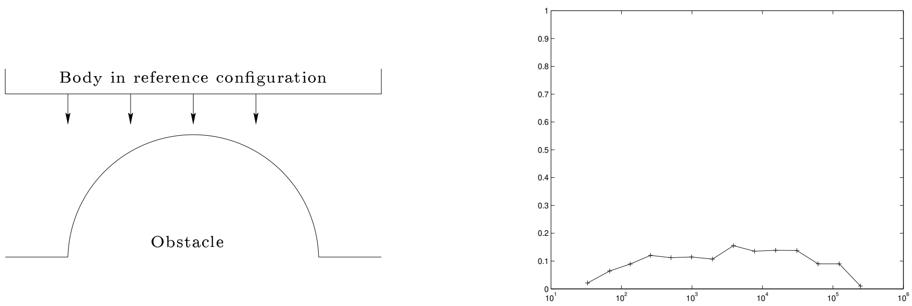

Figure 2:

$u_0^0 = 0$ and $u_{k+1}^0 = u_k^{\nu_0}$, $k > 0$. We obtain similar results as for related linear problems. In particular, the asymptotic convergence rates seem to saturate with increasing refinement.

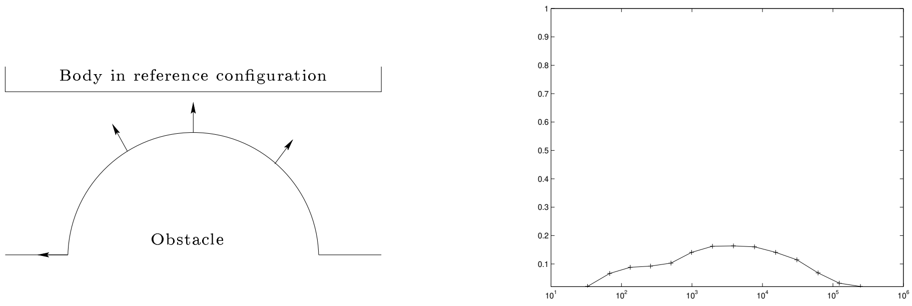

Figure 3:

In our next experiment, we investigate the robustness of our multigrid method with respect to varying normals. For this reason, we select approximate normals $n_\ell = n_C$ with n_C denoting the inner normal of the rigid obstacle, see Figure 3. In this case, only tangential corrections are allowed at the contact boundary so that varying fine grid normals have to be represented by appropriate coarse grid functions. Sliding of the body is possible along a curved obstacle only. Nevertheless, we still observe similar asymptotic convergence rates as illustrated in Figure 3.

Numerical results for $d = 3$ space dimensions will be contained in a forthcoming paper.

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