Collision states in quantum chromo(flavor)-dynamics.

A.H. Völkel
Institut für Mathematik I
Freie Universität Berlin
14195 Berlin, Germany

Abstract: In the color neutral sector of flavored gauge quantum chromo dynamics with color confining and flavor symmetry breaking interaction (i.e. time translations) asymptotic incoming and outgoing many particle collision states are shown to exist as strong Haag-Ruelle limits for every given color neutral one particle state provided its excitation operator satisfies besides the Haag-Ruelle-Hepp (or Herbst) condition a positivity condition on its two-point function and the vacuum is an eigenstate of the metric operator.

I: Introduction

In gauge quantum chromo(flavor)-dynamics of quarks and gluons color confinement enforces the replacement of the unitary representation of the translation group at least by non-isometric or even unbounded ones with respect to the Hilbert norm[1,2,3] remaining isometric only with respect to the inner product of an indefinite metric. Otherwise nothing could prevent colored quarks to manifest themselves on the same footing with the uncolored hadrons as asymptotic collision states, if such states exist at all. Moreover depending on the indefinite metric the standard representation defined by the substitution law $\varphi^{\cdots}_{\cdots}(x) \to \varphi^{\cdots}_{\cdots}(x+y)$ for the basic fields must be replaced by other ones[4]. For if the translational invariant vacuum is an eigenvector of the metric operator then due to Theorem 4.2 in Ref.[5] isometry with respect to the indefinite metric together with the standard substitution law implies unitarity with respect to the Hilbert norm.

Furthermore in the standard representation of the translation group the breaking of further interval symmetries like $SU(?)-$flavor symmetry is usually treated in a perturbative way via adding some suitable breaking term to the a priori symmetry conserving Lagrangian respectively Hamiltonian[6,7,8]. If however the Hamiltonian is considered to be the infinitesimal generator of the time translations one should expect a backlash on their representation when adding new terms to the Hamiltonian. Vice versa it should be possible to obtain the symmetry breaking Hamiltonian from a symmetry conserving, that means form invariant, Lagrangian by calculating the Noether energy-momentum density tensor within a representation of the translation group which is chained to the internal symmetry group to be broken (and therefore must obey a different substitution law). Indeed, the Hamiltonian breaking the $SU(3)$ flavor symmetry of the chromo-dynamic Lagrangian down to $SU(2) \times U(1)$ symmetry as observed in the world of hadrons has been derived via this mechanism in Ref.[4].

More explicite, by inspection the classical color $SU(3)$ local gauge and flavor $SU(3)$ global form invariant chromo-dynamic Lagrangian $L(\varphi^{8,1}(x)_m, \varphi^{3,3}(x)_m)$ of vector (color octet, flavor singlet) gluon fields $\varphi^{8,1}_{\mu,c}(x)_m$ and Dirac (color and flavor triplet) quark fields $\varphi^{3,3}_{\mu,c,f}(x)_m$ and their Dirac adjoint ones is form invariant under the following group of substitutions satisfying the composition law of the Poincaré group $(x, m) \to (\hat{x}, \hat{m}) := (\Lambda(\alpha)x + y, \Lambda(\alpha)m), \alpha \in SL(2, \mathbb{C}$ and

$$\varphi^{n,d}_{\mu,c,f}(x)_m \to \hat{\varphi}^{n,d}_{\mu,c,f}(\hat{x})_{\hat{m}} := exp.[Q_{\mathcal{C}}(c)\gamma(\Lambda(\alpha)m \cdot y)] \times$$
$$\times R^d(exp.i/2(\wp(\Lambda(\alpha) \cdot m)y)^a_f S^{(n,d)}(\alpha)^\nu_\mu \varphi^{n,d}_{\nu,c,a}(x)_m$$

Here $S^{(n,d)}$ denotes the 4-dim. representation of the Lorentz group for vector respectively spinor fields, $Q_{\mathcal{C}}(c)$ the color charge carried by the field $\varphi^{n,d}_{c,f}$ and R^d a d-dim. representation of the flavor group. $(\wp(m) \cdot y) := (m \cdot y)[\sigma_{(0,0,0)} + 1/\sqrt{3}(2/\delta - 5/3)\tau_{(0,0,0)}]$ is a traceless hermitian matrix from the complexified Lie-algebra of $SU(3)$ with σ and τ specified in equation (10) below. γ and $\delta \neq 0$ are real parameters, and the four-vector parameter m runs over sum Lorentz orbit $\mathcal{O}_{\mathcal{L}} \subset \mathbb{R}_4$ endowed with the invariant measure $d\mu_{\mathcal{L}}(m)$. The standard representation of the Poincaré group is contained in this sceme as the special case belonging to the exceptional orbit $\mathcal{O}^o_o$ which consists of only one point $m = O$ (apex

of the light cone) with the invariant measure $d\mu_o^o(m) = \delta^4(m)d^4m$ given by the Dirac $\delta-$function.
The important point which makes orbits $\mathcal{O}_{\mathcal{L}}$ different from the exceptional one $\mathcal{O}_o^o$ respectively substitution laws for translations different from the standard one acceptable for physical considerations is that deriving from the total action

$$A := \int d^4x d\mu_{\mathcal{L}}(m) L(\varphi^{8,1}(x)_m, \varphi^{3,3}(x)_m)$$

via the principles of stationary or least action the Euler-Lagrange field equation (Hamilton's Principle) or the conserved Noether currents due to some form invariance of L(Noether's Theorem) then it turns out that they are the same for all Lorentz orbits (apart from the dependence of the fields on the parameter m) with one and only one exception: The energy-momentum density tensor and hence the Hamiltonian inherits from the generalized substitution law for the translations in addition to the standard canonical one the internal symmetry breaking terms mentioned above[4]. The reason for the independence from the orbits is that the Lagrangian contains no derivatives of the fields with respect to the orbit parameters m. Needless to mention that in the step from the densities to the total charges (for instance the total Hamiltonian) the integration over the Lorentz orbit is to be included.
Which of the possible Lorentz orbits respectively substitution laws seems to be selected by nature for the quark-gluon world of hadrons ? - First of all after the translation of the up to this point classical sceme into general gauge quantum field theory (see section II) the exponential color factor in the substitution law for the translation operators bans all states with total color charge unequal to zero from the physical sectors of the Hilbert space [see Ref.[4] and section II below]; i.e. color confinement holds provided $\gamma \neq 0$ and the Lorentz orbit is not the exceptional one $\mathcal{O}_o^o$. The price to be paid is the non-isometry of the translation operators with respect to the Hilbert norm. -Secondly the flavor factor $R^d(exp.i/2(\wp(m)\cdot y))$ enforces the breaking of the $SU(3)-$flavor symmetry for all orbits except the exceptional one $\mathcal{O}_o^o$ and the special structure of $\wp(m)$ is the simplest one within a slightly larger class completely determined by the request that the $SU(2)\times U(1)$ flavor (isospin-hypercharge) subgroup remains the strict flavor symmetry observed in the world of hadrons together with the Goldstone theorem for the chiral limit case of vanishing quark masses which states that chiral $SU(3)-$flavor-symmetry is spontaneouly broken down to chiral $SU(2)\times U(1)-$symmetry with the $K-$mesons serving as Goldstone particles of (due to the flavor R^d-factor) finite mass $M(K)$. Once $\wp(m)$ is fixed in this way then in addition the Goldstone condition restricts the admissable Lorentz orbits to be one of the "mass hyperboloids" $\mathcal{O}_\delta^{\pm} = \{m \in I\!R_4 | m^2 = \delta^2 M(K)^2 \wedge m^0 \gtrless 0\}$ endowed with the invariant measure $d\mu_\delta^{\pm}(\vec{m}) = \theta(\pm m^o)\delta(m^2 - \delta^2 M(K)^2)d^4m$. [For details see Ref.[4]]. - However the relevance of any Lorentz orbit and the corresponding representation of the translation respectively Poincaré group (standard or not) for the description of hadrons lives and dies with the existence respectively non-existence of asymptotic incoming and outgoing collision states and an unitary scattering matrix in the color neutral sector.
In the present note we prove along the lines familiar from Wightman theories[9-16] on the

base of local commutativity, finite quark masses, the Haag-Ruelle-Hepp (or Herbst) condition on the existence of single (quasi-) particle excitations and an additional positivity property of their two-point functions that there exist for any Lorentz orbit $\mathcal{O}^{\pm}_{\delta}$ (and the exceptional one $\mathcal{O}^{o}_{o}$ as a special case) and every given color neutral single particle flavor multiplet of (pseudo-) scalar mesons or Fermi-Dirac baryons asymptotic Hilbert spaces of incomming and outgoing many particle collision states carrying the *standard* unitary representation of the Poincaré group. They are unitary equivalent to the familiar Fock space over the corresponding one-particle $\mathcal{L}^2-$spaces with the one dimensional vacuum subspace replaced by an infinite dimensional $\mathcal{L}^2(\mathcal{O}^{\pm}_{\delta}, d\mu^{\pm}_{\delta}(\vec{q}))-$space of translational invariant states (vacuum degeneracy) transforming under Lorentz transformations like a scalar particle of "(orbit-)mass" $|\delta| \cdot M(K)$.
The key to the proofs is the observation (section II) that the restriction of the translation operator to the color neutral subspace is isometric with respect to the Hilbert norm and therefore its closure unitary and that with the possible exception of the isospin and hypercharge zero meson channel there excist finite mass gaps above the vacuum if the quark masses are larger than zero.
The new problems to be solved compared to the Wightman theories emerge from the indefinite metric besides the Hilbert metric, the presence of the additional orbit variables m shifting the support in momentum space and the non-isometry or even unboundedness of the Lorentz boosts with respect to the Hilbert norm[17] even in the color neutral subspace. All three together request the reinvestigation and several modifications of the whole construction known in Wightman theorie from the cluster theorems (section III) through the construction of the excitation operators (section IV) to newly arising positivity problems for the metric inner products of the asymptotic limits (section V).
It should be remarked in passing that the flavor group $SU(3)$ has been chosen for simplicity only. All results can easily be generalized to higher order flavor groups $SU(n), n =?$.

II: Basic Assumptions and Conclusions

Basic and composite fields: For every m from the Lorentz orbit $\mathcal{O}^{\pm}_{\delta}, \delta \in I\!R\backslash\{0\}$ and f from the countably normed Schwarz space $S(I\!R_4)$ $\varphi^{n,d}_{\nu,c,a}(f)_m$ with $(n,d) \in \{(8,1),(3,3)\}, \nu \in \{0,1,2,3\}$ are closed linear operators with a common dense subdomain $D(m)$ in a Hilbert space $(\mathcal{H}(m), (\cdot|\cdot)_m, \eta_m)$ with η_m a bounded selfadjoint operator with bounded inverse generating the (non-degenerate) indefinit metric $< \cdot,\cdot >_m := (\cdot|\eta_m \cdot)_m$, which is nonnegative on some maximal proper subspace $H(m)$ und zero on a subset $\mathcal{H}_0(m).\eta_m, \eta_m^{-1}$, the basic fields $\varphi^{n,d}_{\nu,c,a}(f)$ and their (Hilbert) adjoint ones $\varphi^{n,d*}_{\nu,c,a}(f) := \varphi^{n,d}_{\nu,c,d}(\bar{f})^*$, ($\bar{f}$ complex conjugate of f) map $D(m)$ into itself and the matrix elements of all fields with states from $D(m)$ are continuous linear functionals on $S(I\!R_4)$.

Remark: Since η_m^{-1} is bounded it can be assumed without loss of generality that $\eta_m^2 = id_{\mathcal{H}(m)}$.[3] On the basis of the polar decomposition $\eta_m = sign(\eta_m) \cdot |\eta_m|$ with $|\eta_m| :=$

$\sqrt{\eta_m\eta_m^*} = \sqrt{\eta_m^2}$ and $sign.(\eta_m)$ an unitary selfadjoint operator and therefor $sign.(\eta_m)^2 = id_{\mathcal{H}(m)}$ one can introduce a new scalar product $(\cdot|\cdot)^{\sim}_m := (\cdot||\eta_m|\cdot)_m$ in the vector space $\mathcal{H}(m)$ which defines an equivalent topology to the original one and therefore the same Hilbert space and moreover we have: $<\cdot,\cdot>_m = (\cdot|\eta_m\cdot)_m = (\cdot|sign(\eta_m)\cdot)^{\sim}_m$.

The upper pair of indices (n,d) denotes the irreducible matrix representation R^d of the $SU(3)$ color-respectively flavor-group with which the field is linked together. Concerning the group $SU(3)$ and its representations we adopt the conventions and (apart from minor changes) the notations of de Swart[6,18,19] : $R^d := D^{(p,q)}$ for $p \geq q$, $R^{d^*} := D^{(p,q)}$ for $p \leq q$ and $2d = [(1+p)^2(1+q)+(1+p)(1+q)^2]$. The lower Latin indices are triples $a = (I, I^3, Y)$ counting rows/columns within R^d and running over the index set Γ_d given by the Okubo rule. I is interpreted to be the total "color/flavor isospin", I^3 its third component and Y the "color/flavor hypercharge". In accordance with the equivalence relation

$$(I, I^3, Y) \in \Gamma_d(\Gamma_{d^*}) \Longleftrightarrow (I, -I^3, -Y) \in \Gamma_{d^*}(\Gamma_d)$$

the negativ of an index triple $a = (I, I^3, Y)$ is by definition the triple $-a := (I, -I^3, -Y)$. Hence the "color/flavor charge" $Q(a) = Q(I, I^3, Y) = I^3 + \frac{1}{2}Y$ satisfies $Q(-a) = -Q(a)$. Finally there exists an important relation between the contragredient representation $\overline{R^d}(\overline{R^{d^*}})$ (by the complex conjugate matrices) and the representation $R^{d^*}(R^d)$. Adopting the notational convention $d^{**} = (d^*)^* = d$ it reads for all d and d^* :

$$R^{d^*}(u)^b_a = (-1)^{Q(b)-Q(a)}\overline{R^d(u)^{-b}_{-a}}, \tag{1}$$

with $a, b \in \Gamma_{d^*}, u \in SU(3)$. In case of the adjoint (octet) representation as well as all other ("diagonal") representations with $R^d = D^{(p,p)} = R^{d^*}$ this relation is an equation between different elements of the same matrix which is called *self-conjugateness.*

$\varphi^{3,3}_{\nu,...}(f)_m, \nu = 0,1,2,3$ are considered to be components of a Fermi-Dirac-field and called *quark* fields whereas $\varphi^{8,1}_{\nu,...}(f)_m, \nu = 0,...,3$ are the components of a vector field and called *gluon* fields. Chained to the (Hilbert-)adjoint operator $\varphi^{n,d*}_{...}(f)_m$ is the $\eta-adjoint$ operator $\varphi^{n,d+}_{...}(f)_m := \varphi^{n,d}_{...}(\bar{f})^+_m$. It is straight forward to show:

$$D_{\varphi^{n,d+}_{...}}(m) = \eta_m^{-1} D_{\varphi^{n,d*}_{...}}(m); \quad \varphi^{n,d+}_{\nu,a,b}(f)_m = \eta_m^{-1}\varphi^{n,d*}_{\nu,a,b}(f)_m\eta_m \tag{2}$$

Since $\Psi \in \eta_m D_{\varphi^{n,d}_{...}}(m)$ implies $\Psi \in D_{\varphi^{n,d+*}}(m)$ and $\eta D(m)$ is dense in $\mathcal{H}(m)$ it follows that $\varphi^{n,d+}_{\nu,a,b}(f)_m$ is closable and its closure will be denoted by the same symbol. The $*-$adjoint field being the quantum mechanical equivalent to the complex conjugate field of the classical Lagrangian level must be chained up with the complex conjugate representation (in comparison to the original field). Hence the relation (1) seems to suggest itself to introduce instead of the $*-$adjoint operators $\varphi^{n,d*}_{...}(f)_m, (n,d) \in \{(8,1),(3,3)\}$ new ones $\varphi^{n^*,d^*}_{...}(f)_m$ being chained up with the representation $R^{n^*} \times R^{d^*}$. This is achieved by:

$$\varphi^{n^*,d^*}_{\nu,a,b}(f)_m := (-1)^{-Q(a)-Q(b)+1/3(t(n)+t(d))}\varphi^{n,d,*}_{\rho,-a,-b}(f)_m\gamma(n,d)^\rho_\nu, \tag{3}$$

$$\gamma(n,d)^\rho_\nu := \begin{cases} (\gamma^0)^\rho_\nu & \text{for } (n,d) = (3,3) \\ g^\rho_\nu & \text{for } (n,d) = (8,1), \end{cases}$$

where γ^0 is the (hermitian) Dirac-matrix and g the Minkowski tensor ($g^\rho_\nu = g^{\rho\nu} = g_{\rho\nu}$). The triality

$$t(n) := \begin{cases} 1 \text{ for } n = 3 \\ 0 \text{ for } n = 8,1 \quad (t(p,q) = (p-q)\text{mod. } 3) \\ -1 \text{ for } n = 3^* \end{cases}$$

is put in to asure the Condon-Shortley phase convention in Clebsch-Gordon gymnastics. The (Fermi-Dirac-) fields $\varphi^{3^*,3^*}_{\dots}(f)_m$ are called *anti-quark-fields*. They are related to the Dirac-adjoint operators $\bar{\varphi}^{3,3}_{\mu,a,b}(f)_m := \varphi^{3,3*}_{\rho,a,b}(f)_m(\gamma^0)^\rho_\mu = (-1)^{-Q(a)-Q(b)-2/3}\varphi^{3^*,3^*}_{\mu,-a,-b}(f)_m$. Corresponding to the self-conjugateness of the diagonal representations $R^d = R^{d^*}$ for $d \in \{1,8\}$ the gluon fields are assumed to form a self-conjugate octet $\varphi^{8,1}_{\dots}(f)_m = \varphi^{8^*,1^*}_{\dots}(f)_m$, that is:

$$\varphi^{8,1*}_{\mu,c}(f)_m = (-1)^{Q(c)} g^\nu_\mu \varphi^{8,1}_{\nu,-c}(f)_m = \eta^{-1}_m \varphi^{8,1+}_{\mu,c}(f)_m \eta_m \tag{4}$$

The occurrence of the Minkowski tensor in the equations (3) and (4) seems artificially flavored. However without it the representation of the Lorentz group below would be $*-$isometric and hence unitarily extendable. For the free electromagnetic field in the Gupta-Bleuler gauge this is known to be not the case[17,20] and the relation between the potential and its $*-$adjoint is just the same as equation (4) (with g!).

It should be pointed out already here that in ervery color confining representation of the translation group (i.e. $\gamma \neq 0$ below) the gluon *octet* like any other translational covariant *color octet* cannot be $\eta-$self-conjugate (this means self-conjugate with respect to the +-operation). In case η commutes with the subquartet of the uncharged ($Q_{\mathcal{C}}(c) = 0; c \in \{(0,0,0),(1,0,0),(1/2,1/2,-1),(1/2,-1/2,1)\}$) members of a color octet, then this subquartet is both $*-$self-conjugate and $\eta-$self-conjugate. However η can never commute with the remaining charged ($Q_{\mathcal{C}}(c) \neq 0$) members unless there is no color confinement ($\gamma = 0$). On the other hand all flavor octets related to particle observables (color singlets !) like conserved flavor currents etc. have to be $\eta-$self-conjugate: $j^{1,8+}_{\mu,a}(f)_m = (-1)^{Q(a)} j^{1,8}_{\mu,-a}(f)_m$, and therefore commute with η_m in case of being also $*-$self-conjugate (without g). [Unfortunately in the self-conjugateness relation (47) in Ref.[4] this has been confused. Except for the flavor current the η_m has to be canceled in the equation (47) and also the two following ones since the Dirac adjoint field $\tilde{\varphi}^{3,3}_{\dots}(f)_m$ has to be linked to the $*-$adjoint and not the +-adjoint of the quark field $\varphi^{3,3}_{\dots}(f)_m$. Furthermore a g has to be added in the relation (47) for the gluon field and color current.]

In correspondence to the Noether currents of color respectively flavor symmetry of the classical Lagrangian level the linkage of fields and internal symmetries will be established via the existence of a $+-$self-conjugate octet $j^{1,8}_{\mu,a}(f)_m$ of conserved flavor currents and a $*-$self-conjugate color octet $j^{8,1}_{\mu,a}(f)_m$ of conserved currents which are either constructable from the basic fields or added to them by hand. They share all domain properties of the basic fields.

Einstein Causality (Locality): For any pair $f, g \in S(I\!R_4)$ with space-like separated support, i.e. $f(x)g(y) = 0$ whenever $(x-y)^2 \geq 0$, and all $\Psi(m) \in D(m)$ the normal commutation relations hold:

$$[\varphi^{n,d}_{...}(f)_m, \varphi^{n',d'}_{...}(g)^{(+)}_m]_\pm \Psi(m) = \mathcal{O}, \quad [j^{...}_{...}(f)_m \varphi^{...}_{...}(g)^{(+)}_m]_- \Psi(m) = \mathcal{O}$$

$$[j^{...}_{...}(f)_m, j^{...}_{...}(g)^{(\dagger)}_m]_- \Psi(m) = \mathcal{O}, \quad \dagger \in \{*, +\}$$

with the anticommutator$[,]_+$ if and only if both (n,d) and (n',d') are from $\{(3,3),(3^*,3^*)\}$.

Vacuum state: In the subset $H(m) \setminus H_0(m)$ with $H_0(m) := H(m) \cap \mathcal{H}_0(m)$ exists a normalized eigenvektor $\Psi_0(m)$ of η_m with $< \Psi_0(m), \Psi_0(m) >_m = 1$ which is cyclic for the polynomial $*-$algebra $\pi(\varphi_m, (j_m))$ spanned by the basic field operators $\varphi^{n,d}_{...}(f)_m, (n,d) \in \{(3,3),(3^*,3^*),(81)\}, f \in S(I\!R_4)$ (and conserved currents). Moreover
$\pi(\varphi_m, (j_m))\Psi_0(m) \cap H(m)$ is dense in $H(m)$.

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Since $H(m)$ is not completely fixed by positive semi-definiteness of the metric and maximality with respect to inclusion[21], $H(m)$ in the following denotes such a closed subspace of $\mathcal{H}(m)$ with $\Psi_0(m) \in H(m)$ and $\overline{\pi(\varphi_m, (j_m))\Psi_0(m) \cap H(m)} = H(m)$.

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Composite fields: In exactly the same way as in Wightman theories composite field operators (completed tensor products)

$$\varphi^{[n]_L[d]_L}_{[\nu]_L[a]_L[b]_L}(f)_m := \left[\bar{\otimes}^L_{k=1} \phi^{n_k,d_k}_{\nu_k,a_k,b_k}\right](f)_m, f \in S(I\!R_{4L})$$

with $\phi^{...}_{...}$ denoting one of the basic fields or currents specified above are obtained on $D(m)$ by means of the strong limits in $(\mathcal{H}(m), (\cdot|\cdot)_m)$

$$\varphi^{[n]_L[d]_L}_{[\nu]_L[a]_L[b]_L}(f)_m \Psi(m) = s - \lim_{j \to \infty} \Pi^L_{k=1} \phi^{n_k,d_k}_{\nu_k,a_k,b_k}(f^j_k)_m \Psi(m)$$

for an arbitrary sequence $(\bigotimes^L_{k=1} f^j_k)_{j \in I\!N}$ converging in $S(I\!R_{4L})$ to f. The dense linear subspace defined by the linear hull $\mathcal{D}_{QL}(m) := L.H.\{[\bar{\otimes}^L_{k=1} \phi^{n_k,d_k}_{\nu_k,a_k,b_k}](f)_m \Psi_0(m) | f \in S(I\!R_{4L});$
$(n_k, d_k) \in \{(3,3),(3^*,3^*),(8,1),(1,8)\}; a_k \in \Gamma_{n_k}, b_k \in \Gamma_{d_k}; \nu_k = 0,...,3; L \in I\!N\}$
is called the subspace of *quasi-local states*; and the subspace $\mathcal{D}_{SL}(m) \subset \mathcal{D}_{QL}(m)$ obtained by restricting the testfunctions from $S(I\!R_{4L})$ to those with compact support is called the subspace of *strictly local states*. Then $\mathcal{D}_{QL}(m)$ belongs to the domain of every (closed!) basic field or current and $\overline{\eta_m \mathcal{D}_{SL}(m)} = \overline{\eta_m \mathcal{D}_{QL}(m)} = \mathcal{H}(m)$. *Additionally η_m is assumed to map $\mathcal{D}_{QL}(m)$ into itself.*
If $\mathcal{J}$ denotes the map $f(x_1,...,x_L) \to (\mathcal{J}f)(x_1,...,x_L) := \overline{f(x_L,...,x_1)}$ of $S(I\!R_{4L})$ onto itself and $[n]_{\mathcal{J}L}$ the inversion of the order of indices in $[n]_L$, that is the transition
$[n]_L = n_1,...,n_L \to [n]_{\mathcal{J}L} = n_L,...,n_1$ then the equations (2) and (3) imply for the composite fields:

$$\begin{array}{l}\varphi^{[n]_L[d]_L*}_{[\mu_L][c]_L[a]_L}(f)_m=\varphi^{[n]_{\mathcal{J}L}[d]_{\mathcal{J}L}}_{[\mu]_{\mathcal{J}L}[c]_{\mathcal{J}L}[a]_{\mathcal{J}L}}(\mathcal{J}f)^*_m\\ \varphi^{[n]_{\mathcal{J}L}[d]_{\mathcal{J}L}}_{[\mu]_{\mathcal{J}L}[c]_{\mathcal{J}L}[a]_{\mathcal{J}L}}(\mathcal{J}f)^*_m=\varphi^{[n^*]_L[d^*]_L}_{[\rho]_L[-c]_L[-a]_L}(f)_m\times\\ \times\Pi^L_{j=1}\gamma(n_j,d_j)^{\rho_j}_{\mu_j}(-1)^{-Q(c_j)-Q(a_j)-1/3(t(n_j)+t(d_j))}\end{array}\tag{5}$$

Observables or composite field operators connected with them have to be $\eta-$symmetric respectively $\eta-$symmetric linear combinations of $\eta-$self-conjugate ones. Such operators are obtained on $D(m)$ via strong limits in the sense above:

$$\begin{array}{l}\varphi^{[n^*]_K[n]_K,[d^*]_K[d]_K}_{[\mu]_K[\nu]_K,[c]_K[e]_K,[a]_K[b]_K}(f)_m\Psi=\Pi^K_{i=1}(-1)^{-Q(c_i)-Q(a_i)+1/3(t(n_i)+t(d_i))}\times\\ \times[\eta_m\varphi^{[n]_{\mathcal{J}K}[d]_{\mathcal{J}K}*}_{[\mu]_{\mathcal{J}K}[-c]_{\mathcal{J}K}[-a]_{\mathcal{J}K}}\eta_m\bar{\otimes}\varphi^{[n]_K[d]_K}_{[\nu]_K[e]_K[b]_K}](f)_m\Psi:=\Pi^K_{i=1}(-1)^{-Q(c_i)-Q(a_i)+1/3(t(n_i)+t(d_i))}\\ \times s-\lim_{j\to\infty}[\eta_m\varphi^{[n]_{\mathcal{J}K}[d]_{\mathcal{J}K}*}_{[\mu]_{\mathcal{J}K}[-c]_{\mathcal{J}K}[-a]_{\mathcal{J}K}}(f^j_1)_m\eta_m\varphi^{[n]_K[d]_K}_{[\nu]_K[e]_K[b]_K}](f^j_2)_m\Psi\end{array}\tag{6}$$

Translation invariance: On $\mathcal{D}_{QL}(m)$ a representation of the translation group $(I\!R_4,+)$ mapping the subspace $\mathcal{D}^H_{QL}(m):=H(m)\cap\mathcal{D}_{QL}(m)$ onto itself is assumed to exist and obtained via linear extension of the operations $\mathcal{T}_m(y),y\in I\!R_4$ defined on all states created by products of the basic and composite fields from $\Psi_0(m)$:

$$\mathcal{T}_m(y)\Psi_0(m):=\Psi_0(m)\tag{7}$$

$$\mathcal{T}_m(y)\varphi^{[n]_L[d]_L}_{[\nu]_L[c]_L[a]_L}(f)_m\Psi_0(m):=(\Pi^L_{k=1}\exp[Q(c_k)(\gamma m\cdot y)])\times$$

$$\times(\Pi^L_{j=1}R^{d_j}(\exp[i/2(\wp(m)y)])^{b_j}_{a_j})\varphi^{[n]_L[d_L]}_{[\nu]_L[c]_L[b]_L}(f_y)_m\Psi_0(m)\tag{8}$$

with $\gamma\in I\!R,f_y(x_1,...,x_L):=f(x_1-y,...,x_L-y)$ and the flavor matrix for $\delta\in I\!R\setminus\{0\}$, given by:

$$\wp(m):=[\sigma_{(0,0,0)}+1/\sqrt{3}\tau_{(0,0,0)}(2/\delta-5/3)]m\tag{9}$$

Here $\tau_a,a\in\Gamma_8$ denote the traceless $(3,3)-$matrices of the self-conjugate fundamental representation of the (complex)Lie-algebra $su_{\mathbb{C}}(3)$ (generating the adjoint representation R^8 and) satisfying the relations [6,18,19] :

$$\begin{array}{l}[\tau_a,\tau_b]_-=-2\sqrt{3}\binom{888_A}{abc}\tau_c,\quad tr.(\tau_a\cdot\tau_b^+)=2\delta_{ab},\tau_a^+=(-1)^{Q(a)}\tau_{-a}\\ \sigma_a:=2\sqrt{5}/3\binom{888_S}{bca}\tau_b\tau_c\end{array}\tag{10}$$

Due to the color factors $\Pi^L_{k=1}\exp[Q(c_k)(\gamma m\cdot y)]$ in the equation (8) this representation is certainly *not* isometric with respect to the Hilbert scalar product $(\cdot|\cdot)_m$ [Ref.[5], Theorem 4.2] unless $\gamma=0$ and by many authors[1−3,17,21]] claimed to be even unbounded also in the

standard case (which however can only be true if $\Psi_0(m)$ is not an eigenvektor of the metric operator !) However it must be assumed to be isometric with respect to the indefinite metric $<\cdot,\cdot>_m$ for μ-almost all m in order that it lifts to a unitary representation of the translation group in the physical Hilbert space, which is unitary equivalent to the direct integral over the Lorentz-orbit of the factor spaces $\overline{(N(m)/(N(m)\cap\mathcal{H}_0(m)),<\cdot,\cdot>_m)}$, where $N(m)$ is a color neutral maximal η_m-semi-normed subspace of $\mathcal{H}(m)$ which can but must not be equal to $H(m)$ (see sections V-VI).
η-isometry in addition implies that the dense subspace $\eta_m\mathcal{D}_{QL}(m)$ is in the domain of the Hilbert adjoint operator $\mathcal{T}_m(y)^*$. Therefore $\mathcal{T}_m(y)$ is closable and its closure $\overline{\mathcal{T}_m(y)}=\mathcal{T}_m(y)^{**}$ again η-isometric. Finally $\eta_m\mathcal{D}_{QL}(m)$ is a core for $\mathcal{T}_m(y)^*$ and $\mathcal{T}_m(y)^*$ $\eta-$isometric. An immediate consequence of tranlational invariance and the invariane of $\Psi_0(m)$ under η_m is the confinement of all colored states provided γ is unequal to zero, [Ref.[4], section IV].

Theorem (Color-Confinement): a) The linear hull of all states $\Psi^{n_1,\dots,n_L}_{c_1\dots c_L}(m)\in H(m)$ with total color charge $Q_{\mathcal{C}}(L)=\sum\limits_{j=1}^{L}Q(c_j)\neq 0$ is contained in $H_0(m)=H(m)\cap\mathcal{H}_0(m)$.

b) For all $f\in S(I\!\!R_{4L})$ and $h\in S(I\!\!R_{4K})$

$$<\varphi^{[n]_L[d]_L}_{\dots[c]_L[a]_L}(f)_m\Psi_0(m),\varphi^{[q]_K[r]_K}_{\dots[e]_K[b]_K}(h)_m\Psi_0(m)>=0$$

unless $\sum\limits_{j=1}^{L}Q(c_j)=-\sum\limits_{k=1}^{K}Q(e_k)$ (Hence all these colored states are in $\mathcal{H}_0(m)$).

$\div$

Let $\mathcal{D}_0(m)=L.H.\{\varphi^{[n_L]\dots}_{\dots[c]_L\dots}(f)_m\Psi_0(m)\in\mathcal{D}_{QL}(m)|f_j\in S(I\!\!R_4),Q_{\mathcal{C}}(L)=\sum\limits_{j=1}^{L}Q(c_j)=0,$ $L\in I\!\!N_0\}$ be the linear hull of all quasi-local states with total color charge zero. An immediate consequence of the color confinement theorem is that the (Hilbert-) closure $\overline{\mathcal{D}_0(m)}$ is an invariant subspace for η_m if and only if $\mathcal{D}_{QL}(m)\setminus\mathcal{D}_0(m)\subseteq\mathcal{D}_0(m)^{\perp}=\overline{\mathcal{D}_0(m)}^{\perp}$ with $\perp$ denoting the orthogonal complement in $(\mathcal{H}(m),(\cdot|\cdot)_m)$. Obviously a sufficient condition and in case there do not exist states with discrete mass zero also a necessary one[4] is that the equations:

$$U_m(t)\phi^{[n]_L[d]_L}_{\dots[c]_L[a]_L}(f)_m\Psi_0(m):=\exp[it\sum_{j=1}^{L}Q(c_j)]\phi^{[n]_L[d]_L}_{\dots[c]_L[a]_L}(f)_m\Psi_0(m)$$

define a $*-$isometric representation of the unitary group $U(1)$ on $\mathcal{D}_{QL}(m)$ and hence a unitary one in $(\mathcal{H}(m),(\cdot|\cdot)_m)$. This in turn is equivalent to the condition, that the total charge constructed in the standard way[22,23] from the $*-$symmetric color current $j^{8,1}_{0,(1,0,0)}(f)_m+1/2j^{8,1}_{0,(0,0,0)}(f)_m$ is a bona fide linear operator on $\mathcal{D}_{SL}(m)$ satisfying the commutation relations below. Hence presuming $\overline{\mathcal{D}_0(m)}$ to be an invariant subspace for η_m, then the restriction $\mathcal{T}_m(y)\lceil\mathcal{D}_0(m)$ satisfies the preconditions of Theorem 4.2 in [VII] which states that $\mathcal{T}_m(y)\lceil\mathcal{D}_0(m)$ is isometric with respect to the Hilbert scalar product $(\cdot|\cdot)_m$ for every $y\in(I\!\!R_4,+)$ and therefore has an unique *unitary extension* $\mathcal{T}^0_m(y)$ onto the closure $\overline{\mathcal{D}_0(m)}$. Being a unitary operator on the Hilbert subspace $(\overline{\mathcal{D}_0(m)}, (\cdot|\cdot)_m)$ of $(\mathcal{H}(m), (\cdot|\cdot)_m)$ it allows a spectral resolution $\mathcal{T}^0_m(y) = \int_{I\!R_4} e^{iqy} dE_m(q)$ with E_m a bounded measure. Now the demand that the physical Hilbert space contains only states with non negative energy and real masses suggests the following support properties of the spectral measures E_m:

Spectrum Condition: For μ-almost all m the support of E_m is contained in the closed forward cone $\overline{V^0_+} = \{q \in I\!R_4 | q_\nu q^\nu \geq 0 \wedge q^0 \geq 0\}$.

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The assumption of a Lorentz covariant shape of the boundary of the spectral support without Lorentz transformations in $(\overline{\mathcal{D}_0(m)}, (\cdot|\cdot)_m)$ being placed to our disposal is supported by the fundamental work of Borchers and Buchholz [Ref.[24] and Refs. therein] on the intimate connection between Einstein causality and spectral support (at least in Wightman theories).
The spectrum condition implies for all $\Phi(m) \in \overline{\mathcal{D}_0(m)}, \Psi(m) \in \mathcal{H}(m), h \in S(I\!R_4)$ the Fourier transform $\tilde{h}(q) = (2\pi)^{-3/2} \int d^4y\ e^{-iq\cdot y} h(y)$ of which vanishes in a neighbourhood of the closed backward cone $\overline{V^0_-} = \{q \in I\!R_4 | q_\nu q^\nu \geq 0, q^0 \leq 0\}$ and μ-almost all m :

$$\int_{I\!R_4} d^4y h(y) (\Psi(m)|\mathcal{T}^0_m(y)\Phi(m))_m = (2\pi)^{3/2} \int_{I\!R_4} \tilde{h}(-q) d(\Psi(m)|E_m(q)\Phi(m))_m = 0 \quad (11)$$

The Haag-Ruelle-construction of asymptotic scattering states[9,10,12–14,16] requests a more refined spectrum condition showing a mass gap above the vacuum $\Psi_0(m)$ at least in those (uncolored) channels of flavored quantum-chromo-dynamics carrying the quantum numbers of known strongly interacting particles (either stable and hence an isolated point or of sufficiently small width in order to be approximated by a discrete point in the continuous mass spectrum). Since the massless gluon fields are flavor scalars, the $SU(2) \times U(1)$ flavor symmetry of the known hadron particle spectrum suggests that in the meson sector the spectral support is governed by the mass $M(\pi)$ of the stable $\pi-$meson and with the possible exception of the $(I, I_3, Y) = (0, 0, 0)$ channel contained in the set

$$\{0\} \cup \{q \in I\!R_4 | q^2 = M(\pi)^2, q^0 > 0\} \cup \overline{V_+^{M(\pi)+\epsilon}}, \quad \epsilon \geq 0 \quad (12)$$

whereas in the baryon sector governed by the mass $M(N)$ of the stable nucleon and contained in

$$\{0\} \cup \{q \in I\!R_4 | q^2 = M(N)^2, q^0 > 0\} \cup \overline{V_+^{M(N)+\epsilon}}, \quad \epsilon \geq 0 \quad (13)$$

with ϵ depending on the channel.
In other words (apart from the $I = Y = 0$ meson sector) *in all channels with the quantum numbers of known hadrons there is a finite mass gap beween the vacuum* $\Psi_0(m)$ *and the rest of the spectrum* consisting of a continuum and discrete "one particle" hyperboloids either isolated from ($\epsilon = M(\pi)$) or contained in the continuous spectrum. Furthermore

there exists *no* accumulation point of discrete "one particle" hyperboloids.

Color- and Flavor-Symmetry: Let f_ϵ and f_R be arbitrary functions from $S(\mathbb{R})$ respectively $S(\mathbb{R}_3)$ with the properties:

$$supp.f_\epsilon \subseteq [-\epsilon,\epsilon] \wedge \int_{\mathbb{R}} dx f_\epsilon(x) = 1$$

$$f_R(x) = \begin{cases} 1 & \text{for} |x| < R \\ f_R(|x|) & \text{for} R \leq |x| < R + \triangle R \\ 0 & \text{for} |x| \geq R + \triangle R \end{cases}$$

Then due to Einstein causality and current conservation for every $f \in S(\mathbb{R}_4)$ with compact support and $\epsilon > 0$ there exists a $R_0(\epsilon, f) > 0$ such that the commutators $[j^{\cdot\cdot\cdot}_{0,a}(f_\epsilon, f_R)_m, \varphi^{\cdot\cdot\cdot}_{\cdots}(f)_m]_-$ become independent from f_ϵ and f_R [swi] for all $R > R_0(\epsilon, f)$.
If

$$j^{\cdot\cdot\cdot}_{0,a\pm}(f_\epsilon, f_R)_m := 1/2(j^{\cdot\cdot\cdot}_{0,-a}(f_\epsilon, f_R) \pm j^{\cdot\cdot\cdot}_{0,a}(f_\epsilon, f_R)) \begin{cases} i^{1/2\pm 1/2} & \text{if } Q(a) = 1 \\ i^{1/2\mp 1/2} & \text{if } Q(a) = 0 \end{cases}$$

are the $*/+$-symmetric combinations of the conserved currents and $\tau_{a\pm}$ the same hermitian combinations of the traceless self-conjugate $\tau-$matrices then the following commutation relations of the basic fields and currents for all $f \in S(\mathbb{R}_4)$ with compact support are a consistent and careful way of expressing the content of the equal time commutation relations between the Noether currents and the basic fields of the $SU(3)_{\mathcal{C}} \times SU(3)_{\mathcal{F}}$ symmetric Lagrangian of chromo-flavor-dynamics which follow on this formal level by blindly applying equal time canonical commutation relations.

$$[j^{8,1}_{0,c\pm}(f_\epsilon, f_R)_m, \varphi^{n,d}_{\mu,a,e}(f)_m]_{-|R>R_0(\epsilon,f)} = i\frac{d}{dt}R^n(e^{-i/2t\tau_{c\pm}})^b_a \varphi^{n,d}_{\mu,b,e}(f)_{m|t=0}$$

$(n,d) \in \{(3,3),(3^*,3^*),(8,1)\}$ and the same ones with $\varphi^{n,d}_{\cdots}$ replaced by $j^{n,d}_{\cdots}$ for $(n,d) = (8,1)$.

$$[j^{1,8}_{0,c\pm}(f_\epsilon, f_R)_m, \varphi^{n,d}_{\mu,a,e}(f)_m]_{-|R>R_0(\epsilon,f)} = i\frac{d}{dt}R^d(e^{-i/2t\tau_{c\pm}})^b_e \varphi^{n,d}_{\mu,a,b}(f)_{m|t=0},$$

$(n,d) \in \{(3,3),(3^*,3^*)\}$ and the same ones for $(n,d) = (1,8)$ with φ replaced by j.
$[j^{8,1}_{0,c\pm}(f_\epsilon, f_R)_m, j^{1,8}_{\mu,a}(f)_m]_- = [j^{1,8}_{0,c\pm}(f_\epsilon, f_R)_m, j^{8,1}_{\mu,a}(f)_m]_-. = [j^{1,8}_{0,c\pm}(f_\epsilon, f_R)_m, \varphi^{8,1}_{\mu,a}(f)_m]_- = 0$, for all $R > R_0(\epsilon, f)$
On the basis of these commutation relations it has been proved in [Ref.[4], section VI] by means of generalized Goldstone theorems:

Theorem (Goldstone): (i) : There exists an $*$-isometric representation $I_m : U(1)^{\mathcal{Q}}_{\mathcal{C}} \times SU(2)^{\mathcal{U}}_{\mathcal{C}} \times SU(3)_{\mathcal{F}} \rightarrow Aut_{\mathcal{D}_{QL(m)}}$ by closable operators with

$$I_m(u,v)\Psi_0(m) = \Psi_0(m)$$
$$I_m(u,v)\varphi^{[n]_L[d]_L}_{[\nu]_L[c]_L[e]_L}(f)_m\Psi_0(m) = \tag{14}$$
$$(\Pi_{k=1}^L R^{n_k}(u)^{a_k}_{c_k} R^{d_k}(v)^{b_k}_{e_k})\varphi^{[n]_L[d]_L}_{[\nu]_L[a]_L[b]_L}(f)_m\Psi_0(m),$$

where $U(1)^{\mathcal{Q}}_{\mathcal{C}} \times SU(2)^{\mathcal{U}}_{\mathcal{C}} := \{e^{i/2t(\tau_{(1,0,0)}+1/\sqrt{3}\tau_{(0,0,0)})}\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix}|t \in I\!R \wedge \alpha \in SU(2)\}$ is the "U-spin" subgroup of the color group $SU(3)_{\mathcal{C}}$, provided there is *no discrete* state of mass zero in $\mathcal{D}_{QL}(m)$ createt by the color currents $j^{8,1}_{0,a}$ with $Q(a) = 0$ from the vacuum.

(ii) :The operator $I_m(u,v)$ commutes with the translations $\mathcal{T}_m(y)$ if and only if $v \in SU(2)^{\mathcal{I}}_{\mathcal{F}} \times U(1)^{\mathcal{Y}}_{\mathcal{F}} := \{e^{i/2t\tau_{(0,0,0)}}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}|t \in I\!R \wedge \alpha \in SU(2)\}$ or $v \in U(1)^{\mathcal{Q}}_{\mathcal{F}} \times SU(2)^{\mathcal{U}}_{\mathcal{F}}$ the "$U-$spin" subgroup of the flavor group $SU(3)_{\mathcal{F}}$.
(iii) : The restrictions $I_m(e,v)$ to the flavor subgroup $SU(3)_{\mathcal{F}}$ are $\eta-$isometric and commute with η_m.

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Remarks: 1.) The representation I_m has a unique unitary extension to $\mathcal{H}(m)$ and $\overline{\mathcal{D}_0(m)}$ as an invariant subspace.

2.) The restriction in (i) to the U-spin subgroup of $SU(3)_{\mathcal{C}}$ comes from the lack of spectral support in the colored sectors and therefore no Goldstone theorem could be proved. In case $\gamma = 0$ (no color-confinement) the translation operators $\mathcal{T}_m(y)$ have a unique unitary extension to $\mathcal{H}(m)$ and the theorem is true for the full color group $SU(3)_{\mathcal{C}}$ if the spectrum condition is assumed to hold also in the unphysical sectors of $\mathcal{H}(m)$. *Even in the general case $\gamma \neq 0$ there can exist a $*$-isometric representation of the color group $SU(3)_{\mathcal{C}}$ on $\mathcal{D}_{QL}(m)$ satisfying the covariance equations* (14) *for all $u \in SU(3)_{\mathcal{C}}$ but commuting with $\mathcal{T}_m(y)$ if and only if u is from the "$U-spin$" subgroup* $U(1)^{\mathcal{Q}}_{\mathcal{C}} \otimes SU(2)^{\mathcal{U}}_{\mathcal{C}}$, if the gluon fields are considered to be massless; i.e. they contribute only to the continuous spectrum of E_m.

3.) Due to the $\eta-$self-conjugateness but possible absence of $*-$self-conjugateness for the flavor currents the Flavor-Symmetry-Theorem of Ref.[4] establishes in the first instance only the existence of a $\eta-$isometric representation $I_m(e,v)$ of the flavor subgroup $SU(3)_{\mathcal{F}}$ with the transformation law (14). But then Theorem 4.2 of Ref.[5] implies its $*-$isometry in addition. On the other side due to the $*-$self-conjugateness of the color currents the Generalized Color-Goldstone-Theorem of Ref.[4] implies the existence of a $*-$isometric representation $I_m(u,e)$ of the color subgroup $U(1)^{\mathcal{Q}}_{\mathcal{C}} \times SU(2)^{\mathcal{U}}_{\mathcal{C}}$ and for simplicity it is assumed to commute with η_m.

4.) In Ref.[4] the detailed proof of part (i) has only be given for the subspace of strictly local states $\mathcal{D}_{SL}(m)$. However there is no problem to extend it to $\mathcal{D}_{QL}(m)$ in the same way as in case of Wightman theories via sesquilinear form limits [Ref.[25], chapt. IV].

Lorentz- and Poincaré-Symmetry: For every Lorentz-transformation $\Lambda(\alpha), \alpha \in SL(2,\mathbb{C})$ a linear η-isometric operator $\mathcal{L}(\alpha)_m$ from $\mathcal{D}_{QL}(m)$ onto $\mathcal{D}_{QL}(\Lambda(\alpha)m)$ is assumed to exist and obtained via linear extension from the covariance equations (including the currents):

$$\begin{aligned}
&\mathcal{L}(\alpha)_m \Psi_0(m) = \Psi_0(\Lambda(\alpha)m) \\
&\mathcal{L}(\alpha)_m \varphi^{[n]_L [d]_L}_{[\nu]_L,\ldots}(f)_m \Psi_0(m) = \\
&(\Pi_{j=1}^{L} S^{(n_j,d_j)}(\alpha)^{\rho_j}_{\nu_j}) \varphi^{[n_L][d]_L}_{[\rho]_L,\ldots}(f_{(\Lambda(\alpha),0)})_{\Lambda(\alpha)m} \Psi_0(\Lambda(\alpha)m)
\end{aligned} \tag{15}$$

with $f_{(\Lambda(\alpha),y)}(x_1,..,x_L) := f(\Lambda(\alpha)^{-1}(x_1-y),...,\Lambda(\alpha)^{-1}(x_L-y))$ and

$$S^{(n,d)}(\alpha)^{\mu}_{\nu} := \begin{cases} S^{1/2}(\alpha^{-1})_{\nu}{}^{\mu} & \text{for } (n,d)=(3,3) \\ \Lambda(\alpha)^{\mu}{}_{\nu} & \text{for } (n,d) \in \{(8,1),(1,8)\} \\ S^{1/2}(\alpha)^{\mu}{}_{\nu} & \text{for } (n,d)=(3^*,3^*) \end{cases}$$

the standard (4,4)-matrices by which Dirac-Fermi-fields respectively (axial-) vector fields transform under Lorentz-transformations. They obviously obey the composition laws:

$$\mathcal{L}(\alpha_2\alpha_1)_m = \mathcal{L}(\alpha_2)_{\Lambda(\alpha_1)m}\mathcal{L}(\alpha_1)_m, \quad \mathcal{L}(\alpha^{-1})_{\Lambda(\alpha)m}\mathcal{L}(\alpha)_m = id_{\mathcal{D}_{QL}(m)} \tag{16}$$

η-isometry means for all $\Psi(m), \Phi(m) \in \mathcal{D}_{QL}(m)$

$$< \mathcal{L}(\alpha)_m\Psi(m), \mathcal{L}(\alpha)_m\Phi(m) >_{\Lambda(\alpha)m} = < \Psi(m), \Phi(m) >_m$$

and it implies that $\eta_{\Lambda(\alpha)m}\mathcal{D}_{QL}(\Lambda(\alpha)m)$ is contained in the domain of the Hilbert adjoint operator $\mathcal{L}(\alpha)^*_m$ and for all $\Phi(m)$ from $\mathcal{D}_{QL}(m)$ we have

$$\mathcal{L}(\alpha)^*_m \eta_{\Lambda(\alpha)m}\mathcal{L}(\alpha)_m\Phi(m) = \eta_m\Phi(m).$$

Hence $\mathcal{L}(\alpha)_m$is closable and its closure will again be denoted by the same symbol. The closure and $\mathcal{L}(\alpha)^*_m$ are again $\eta-$isometric and $\eta_{\Lambda(\alpha)m}\mathcal{D}_{QL}(\Lambda(\alpha)m)$ is a core for $\mathcal{L}(\alpha)^*_m$. Calculating the action of the $*$-adjoint Lorentz operators $\mathcal{L}(\alpha)^*_m$ it turns out that *the rotations $\mathcal{L}(u)_m, u \in SU(2)$ are $*-$ isometric on $\mathcal{D}_{QL}(m)$ and hence their closure unitary. But the Lorentz boosts are not.* Finally $\mathcal{T}_m(y)$ and $\mathcal{L}(\alpha)_m$ satisfy on $\mathcal{D}_{QL}(m)$ the commutation relations

$$\mathcal{T}_{\Lambda(\alpha)m}(y)\mathcal{L}(\alpha)_m\Psi(m) = \mathcal{L}(\alpha)_m\mathcal{T}_m(\Lambda(\alpha^{-1})y)\Psi(m)$$

and therefore

$$\mathcal{U}(\alpha,y)_m : \mathcal{D}_{QL}(m) \to \mathcal{D}_{QL}(\Lambda(\alpha)m), \Psi(m) \to \mathcal{U}(\alpha,y)_m\Psi(m) := \mathcal{T}_{\Lambda(\alpha)m}(y)\mathcal{L}(\alpha)_m\Psi(m) \tag{17}$$

defines closable operators satisfying the semidirect product composition law of the Poincaré-group: $\mathcal{U}(\alpha_2,y_2)_{\Lambda(\alpha_1)m}\mathcal{U}(\alpha_1,y_1)_m = \mathcal{U}(\alpha_2\alpha_1, y_2+\Lambda(\alpha_2)y_1)_m$

Now Poincaré invariance of the observables requires the physical Hilbert space to be defined via the direct integral $\int_{\oplus\mathcal{O}^{\pm}_{\delta}} d\mu^{\pm}_{\delta}(m)\overline{N(m)/N_0(m)}$ with $N_0(m) = N(m) \cap \mathcal{H}_0(m)$. $N(m)$ is the color neutral maximal η_m-semi-normed subspace of $\mathcal{H}(m)$ mentioned at the beginning and being specified in detail in section VI.
η-isometry implies the important smoothness properties of the following Lemma:

Lemma I: For all $\Psi, \Phi : \mathcal{O}^{\pm}_{\delta} \to \cup_{m\in\mathcal{O}^{\pm}_{\delta}}\mathcal{D}_{QL}(m)$ the functions

$$F_{\Psi,\Phi} : \mathcal{O}^{\pm}_{\delta} \to \mathbb{C}, m \to < \Phi(m), \Psi(m) >_m$$

are $\mathcal{C}^{\infty}$-functions and together with all derivatives polynomical bounded ($\mathcal{O}_{\mathcal{M}}(\mathbb{R}_3)$-functions).

$$\div$$

Proof: In the parametrization of the orbits $\mathcal{O}^{\pm}_{\delta}$ by the boost parameters $\vec{m} \in \mathbb{R}_3$

$$\mathcal{O}^{\pm}_{\delta} = \{\Lambda(\alpha^{\pm}_{|\delta|M(K)}(\vec{m}))(\pm|\delta|M(K),\vec{0})^{tr}|\vec{m} \in \mathbb{R}_3\}$$

where $\alpha^{\pm}_{\kappa}(\vec{m}) = [2\kappa(\kappa + \sqrt{\kappa^2 + \vec{m}^2})]^{-1/2}[(\kappa + \sqrt{\kappa^2 + \vec{m}^2})\sigma^0 \mp \vec{m}\vec{\sigma}]$
with $\kappa := |\delta|M(K), \sigma^0 = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\sigma^k, k = 1,2,3$ the hermitian traceless Pauli matrices is the unique hermitian and positive definite solution of the boost equation

$$\alpha^{\pm}_{\kappa}(\vec{m})\begin{pmatrix} \pm\ \kappa & 0 \\ 0 & \pm\ \kappa\end{pmatrix}\alpha^{\pm}_{\kappa}(\vec{m}) = \pm\kappa\alpha^{\pm}_{\kappa}(\vec{m})^2 = \pm\sqrt{\kappa^2 + \vec{m}^2}\sigma^0 - \vec{m}\vec{\sigma},$$

the functions

$$\alpha^{\pm}_{\kappa}(\vec{m}) \text{ and } \Lambda(\alpha^{\pm}_{\kappa}(\vec{m}))^{\nu}_{\mu} = 1/2tr.(\sigma^{\nu}\alpha^{\pm}_{\kappa}(\vec{m})\sigma^{\mu}\alpha^{\pm}_{\kappa}(\vec{m}))$$

are obviously $\mathcal{C}^{\infty}-$functions and together with all derivatives polynomial bounded.
For $\Phi_{[\mu]_K}(m) = \varphi^{[n']_K[d']_K}_{[\mu]_K,\dots}(h)_m\Psi_0(m)$ and $\Psi_{[\nu]_L}(m) = \varphi^{[n]_L[d]_L}_{[\nu]_L,\dots}(f)_m\Psi_0(m)$ Lorentz covariance and η-isometry imply for all $m \in \mathcal{O}^{\pm}_{\delta}$:
$< \Phi_{[\mu]_K}(m), \Psi_{[\nu]_L}(m) >_m=$
$< \mathcal{L}(\alpha_m^{-1})_{\Lambda(\alpha_m)m_0}\Phi_{[\mu]_K}(\Lambda(\alpha_m)m_0), \mathcal{L}(\alpha_m^{-1})_{\Lambda(\alpha_m)m_0}\Psi_{[\nu]_L}(\Lambda(\alpha_m)m_0) >_{m_0}$
$= (\Pi^K_{k=1}\overline{S^{(n'_k,d'_k)}(\alpha_m^{-1})_{\mu_k}^{\lambda_k}})(\Pi^L_{j=1}S^{(n_j,d_j)}(\alpha_m^{-1})^{\rho_j}_{\nu_j})\times$
$\times < \varphi^{[n']_K[d']_K}_{[\lambda]_K,\dots}(h_{(\Lambda(\alpha_m^{-1}),0)})_{m_0}\Psi_0(m_0), \varphi^{[n]_L[d]_L}_{[\rho]_L,\dots}(f_{(\Lambda(\alpha_m^{-1}),0)})_{m_0}\Psi_0(m_0) >_{m_0}$
where $m_0 := (\pm|\delta|M(K),\vec{0})^{tr}$ and $\alpha_m = \alpha^{\pm}_{\kappa}(\vec{m})$. The first row on the right hand side of the last equality is just a product of (Jacobi-) polynomials in the elements of the matrix α_m and therefore a polynomial bounded $\mathcal{C}^{\infty}$-function in the orbit parameters $\vec{m}$. The last row is also a polynomial bounded $\mathcal{C}^{\infty}$-function (even $\mathcal{O}_{\mathcal{M}}$-function) in these parameters since the vacuum expectation values of fields from $\pi(\varphi_m,(j_m))$ are tempered distributions.
$\square$

Remark: Since $F_{\Psi,\Phi}$ is continuous on $\mathcal{O}^{\pm}_{\delta}$ it is μ-measurable for all $\Psi(m), \Phi(m) \in \mathcal{D}_{QL}(m)$. But then it follows that $F_{\Psi,\Phi}$ is also μ-measurable for all $\Psi(m), \Phi(m) \in \mathcal{H}(m)$ since

$\mathcal{D}_{QL}(m)$ is dense in $\mathcal{H}(m), \eta_m$ continuous and the vectorspace of measurable functions closed with respect to point-wise convergence.

$\div$

Since all $\Lambda(\lambda)$ with λ from the subgroup

$$SU(2)_m := \{\alpha_\kappa^\pm(\vec{m})u\alpha_\kappa^\pm(\vec{m})^{-1} \in SL(2,\mathbb{C}|u \in SU(2)\}, \kappa = |\delta|M(K)$$

are rotations with the point $m \in \mathcal{O}_\delta^\pm$ being a fixpoint, the corresponding Lorentz operators $\mathcal{L}(\alpha_\kappa^\pm(\vec{m})u\alpha_\kappa^\pm(\vec{m})^{-1})_m$ map $\mathcal{D}_{QL}(m)$ onto itself: $\forall_{\lambda\in SU(2)_m}\mathcal{D}_{QL}(\Lambda(\lambda)m) = \mathcal{D}_{QL}(m)$.
Hence for every $m \in \mathcal{O}_\delta^\pm$ the map
$\mathcal{U}_m^R : SU(2) \times \mathbb{R}_4 \to Aut_{\mathcal{D}_{QL}(m)}, (v,y) \to \mathcal{U}_m^R(v,y) := \mathcal{U}(\alpha_\kappa^\pm(\vec{m})v\alpha_\kappa^\pm(\vec{m})^{-1}, y)_m$
is a η_m-isometric representation of the inhomogeneous rotation group $SU(2)\times)\mathbb{R}_4$ on $\mathcal{D}_{QL}(m)$ with $\mathcal{D}_0(m)$ as an invariant subspace. $\mathcal{U}_m^R(v,0)$ is even unitary.

$\div$

III Cluster Theorems

The further considerations can be restricted to the case $\gamma \neq 0$ because the case $\gamma = 0$ is managed in exactly the same way just by replacing the pair $(\mathcal{D}_0(m), \mathcal{T}_m^0(y))$ by $(\mathcal{D}(m), \mathcal{T}_m(y))$.
All cluster theorems known to hold in Wightman theories respectively their algebraic versions[10−14,26−29] and even in indefinite metric spaces[1,10] rest on Einstein causality and the assumption that the spectrum condition (respectively some obvious modification of it[1,3]) holds for *all* quasi-local states from $\mathcal{D}_{QL}(m)$. In the case at hand the latter cannot be true any more for the coloured states due to the exponential color factor in $\mathcal{T}_m(y)$. However since in contrast to $\mathcal{T}_m(y)$ itself the extension $\mathcal{T}_m^0(y)$ of the restriction $\mathcal{T}_m(y)\rceil\mathcal{D}_0(m)$ onto the Hilbert subspace $\overline{(\mathcal{D}_0(m)}, (\cdot|\cdot)_m)$ of $(\mathcal{H}(m), (\cdot|\cdot)_m)$ is again unitary, it is rather obvious that the usual cluster properties hold for the truncated vacuum expectation values of products of translated quasi local fields with total color charge zero. In order to shorten the notation all color and space-time indices will be dropped or replaced by dots whenever possible without loss of clearity.
For $\varphi^{[n^i]_{K_i}[d^i]_{L_i}}_{\dots,[c^i]_{K_i}[a_i]_{K_i}}(f_i)_m, i = 1,2$ with total color charg $Q_\mathcal{C}(K_i) = \sum\limits_{k=1}^{K_i} Q(c_k^i) = 0$ the spectrum, that is the support of the Fourier transform $\mathcal{W}_{\dots}^{\dots}(\dots|q)_m^T := (2\pi)^{-5/2}\int d^4xe^{iqx}\mathcal{W}_{\dots}^{\dots}(\dots|x)_m^T$ of the polynomial bounded $\mathcal{C}^\infty-$functions

$$\mathcal{W}_{\dots}^{K_1^*K_2}(f_1\otimes f_2|x)_m^T := (\varphi^{\dots[d^1]_{K_1}}_{\dots[a^1]_{K_1}}(f_1)_m\Psi_0(m), \mathcal{T}_m^0(x)\varphi^{\dots[d^2]_{K_2}}_{\dots[a^2]_{K_2}}(f_2)_m\Psi_0(m))_m$$

$$-(\varphi^{\dots[d^1]_{K_1}}_{\dots[a^1]_{K_1}}(f_1)_m\Psi_0(m), \Psi_0(m))_m(\Psi_0(m), \varphi^{\dots[d^2]_{K_2}}_{\dots[a^2]_{K_2}}(f_2)_m\Psi_0(m))_m$$

respectively

$$\mathcal{W}_{\dots}^{K_1^+K_2}(f_1\otimes f_2|x)_m^T :=< \Psi_0(m)|\varphi^{\dots[d^1]_{K_1}}_{\dots[a^1]_{K_1}}(f_1)_m^+\mathcal{T}_m^0(x)\varphi^{\dots[d^2]_{K_2}}_{\dots[a^2]_{K_2}}(f_2)_m\Psi_0(m) >_m$$

$-(\Psi_0(m)|\varphi^{...[d^1]_{K_1}}_{...[a^1]_{K_1}}(f_1)^*_m\Psi_0(m))_m(\Psi_0(m)|\varphi^{...[d^2]_{K_2}}_{...[a^2]_{K_2}}(f_2)_m\Psi_0(m))_m$
obviously is contained in the closed backward cone $\overline{V^0_-}$ due to the spectrum condition. Since the contribution of the unique translational invariant vacuum has been subtracted the spectrum of the function $\mathcal{W}^{K_1\cdot\cdot K_2}_{...}(f_1\otimes f_2|x)^T_m$ does not contain parts with point support in $q=0$ (,that is no δ^4- singularities). But then Einstein-causality, unitarity of $\mathcal{T}^0_m(x)$ and the spectrum condition imply the standard cluster properties[11,14] :

Cluster-Theorem I: Let $G'_{K_1K_2}\subset I\!R_4$ be the set of all points space-like to the set

$$G_{K_1K_2}:=\{y_i-z_i\in I\!R_4|(y_1,...,y_{K_1})\in supp.f_1\wedge(z_1,...,z_{K_2})\in supp.f_2\},$$

$G^0_{K_1K_2}$ the complement of $G'_{K_1K_2}\cap\{x\in I\!R_4|x^0=0\}$ in the hyperplane $\{x\in I\!R_4|x^0=0\}$, $\hat{G}^0_{K_1K_2}$ its convex closure and $V(\hat{G}^0_{K_1K_2})$ the volume of $\hat{G}^0_{K_1K_2}$. Then $W^{K_1\cdot\cdot K_2}_{...}(f_1\otimes f_2|x)^T_m$ satisfies the following bounds respectively limits:

(i): If f_1 *and* f_2 have *compact* support, then for all x with $[x]:=\inf.\{\sqrt{(\vec{x}-\vec{\eta})^2-(x^0)^2}\geq\epsilon>1|\vec{\eta}\in\hat{G}^0_{K_1K_2}\}$ (space-like to the *compact* set $\hat{G}^0_{K_1K_2}$) :
$|\mathcal{W}^{K_1\cdot\cdot K_2}_{...}(f_1\otimes f_2|x)^T_m|\leq V(\hat{G}^0_{K_1K_2})[x]^{-3/2}\times$

$$\times\begin{cases} e^{-|\kappa|[x]}(1+\frac{|x^0|}{[x]}) & \text{if spectrum } \mathcal{W}^{K_1\cdot\cdot K_2}_{...}(...|x)^T_m\subseteq\overline{V^\kappa_-},\kappa>0\\ [x]^{-1/2}(1+\frac{|x^0|}{[x]}) & \text{if spectrum } \mathcal{W}^{K_1\cdot\cdot K_2}_{...}(...|x)^T_m\subseteq\overline{V^0_-}\end{cases}$$

(ii): If $f_1\in S(I\!R_{4K_1})$ and/or $f_2\in S(I\!R_{4K_2})$ then

$$\lim_{\sqrt{-x^2}\to+\infty}|(\sqrt{-x^2})^n\mathcal{W}^{K_1\cdot\cdot K_2}_{...}(f_1\otimes f_2|x)^T_{m|_{|x^0|\leq\lambda|\vec{x}|}}|=0$$

for any $\lambda\in]0,1[$ and for all $n\in I\!N$ if the spectrum of $\mathcal{W}^{K_1\cdot\cdot K_2}_{...}(...|x)^T_m$ is contained in $\overline{V^\kappa}$, $\kappa>0$ but for $n=2-\epsilon,\epsilon>0$ only if the spectrum is contained in $\overline{V^0_-}$.

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In the general case of the product of n uncolored fields it follows (along the usual lines Ref.[12]) from the spectrum condition that the polynomial bounded $\mathcal{C}^\infty-$function

$$\begin{aligned}&\mathcal{W}^{[d^1]_{K_1}...[d^n]_{K_n}}_{[a^1]_{K_1}...[a^n]_{K_n}}(\otimes^n_{j=1}f^j|x_1,...,x_n)_m:=\\&(\Psi_0(m)|\Pi^n_{j=1}\mathcal{T}^0_m(x_j)\varphi^{...[d^j]_{K_j}}_{...[a^j]_{K_j}}(f^j)^{(+)}_m\mathcal{T}^0_m(x_j)\Psi_0(m))_m\end{aligned}\tag{18}$$

with $f^j\in S(I\!R_{4K_j})$ has its spectrum contained in the set

$$\{k\in I\!R_{4n}|\sum^n_{j=1}k_j=O\wedge\ \forall_{r\in\{2,...,n\}}\sum^n_{j=r}k_j\in\overline{V^\kappa_-}\cup\{O\},\kappa\geq0\}\tag{19}$$

Unitarity of $\mathcal{T}^0_m$ and invariance of $\Psi_0(m)$ then implies $\mathcal{W}^{\cdots}_{\cdots}(...|x_1,...,x_n)_m$ to depend only on the $n-1$ difference variables $\xi_j = x_{j+1}-x_j, j=1,...,n-1$:

$$\mathcal{W}^{\cdots}_{\cdots}(...|x_1,...,x_m)_m = W^{\cdots}_{\cdots}(...|x_2-x_1,x_3-x_2,...,x_n-x_{n-1})_m$$

which means for the Fourier transforms:
$\hat{\mathcal{W}}^{\cdots}_{\cdots}(...|k_1,...,k_n)_m = (2\pi)^{3/2}\delta^4(\sum_{j=1}^n k_j)\hat{W}^{\cdots}_{\cdots}(...|\sum_{j=2}^n k_j,...,\sum_{j=n-1}^n k_j,k_n)_m$
and therefore:

$$\text{spectrum } W^{[d^1]_{K_1}\cdots[d^n]_{K_n}}_{[a^1]_{K_1}\cdots[a^n]_{K_n}}(\otimes^n_{j=1}f^j|\xi_1,...,\xi_{n-1})_m \tag{20}$$

$$\subseteq \{q\in I\!R_{4(n-1)}|\forall_{j\in\{1,...,n-1\}}q_j\in\overline{V^\kappa_-}\cup\{O\},\kappa\geq 0\}$$

The lowest mass κ depends on the fields contained in the function $W^{\cdots}_{\cdots}(...|...)_m$.
There is some remark to be made concerning the implications of the flavor exponentials $R^d(exp.[i/2(\wp(m)y)])$ in the substitution law (8) of the translation operators for the support of the vacuum expectation values (Wightman distributions) in momentum space: In contrast to the spectrum of the functions

$$\mathcal{W}^{\cdots}_{\cdots}(\otimes^n_{j=1}f^j|x_1,...,x_n)_m \text{ respectively } W^{\cdots}_{\cdots}(\otimes^{n-1}_{j=1}f^j|\xi_1,...,\xi_{n-1})_m,$$

which is just the support of the spectral measure E_m of $\mathcal{T}^0_m$ mirrored in the origin, the support of the Fourier transforms of the Wightman distributions

$$\tilde{\mathcal{W}}^{\cdots}_{\cdots}(\otimes^n_{j=1}\tilde{f}^j)_m := \mathcal{W}^{\cdots}_{\cdots}(\otimes^n_{j=1}f^j)_m = \mathcal{W}^{\cdots}_{\cdots}(\otimes^n_{j=1}f^j|O,...,O)_m$$

in addition is displaced by the vectors

$$p_m(d,a)_\mu := -i\partial_\mu R^d(exp.[i/2(\wp(m)\cdot y)]^a_{a|y=0} \tag{21}$$

A straight forward calculation exploiting the substitution law (8) and the spectrum condition results in:

$$\begin{array}{l} supp.\tilde{\mathcal{W}}^{[d]_K}_{[a]_K}(\tilde{f})\subseteq\{k\in I\!R_{4K}|\sum^K_{j=1}(k_j-p_m(d_j,a_j))=O\ \ \wedge \\ \forall_{r\in\{2,...,K\}}\sum^K_{j=r}(k_j-p_m(d_j,a_j))\in\overline{V^\kappa_-}\cup\{O\},\kappa\geq 0\} \end{array} \tag{22}$$

and the corresponding one for $\tilde{W}^{[d]_K}_{[a]_K}(\tilde{f})$.

The last step to the analogon of the Cluster Theorem I for many clusters is the introduction of truncated vacuum expectation values via substraction of the intermediate vacuum contributions in such a way that the locality properties are preserved for them but their spectrum does not contain any δ^4-singularities at the apex of the spectral cone anymore. They are recursively defined by[9,10,14]

$$
\begin{aligned}
&\mathcal{W}^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|x_1,...,x_n)^T_m := \mathcal{W}^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|x_1,...,x_n)_m \\
&-\textstyle\sum_{part.}(-1)^{N_F(part.)}\mathcal{W}^{[d^1]_{K_1}[d^{j_1}]_{K_{j_1}}\dots[d^{j_i}]_{K_{j_i}}}_{[a^1]_{K_1}[a^{j_1}]_{K_{j_1}}\dots[a^{j_i}]_{K_{j_i}}}(f^1\otimes f^{j_1}\otimes\dots\otimes f^{j_i}|x_1,...,x_n)^T_m\times \\
&\times\mathcal{W}^{[d^{j_i+1}]_{K_{j_{i+1}}}\dots[d^{j_{i+r}}]_{K_{j_{i+r}}}}_{[a^{j_i+1}]_{K_{j_{i+1}}}\dots[a^{j_{i+1}}]_{K_{j_{i+1}}}}(\otimes_{k=1}^r f^{j_{i+k}}|x_{j_{i+1}},...,x_{j_{i+r}})^T_m\times ....,
\end{aligned}
\tag{23}
$$

where the *sum* extends over all partitions $\{1,j_1,...,j_i\}\{j_{i+1},...,j_{i+r}\}...$ of the set $\{1,...,n\}$ into disjoint subsets such that within every truncated function on the right hand side the order of indices is the same as in $\mathcal{W}^{[d_1]_{K_1}\dots[d^n]_{K_n}}_{[a_1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|x_1,...,x_n)^T_m$ and moreover $1<j_{i+1}<j_{i+r+1}<...N_F(part)$ is the number of interchanges of Fermi-fields in going from $1,...,n$ to $1,j_1,...,j_{n-1}$.
The spectrum of the truncated functions satisfies the restriction:

$$
\text{spectrum}\ \ \mathcal{W}^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|x_1,...,x_n)^T_m \tag{24}
$$

$$
\subseteq\{k\in I\!R_{4n}|\sum_{j=1}^n k_j=0\wedge\forall_{r\in\{2,...,n\}}\sum_{j=r}^n k_j\in\overline{V^\kappa_-},\kappa\geq 0\}
$$

respectively

$$
\text{spectrum}\ \ W^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|\xi_1,...,\xi_{n-1})^T_m \tag{25}
$$

$$
\subseteq\{q\in I\!R_{4(n-1)}|\forall_{j\in\{1,...,n-1\}}q_j\in\overline{V^\kappa_-},\kappa\geq 0\}.
$$

Even in those cases for which κ is zero the spectrum of the truncated functions in contrast to the original ones have no δ^4-singularities at the apex of $\overline{V^0_-}$. Now from Einstein causality, the unitarity of $\mathcal{T}^0_m$ and the spectral support (24) respectively (25) it follows in case of a mass gap $\kappa>0$ in literally the same way as in Wightman theories[10,14]:

Cluster Theorem II: Let D^r_x be a derivation monomial of total degree r, $d(\vec{x}):=\max\{|\vec{x}_i-\vec{x}_j||i,j\in\{1,...,n\}\}$ the diameter of the configuration $\{\vec{x}_1,...,\vec{x}_n\}$ and the spectrum of $\mathcal{W}^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[d^n]_{K_n}}(\otimes_{j=1}^n f^j|x_1,...,x_n)^T_m$ satisfy the condition (22) for some $\kappa>0$ then the following limits hold for all $r\in I\!N$:

(i) If $f^1,...,f^n$ all have compact support, then

$$
\lim_{d(\vec{x})\to\infty} d(\vec{x})^{3/2}e^{\kappa d(x)/(n-1)}D^r_x\mathcal{W}^{\dots}_{\dots}(\otimes_{j=1}^n f^j|x_1,...,x_n)^T_{m|_{x_1^0=x_2^0=...=x_n^0}}=0
$$

(ii) If for some $j\in\{1,...,n\}$ $f^j\in S(I\!R_{4K_j})$ has non-compact support, then for every $N\in I\!N$:

$$
\lim_{d(\vec{x})\to\infty} d(\vec{x})^N D^r_x\mathcal{W}^{\dots}_{\dots}(\otimes_{j=1}^n f^j|x_1,...,x_n)^T_{m|_{x_1^0=x_2^0=...=x_n^0}}=0
$$

÷

An immediate consequence reads:

Corallary: If the spectrum of the function $W^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|\xi_1,\dots,\xi_{n-1})_m$ has a mass gap (, that means the condition (20) respectively (25) holds for some $\kappa > 0$), then the truncated function $W^{[d^1]_{K_1}\dots[d^n]_{K_n}}_{[a^1]_{K_1}\dots[a^n]_{K_n}}(\otimes_{j=1}^n f^j|\xi_1,\dots,\xi_{n-1})^T_{m|_{\xi_1^0=\xi_2^0=\dots=\xi_{n-1}^0=0}}$ is an element from $S(I\!R_{3(n-1)})$.

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IV Quasi-Particle Excitations

In this section so called quasi-particle excitations are contructed by means of the spectrum condition and the transformation laws under the various symmetries; that is composite local operators creating from the vacua $\Psi_0(m), m \in \mathcal{O}_\delta^\pm$ states in $\mathcal{D}_0(m)$ from which the (real) one particle states manifesting themselves as discrete (and up to parity-doubling) irreducible unitary representations $[m, s]$ of the Poincaré group in the physical Hilbert space $\int_{\oplus\mathcal{O}_\delta^\pm} d\mu_\delta^\pm(m)\overline{N(m)/N_0(m)}$ are constructed in the following sections.
Only the deviations from the standard construction of Wightman theories $(d\mu_\delta^\pm(m) = \delta^{(4)}(m)d^4m, \eta_m = id_{\mathcal{H}(m)})$ due to the generalized transformation laws (8), (15) and (17) with respect to the Poincaré-Group and especially due to the flavor matrix

$$R^d(\exp.[i/2(\wp(m)\cdot x)])^a_b = \delta^a_b e^{i(p_m(d,a)\cdot x)} \tag{26}$$

with

$$p_m(d,a)_\mu := -i\partial_\mu R^d(\exp.[i/2(\wp(m)\cdot x)])^a_{a|x=0}$$

in (8) are treated. Explicitely we have independent from the flavor isospin (I, I^3) and the dimension $d \in \{3, 3^*, 8\}$:

$$\begin{array}{l} p_m(3,(I,I^3,Y)) = p_m(8,(I,I^3,Y)) = p(m,Y) = 1/\delta Y m \\ p_m(3^*,(I,I^3,Y)) = -p_m(3,(I,-I^3,-Y)) = 1/\delta Y m \end{array} \tag{27}$$

For the sake of shortness we restrict the explicite constructions to the pseudo-scalar meson sector $s^P = 0^-$. The missing links in the $s^P = 1/2^+$ baryon and other sectors can be managed with the obvious modifications along the same lines. In misuse of the word spin the *sector* or *channel with spin s* in $\mathcal{D}_0(m)$ is defined as the eigenspace of the Casimir operator of the rotation group $SU(2)$ in its unitary representation U^R_m on $\mathcal{D}_{QL}(m)$ constructed at the end of section II.
Since in contrast to non-relativistic quantum mechanics (where the normalized solutions of the time independent Schrödinger equation determine all stable particles) in the present

case like in very axiomatic quantum field theory a complete specification of the dynamics is not at hand, some further assumption on the spectral measure $E_m(q)$ and a semi-normed maximal subspace of $(H(m), <\cdot,\cdot>_m)$ has to be put in. In order not to burden the investigation of the problems arising from the indefinite metric and the generalized transformation laws we consider only the generalization of the original Haag-Ruelle assumption[9,10–14] due to Hepp[14]. The extension to the further relaxation of it due to Herbst[16] is then rather obvious.

Haag-Ruelle-Hepp Condition: In the flavor channel $(I, Y, s^P), s^P \in \{0^-, 1/2^+, \bar{1/2}^+\}$ there exists for some mass $M = M(I, Y, s^P)$ a set of $(2I+1)(2s+1)^2$ composite local field operators $\phi^{(I,Y,s^P)}_{M,\lambda,I^3}(f_{p(m,Y)})_m \in \pi(\varphi_m, (j_m))$ with the properties:

(i): For all h with $\tilde{h}$ a polynomial bounded $\mathcal{C}^\infty-$ function from $\mathcal{O}_{\mathcal{M}}(I\!R_4)$ vanishing outside some sufficiently small neighbourhood $V_-^{M-\epsilon} \setminus \overline{V_-^{M+\epsilon}}, \epsilon > 0$ of the mass hyperboloid $q^2 = M^2, q^0 < 0$ and $\tilde{h}(-\sqrt{\vec{q}^2 + M^2}, \vec{q}) = 1$:

$$\int_{I\!R_4} d^4x \mathcal{T}^0_m(x) \phi_{\ddot{M},...}(f_{p(m,Y)})^+_m \Psi_0(m)(\Box_x + M^2) h(x) = \mathcal{O} \tag{28}$$

(ii): For all $(\alpha, u, (r,t))$ from the direct product $SL(2, \mathbb{C}) \times (U(1)^{\mathcal{Q}}_{\mathcal{C}} \times SU(2)^{\mathcal{U}}_{\mathcal{C}}) \times (SU(2)^{\mathcal{J}}_{\mathcal{F}} \times U(1)^{\mathcal{Y}}_{\mathcal{F}})$:

$$\begin{aligned} &\mathcal{L}(\alpha)_m I_m(u,(r,t)) \phi^{(I,Y,s^P)}_{M,\lambda,I^3}(f_{p(m,Y)})^+_m (\mathcal{L}(\alpha)_m I_m(u,(r,t)))^{-1} = \\ &\phi^{(I,Y,s^P)}_{M,\nu,\hat{I}^3}(f_{\Lambda(\alpha)p(m,Y)})^+_{\Lambda(\alpha)m} \overline{S^{(n,n)}(\alpha)^\nu_\lambda} D^I(r)_{\hat{I}^3 I^3} e^{iYt} \end{aligned} \tag{29}$$

where D^I is the $(2I+1)-dim.$ irreducible unitary matrix representation of $SU(2)$ and $S^{(n,n)}, n \in \{1, 3, 3^*\}$ the standard $(2s+1)^2 - dim.$ (parity doubled) matrix representation of the Lorentz group of section II.

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It is straight forward to construct from $\pi(\varphi_m, (j_m))$ composite field operators with the transformation properties (ii). Let $S^{[n]_r;d}_{M(I,Y,s^P)}(I\!R_{4(r+1)}), r \geq 2(s+1)$ denote the set of all functions $f^{[\mu]_r[c]_r[a]_r}_{M,q,\lambda,b}(x_1,...,x_r) := f^{[\mu]_r[c]_r[a]_r}_{M,\lambda,b}(x_1,...,x_r q)$ from $S(I\!R_{4(r+1)})$ with the properties:

(I): The (partial) Fouriertransform

$$\tilde{f}_{\ddot{M},q,\lambda,b}(k_1,...,k_r) := (2\pi)^{-3/2r} \int_{I\!R_{4r}} dx e^{-i\sum_{j=1}^r (x_j k_j)} f_{\ddot{M},\lambda,b}(x_1,...,x_r,q) \tag{30}$$

vanishes for all $(k_1 + ... + k_r + q) \notin \overline{V_+^{M-\epsilon}} \setminus V_+^{M+\epsilon}, M = M(I,Y,s^P)$ and some $0 < \epsilon \leq 1/2 \min.\{M, |M - M(\tilde{I}, \tilde{Y}, \tilde{s}^P)||M(\tilde{I}, \tilde{Y}\tilde{s}^P) \neq M\}$, and is unequal to zero for some $q + k_1 + ... + k_r$ on the hyperboloid $(q + k_1 + ... + k_r)^2 = M^2$.

$$(II): \quad \Pi_{i=1}^{r} S^{(n_i,n_i)}(\alpha)^{\mu_i}_{\nu_i} R^{n_i}(u)^{c_i}_{e_i} R^{n_i}(v)^{a_i}_{g_i} \tilde{f}^{[\nu]_r[e]_r[g]_r}_{M,\lambda,a}(\Lambda(\alpha^{-1})k_1,...,\Lambda(\alpha^{-1})k_r,q) = S^{(n,n)}(\alpha)^{\rho}_{\lambda} \overline{R^d(v)^b_a} \tilde{f}^{[\mu]_r[c]_r[a]_r}_{M,\rho,b}(k_1,...,k_r,\Lambda(\alpha)q) \tag{31}$$

$\div$

The solutions of these conditions are known[14] and obtained in the restframe of $(k_1+...+k_r+q)$ via integration with the product of the Haar measures over the compact color and flavor group and the compact subgroup of rotations of the Lorentz group.
In accordance with the general belief that pseudo-scalar mesons should be composed of quarks and anti-quarks respectively the baryons of three quarks we consider only the cases $r=2(s+1)$ and $d=8$ with $(n_1,n_2)=(3^*,3)$ for $s^P=0^-$ and $n_1=n_2=n_3=3$ for $s^P=1/2^+$.
In the former case the functions

$$f^{\mu_1\mu_2,c_1c_2,a_1a_2}_{M,q,a}(x_1,x_2) = \gamma_5^{\mu_1\mu_2}\begin{pmatrix}3^*3 & 1\\ c_1\ c_2 & 0\end{pmatrix}(-1)^{-Q(a)}\begin{pmatrix}3^* & 3 & 8\\ a_1 & a_2 & -a\end{pmatrix} f_M(x_1,x_2,q)$$

with $\gamma^5=\gamma^0\gamma^1\gamma^2\gamma^3$ and f_M a Lorentz invariant function from $S(\mathbb{R}_{12})$ with the support properties (I) are from $S^{3^*,3;8}_{M(I,Y,0^-)}(\mathbb{R}_{12})$.
Now consider the composite fields build with them:

$$\varphi^{3^*3,3^*3}_{\mu_1\mu_2,c_1c_2,a_1a_2}(f^{\mu_1\mu_2,c_1c_2,a_1a_2}_{M,p(m,Y),a})_m := (\eta_m\varphi^{3^*3^*}_{\mu_1,c_1,a_1}\eta_m\bar{\otimes}\varphi^{33}_{\mu_2,c_2,a_2})(f^{\mu_1\mu_2,c_1c_2,a_1a_2}_{M,p(m,Y),a})_m = \binom{3^*3\ 1}{c_1c_2 0}(-1)^{-Q(a)}\binom{3^*3\ 8}{a_1a_2 -a}\gamma_5^{\mu_1\mu_2}\varphi^{3^*3,3^*3}_{\mu_1\mu_2,c_1c_2,a_1a_2}(f_{M,p(m,Y)})_m, \tag{32}$$

where summation over $\mu_1,\mu_2\in\{0,...,3\}; c_1,a_1\in\Gamma_{3^*}$ and $c_2,a_2\in\Gamma_3$ is understood. Then from the transformation properties (14), (15), (30) and the commutation relations $[I_m(u,v),\eta_m]_-=0$ respectively $\mathcal{L}(\alpha)_m\eta_m=\eta_{\Lambda(\alpha)m}\mathcal{L}(\alpha^{-1})^*_{\Lambda(\alpha)m}$ it follows via the limits (6) for all $(\alpha,u,v)\in SL(2,\mathbb{C})\times(U(1)^{\mathcal{Q}}_{\mathcal{C}}\times SU(2)^{\mathcal{U}}_{\mathcal{C}})\times(SU(2)^{\mathcal{I}}_{\mathcal{F}}\times U(1)^{\mathcal{Y}}_{\mathcal{F}})$:

$$\begin{aligned}&\mathcal{L}(\alpha)_m I_m(u,v)\varphi^{3^*3,3^*3}_{\mu_1\mu_2,c_1c_2,a_1a_2}(f^{\mu_1\mu_2,c_1c_2,a_1a_2}_{M,p(m,Y),a})_m(\mathcal{L}(\alpha)_m I_m(u,v))^{-1}\\ &=\overline{R^8(v)^b_a}\varphi^{3^*3,3^*3}_{\mu_1\mu_2,c_1c_2,a_1a_2}(f^{\mu_1\mu_2,c_1c_2,a_1a_2}_{M,\Lambda(\alpha)p(m,Y),b})_{\Lambda(\alpha)m}\end{aligned} \tag{33}$$

Since for $(r,t)\in SU(2)^{\mathcal{I}}_{\mathcal{F}}\times U(1)^{\mathcal{Y}}_{\mathcal{F}}$ the representation $R^8(r,t)$ decomposes into the direct sum:

$$R^8(r,t)=\oplus_{(I,Y)}D^I(r)e^{iYt}; (I,Y)\in\{(1/2,1),(1,0),(0,0),(1/2,-1)\}$$

the operator

$$\begin{aligned}\phi^{(I,Y,0^-)}_{M,I^3}(f_{p(m,Y)})_m \ :=\ &(-1)^{I^3+1/2Y}\begin{pmatrix}3^* & 3 & 8\\ a_1 & a_2 & -(I,I^3,Y)\end{pmatrix}\begin{pmatrix}3^* & 3 & 1\\ c_1 & c_2 & 0\end{pmatrix}\\ &\gamma_5^{\mu_1\mu_2}\varphi^{3^*3,3^*3}_{\mu_1\mu_2,c_1c_2,a_1a_2}(f_{M,p(m,Y)})_m\end{aligned} \tag{34}$$

satisfies the condition (iii).
In the same way in the $1/2^+-$baryon or $1\bar{/}2^+-$antibaryon sectors operators $\phi^{(I,Y,s+)}_{M,\lambda,I^3}(f_{p(m,Y)})_m$, $s \in \{1/2, 1\bar{/}2\}$ satisfying the condition (ii) are obtained from the threefold composite fields $\varphi^{[n]_3[n]_3}_{[\mu]_3[c]_3[a]_3}(f^{[\mu]_3[c]_3[a]_3}_{M,p(m,Y),\lambda,b})_m$ with testfunction $f^{\cdots}_{\cdots}$ from $S^{[3]_3;8}_{M(I,Y,1/2^+)}(I\!R_{16})$ respectively $S^{[3^*]_3;8}_{M(I,Y,1\bar{/}2^+)}(I\!R_{16})$.
Hence the crux of the *HRH*-condition is the part (i): In the pion and (anti-)nucleon channels $(1,0,0^-)$ respectively $(1/2,1,\overset{(-)}{1/2}{}^+)$ the lowest mass is an isolated point in the observed four momentum squared spectrum. It suggests that in these channels the Haag-Ruelle condition holds for the spectral measure of $\mathcal{T}^0_m(x)$. If $P^{(I,Y,s^P)}_m$ denotes the orthogonal projector onto the $(I,Y,s^P)-$sector in $\overline{\mathcal{D}_0(m)}$ it reads:

$$\begin{aligned} &\int_{I\!R_4} \tilde{h}(-q) d(\Phi(m), E_m(q) P^{(I,Y,s^P)}_m \Psi(m))_m \\ &= \int_{I\!R_4} d^4q \Theta(q^0)\delta(q^2 - M(I,Y,s^P)^2)(\Phi(m), E^{(I,Y,s^P)}_m(\omega_M(\vec{q}),\vec{q})\Psi(m))_m \end{aligned} \tag{35}$$

with $\omega_M(\vec{q}) := \sqrt{M(I,Y,s^P)^2 + \vec{q}^2}$ and $\tilde{h}$ defined above (for all ϵ smaller than the pion mass $M(\pi)$). However in these cases it follows at once in the standard way[13,14] that the operators

$$\phi^{(I,Y,s^P)}_{M,\lambda,I^3}(x,p(m,Y))_m := \mathcal{T}^0_m(x)\phi^{(I,Y,s^P)}_{M,\lambda,I^3}(f_{p(m,Y)})_m \mathcal{T}^0_m(-x) = e^{-ip(m,Y)x}\phi^{(I,Y,s^P)}_{M,\lambda,I^3}((f_{p(m,Y)})_x)_m \tag{36}$$

destroy the vacuum $\Psi_0(m)$ and their $\eta-$adjoint companions $\phi^{\cdots}_{\cdots}(x,p(m,Y))^+_m$ satisfy the condition (i) for all $\tilde{h} \in \mathcal{O}_{\mathcal{M}}(I\!R_4)$. In channels where the spectral property (35) is not valid the part (i) of the *HRH*-condition *for all* $\tilde{h} \in \mathcal{O}_{\mathcal{M}}(I\!R_4)$ *and* $f_{p(m,Y)} \in S^{[n]_r;d}_{M(I,Y,s^P)}(I\!R_{4(r+1)})$ *with sufficiently small* $\epsilon > 0$ is taken as an input. In any case then condition (i), the spectrum condition and the support properties (30) of the testfunctions $f_{\ddot{M},\ldots}(x_1,x_2,p(m,Y))$ imply besides

$$\phi_{\ddot{M},\ldots}(x,p(m,Y))_m\Psi_0(m) = \mathcal{O} \wedge (\Box_x + M^2)\phi_{\ddot{M},\ldots}(x,p(m,Y))^+_m\Psi_0(m) = 0 \tag{37}$$

that $\phi_{\ddot{M},\ldots}(x,p(m,Y))^+_m\Psi_0(m)$ is a *positive frequency solution* of the Klein-Gordon equation and for $s \in \{1/2, 1\bar{/}2\}$ morover a positive frequency solution of the Dirac equation

$$\phi^{(I,Y,s^P)}_{M,I^3}(x,p(m,Y))^+_m\gamma^0\Psi_0(m)(i\gamma^\mu \overleftarrow{\partial}_\mu + M) = \mathcal{O}, \tag{38}$$

which may justify to call these operators *quasi-particle excitations.*
All these properties can be immediately read off from the equations:

$$
\begin{aligned}
&\int_{I\!R_4} dx g(x)\phi^{(I,Y,s)}_{M,\nu,I^3}(x,p(m,Y))^{(+)}_m\Psi_0(m) = \\
&(2\pi)^{3/2}\int_{I\!R_4} dE_m(k)\tilde{g}(-k)\tilde{\phi}^{(I,Y,s)}_{M,\nu,I^3}(\tilde{f}_{p(m,Y)})^{(+)}_m\Psi_0(m) \\
&= (2\pi)^{3/2}\sum_{\sigma=-s}^{+s}\int_{I\!R_{8(s+1)}} dk\tilde{\phi}^{(I,Y,s)}_{M,I^3,\sigma}(k_1,...,k_{2(s+1)})^{(+)}_m\Psi_0(m)u^\sigma_{\sqrt{q^2}}(\vec{q})^{(c)}_\nu{}_{|q=\sum_{j=1}^{2(s+1)}k_j+p(m,Y)}\times \\
&\tilde{g}((-)[p(m,Y)+\sum_{j=1}^{2(s+1)}k_j])\tilde{f}_M(k_1,...,k_{2(s+1)},p(m,Y))^{(c)}
\end{aligned}
\tag{39}
$$

with $\tilde{f}_M(...)^c := \overline{\tilde{f}_M(...)}$ and the appropriate choice of $\tilde{g}$ from $\mathcal{O}_{\mathcal{M}}(I\!R_4)$ respectively $\tilde{g}(k) = (M^2-k^2)\tilde{h}(k)$ and taking into account the support properties of $\tilde{f}_M$ and $E_m(k)$. $\tilde{f}_M(...)$ is Lorentz invariant with the support properties of condition (I) and $u^\sigma_M, \sigma \in \{1/2,-1/2\}$ are the standard positive frequency solutions of the Dirac equation $(M-\gamma_\mu q^\mu)u^\sigma_M(\vec{q})_{|q^0=\omega_M(\vec{q})} = 0$ (being deleted for $s=0$). All coupling coefficients and summations which arise when solving conditions (II) and not being needed below are absorbed in $\tilde{\phi}^{...}_{...}(k_1,...,k_{2(s+1)})_m$. In the anti-baryon channel $s=\overline{1/2}$ u^σ_M has to be replaced by the charge-conjugate spinor $u^\sigma_{\bar{M}}(\vec{q}) := C(v^\sigma_M(\vec{q})^+\gamma^o)^{tr.}$ with v^σ_M a negative frequency solution of the Dirac equation $(M+\gamma_\mu q^\mu)v^\sigma_M(\vec{q})_{|q^o=\omega_M(\vec{q})} = 0$ and C the antisymmetric, unitary charge conjugation matrix fixed by $\gamma_\mu = -C(\gamma_\mu)^{tr.}C^{-1}$. In the following this replacement for the antibaryon channel is always understood without stating it.
Moreover it follows from the equations (17), (27), (29) and (36) that the quasi-particle excitations transform under Poincaré transformations according to:

$$
\begin{aligned}
&\mathcal{U}(\alpha,y)_m\phi^{...}_{M,\lambda,...}(x,p(m,Y))^+_m\mathcal{U}(\alpha,y)^{-1}_m = \\
&\overline{S^{(n,n)}(\alpha)^\nu}_\lambda\phi^{...}_{M,\nu,...}(\Lambda(\alpha)x+y,p(\Lambda(\alpha)m,Y))^+_{\Lambda(\alpha)m}
\end{aligned}
\tag{40}
$$

An important consequence of the properties of the quasi-particle excitations for the following limiting procedures is, that the Haag-Ruelle-operators

$$
B^{(I,Y,0)}_{M,I^3}(\hat{f},t)_m := i\int_{x^0=t} d^3x\overline{f^+_M(x)}\overleftrightarrow{\partial_0}\,\phi^{(I,Y,0^-)}_{M,I^3}(x,p(m,Y))_m \tag{41}
$$

with $\overleftrightarrow{\partial_0}=\partial/\overrightarrow{\partial}x^0 - \partial/\overleftarrow{\partial}x^0$ and $f^+_M(x) = (2\pi)^{-3/2}\int_{I\!R_4} dq\Theta(q^0)\delta(q^2-M^2)e^{-i(q\cdot x)}\hat{f}(\vec{q}), \hat{f} \in S(I\!R_3)$ a smooth positive frequency solution of the Klein-Gordon equation satisfy for $\dagger \in \{*,+\}$ the equations:

$$
B^{(I,Y,0)}_{M,I^3}(\hat{f},t)_m\Psi_0(m) = \frac{d}{dt}B^{(I,Y,0)}_{M,I^3}(\hat{f},t)^\dagger_m\Psi_0(m) = \mathcal{O}; \tag{42}
$$

that means the two-point functions of them are *time-independent*. The same equations hold in the spin 1/2 case for the corresponding operators

$$
B^{(I,Y,1/2)}_{M,I^3}(\hat{f}u^\sigma_M,t)_m = B^{(I,Y,1/2)}_{M,I^3,\sigma}(\hat{f},t)_m := \int_{x^0=t} d^3x\overline{f^+_{M,\sigma}(x)^\lambda}\phi^{(I,Y,1/2^+)}_{M,\lambda,I^3}(x,p(m,Y))_m \tag{43}
$$

with $f^+_{M,\sigma}(x) = (2\pi)^{-3/2}\int_{\mathbb{R}_4} dq\Theta(q^0)\delta(q^2-M^2)e^{-i(q\cdot x)}u^\sigma_M(\vec{q})\hat{f}(\vec{q}), \hat{f}\in S(\mathbb{R}_3)$.

Last not least we have to introduce some positivity condition on the two-point function $< B^{\cdots}_{\cdots}\hat{f},t)^+_m\Psi_0(m), B^{\cdots}_{\cdots}(\hat{f},t)^+_m\Psi_0(m) >_m$ of the quasi-particle excitations, which in quantum field theories with flavor symmetry non-breaking and/or color non-confining translation operators $d\mu^\pm_\delta(m) = \delta^4(m)d^4m)$ is a consequence of the presupposed existence of a one-particle state with given quantum numbers $(M, s^P, \sigma, I, I^3, Y)$ in the underlying Hilbert space as has been shown by Araki and Haag[15]. The key to this condition and the construction of the asymptotic collision states is the following Lemma.

Lemma II: Let f_M and $f_{\hat{M}}$ satisfy the support condition (I) with $(\bar{V}^{M-\epsilon}_+\setminus V^{M+\epsilon}_+)\cap(\bar{V}^{\hat{M}-\epsilon}_+\setminus V^{\hat{M}+\epsilon}_+)=\phi$. Then there exists a $\mathcal{O}_{\mathcal{M}}(\mathbb{R}_6)-$function
$\rho^{(I,Y,s)}_M : \mathbb{R}_3\times\mathcal{O}^\pm_\delta\to\mathbb{R}, (\vec{q},m)\to\rho^{(I,Y,s)}_M(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m$ with the properties:

(i) $\forall_{m\in\mathcal{O}^\pm_\delta}\rho^{\cdots}_{\cdots}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m\in S(\mathbb{R}_3)\wedge\forall_{\vec{q}\in\mathbb{R}_3}\rho^{\cdots}_{\cdots}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m\in S(\mathbb{R}_3)$

(ii) (Lorentz invariance):
$\forall_{\beta\in SL(2,\mathbb{C})}\rho^{\cdots}_{\cdots}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m = \rho^{\cdots}_{\cdots}((\omega_M(\vec{q}),\vec{q})\cdot\Lambda(\beta)^{tr};f_{p(\Lambda(\beta)m,Y)})_{\Lambda(\beta)m}$

such that for all $\tilde{g},\tilde{h}\in\mathcal{O}_{\mathcal{M}}(\mathbb{R}_4)$ and all $m\in\mathcal{O}^\pm_\delta$:

$$\begin{array}{l}< \phi^{(\hat{I},\hat{Y},\hat{s})}_{\hat{M},\nu,\hat{I}^3}(h,p(m,Y))^+_m\Psi_0(m), \phi^{(I,Y,s)}_{M,\mu,I^3}(g,p(m,Y))^+_m\Psi_0(m) >_m= \\ \delta_{M\hat{M}}\delta_{s\hat{s}}\delta_{Y\hat{Y}}\delta_{I\hat{I}}\delta_{I^3\hat{I}^3}\int_{\mathbb{R}_3}\frac{d\vec{q}}{2\omega_M(\vec{q})}\tilde{h}(\omega_M(\vec{q}),\vec{q})\tilde{g}(\omega_M(\vec{q}),\vec{q})^c\rho^{(I,Y,s)}_M(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m \\ \begin{cases}1 & \text{for } s=0\\ \sum_{\epsilon=-s}^{+s} u^\epsilon_M(\vec{q})_\nu u^\epsilon_M(\vec{q})^c_\mu & \text{for } s=1/2\end{cases}\end{array}\tag{44}$$

$\div$

Proof: The diagonality in I, Y and s is obvious due to the HRH-condition (ii) and $\eta-$isometry, that is by construction of the quasi-particle excitations. The diagonality in M follows directly from equation (39) and the assumed support of f_M and $f_{\hat{M}}$. Hence it suffices to consider the matrix element in (44) for $M=\hat{M}$ and $(\hat{I},\hat{Y},\hat{s})=(I,Y,s):=J$. We present the proof for $s=1/2$, since the case $s=0$ is then obvious.

Translational invariance implies the existence of a tempered distribution $\tilde{V}^{J,M}_{\hat{I}^3I^3,\hat{\sigma}\sigma}(\ldots)_m\in S'(\mathbb{R}_{20})$ such that:
$< \tilde{\phi^J}_{M,\hat{I}^3,\hat{\sigma}}(k_1,k_2,k_3)^+_m\Psi_0(m), \tilde{\phi}^J_{M,I^3,\sigma}(q_1,q_2,q_3)^+_m\Psi_0(m) >_m=$
$(2\pi)^{3/2}\tilde{V}^{J,M}_{\hat{I}^3I^3,\hat{\sigma}\sigma}(q_1+q_2+q_3-k_2-k_3, q_1+q_2+q_3-k_3, q_1+q_2+q_3, q_2+q_3, q_3)_m\times$
$\delta(q_1+q_2+q_3-k_1-k_2-k_3)$

Then via some variable transformations it follows from equation (39) for all g with $\tilde{g}\in\mathcal{O}_{\mathcal{M}}(\mathbb{R}_4)$
$< \phi^J_{M,\hat{\lambda},\hat{I}^3}(0,p(m,Y)^+_m\Psi_0(m), \phi^J_{M,\lambda,I^3}(g,p(m,Y))^+_m\Psi_0(m) >_m$

$$= \sum_{\hat{\sigma},\sigma=-s}^{+s}\tilde{V}^{J,M}_{\hat{I}^3I^3,\hat{\sigma}\sigma}(\tilde{f}_{p(m,Y)}\circ(u^{\hat{\sigma}}_{\hat{\lambda}}\tilde{g}^c u^{\sigma c}_\lambda)\circ\tilde{f}^c_{p(m,Y)})_m \tag{45}$$

with

$$(\tilde{f}_{p(m,Y)} \circ (u^{\hat{\sigma}}_{\hat{\lambda}} \tilde{g}^c u^{\sigma c}_{\lambda}) \circ \tilde{f}^c_{p(m,Y)})(v_1, v_2, q, z_1, z_2) :=$$
$$(2\pi)^{9/2} \tilde{f}_M(v_1, v_2 - v_1, q - v_2, p(m,Y)) u^{\hat{\sigma}}_{\sqrt{(q+p(m,Y))^2}}(\vec{q} + p(\vec{m}, Y))_{\hat{\lambda}} \times$$
$$\tilde{g}(q + p(m,Y)^c u^{\sigma}_{\sqrt{(q+p(m,Y))^2}}(\vec{q} + p(\vec{m}, Y))^c_{\lambda} \tilde{f}_M(q - z_1, z_1 - z_2, z_2, p(m,Y))^c \tag{46}$$

Now the HRH-condition (i) reads:
$\sum_{\hat{\sigma},\sigma=-s}^{+s} \tilde{V}^{J,M}_{\hat{I}^3 I^3, \hat{\sigma}\sigma}(\tilde{f}_{p(m,Y)} \circ (u^{\hat{\sigma}}_{\hat{\lambda}} \tilde{g}^c_{\Box+M^2} u^{\sigma c}_{\lambda}) \circ \tilde{f}^c_{p(m,Y)})_m = 0$
for *all* $\tilde{g}_{\Box+M^2}(q) := (M^2 - q^2)\tilde{h}(q)$ with $\tilde{h} \in \mathcal{O}_{\mathcal{M}}(I\!R_4)$
This implies via the support property (i) of f_M (and suitable choices of $\tilde{h}$) for all $\sigma, \hat{\sigma} \in \{1/2, -1/2\}$ and $\tilde{g} \in \mathcal{O}_{\mathcal{M}}(I\!R_4)$:

$$\tilde{V}^{J,M}_{\hat{I}^3 I^3, \hat{\sigma}\sigma}(\tilde{f}_{p(m,Y)} \circ (u^{\hat{\sigma}}_{\hat{\lambda}} \tilde{g}^c u^{\sigma c}_{\lambda}) \circ \tilde{f}^c_{p(m,Y)})_m =$$
$$\int_{I\!R_4} d^4q \Theta(q^0) \delta(q^2 - M^2) \rho^{J,M}_{\hat{I}^3 I^3, \hat{\sigma}\sigma}(\omega_M(\vec{q}), \vec{q}; f_{p(m,Y)})_m u^{\hat{\sigma}}_{\sqrt{q^2}}(\vec{q})_{\hat{\lambda}} u^{\sigma}_{\sqrt{q^2}}(\vec{q})^c_{\lambda} \tilde{g}(q)^c \tag{47}$$

with $\rho^{\dots}_{\dots}(\omega_M(\vec{q}), \vec{q}; f_{p(m,Y)})_m = \rho^{\dots}_{\dots}((M, \vec{0}) \cdot \Lambda(\alpha^+_M(q))^{tr}; f_{p(m,Y)})_m$ being a $\mathcal{O}_{\mathcal{C}}(I\!R_3)-$distribution on the forward mass shell $\mathcal{Q}^+_M := \{q \in I\!R_4 | q^2 = M^2 \wedge q^0 > 0\}$.
Feeding equation (47) back into (45) and taking the special $\mathcal{O}_{\mathcal{M}}(I\!R_4)-$function $\tilde{g}_x(q) := (2\pi)^{-3/2} e^{-iqx}$ or equivalently $g_x(y) = \delta^4(x - y)$ we obtain:

$$< \phi^J_{M,\hat{\lambda},\hat{I}^3}(0, p(m,Y))^+_m \Psi_0(m), \phi^J_{M,\lambda,I^3}(x, p(m,Y))^+_m \Psi_0(m) >_m =$$
$$(2\pi)^{-3/2} \sum_{\hat{\sigma},\sigma=-s}^{+s} \int_{I\!R_3} \frac{d\vec{q}}{2\omega_M(\vec{q})} \rho^{J,M}_{\hat{I}^3 I^3, \hat{\sigma}\sigma}(\omega_M(\vec{q}), \vec{q}; f_{p(m,Y)})_m u^{\hat{\sigma}}_M(\vec{q})_{\hat{\lambda}} u^{\sigma}_M(\vec{q})_{\lambda} e^{i\omega_M(\vec{q})x - i\vec{q}\vec{x}} \tag{48}$$

Due to the cluster theorems of section III the left hand side is for $x^0 = 0$ and every fixed $m = (\pm\omega_{|\delta|M(K)}(\vec{m}), \vec{m})^{tr} \in \mathcal{O}^{\pm}_{\delta}$ from $S(I\!R_3)$. Therefore the right hand side must be equal to the ordinary Fourier transform of a function from $S(I\!R_3)$; in other words the $\mathcal{O}_{\mathcal{C}}(I\!R_3)-$distribution

$$\sum_{\hat{\sigma},\sigma=-s}^{+s} \rho^{J,M}_{\hat{I}^3 I^3, \hat{\sigma}\sigma}(\omega_M(\vec{q}), \vec{q}; f_{p(m,Y)})_m u^{\hat{\sigma}}_M(\vec{q})_{\hat{\lambda}} u^{\sigma}_M(\vec{q})^c_{\lambda}$$

is in fact a strongly decreasing $\mathcal{C}^{\infty}-$function, ergo from $S(I\!R_3)$. Since $u^{\sigma}_M(\vec{q})_{\lambda}$ is a $\mathcal{O}_{\mathcal{M}}(I\!R_3)-$function it follows by means of the orthonormality relations $\sum_{\lambda} u^{\hat{\sigma}}_M(\vec{q})_{\lambda} u^{\sigma}_M(\vec{q})^c_{\lambda} = 2\omega_M(\vec{q})\delta_{\hat{\sigma}\sigma}$ that $\rho^{J,M}_{\hat{I}^3 I^3, \hat{\sigma}\sigma}(\dots)_m$ is itself from $S(I\!R_3)$.
Lorentz covariance (equation (40)), $\eta-$isometry of $U(\beta, 0)_m$ and equation (48) imply the Lorentz invariance of $\rho^{\dots}_{\dots}(\dots)_m$:

$$\begin{aligned}\rho^{\cdots}_{\cdots}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m &= \rho^{\cdots}_{\cdots}((\omega_M(\vec{q}),\vec{q})\cdot\Lambda(\beta)^{tr};f_{p(\Lambda(\beta)m,Y)})_{\Lambda(\beta)m}\\ &= \rho^{\cdots}_{\cdots}((M,\vec{0})\cdot\Lambda(\beta\alpha^{+}_{M}(\vec{q}))^{tr};f_{p(\Lambda(\beta)m,Y)})_{\Lambda(\beta)m}\end{aligned} \tag{49}$$

Taking for β the inverse $\alpha^{\pm}_{|\delta|M(K)}(\vec{m})^{-1}$ of the boost on the Lorentz orbit $\mathcal{O}^{\pm}_{\delta}$ sending m to rest: $\Lambda(\alpha^{\pm}_{|\delta|M(K)}(\vec{m})^{-1})m = \overset{\circ}{m} = (\pm|\delta|M(K),\vec{0})^{tr}$ we get:
$\rho^{\cdots}_{\cdots}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m = \rho^{\cdots}_{\cdots}((\omega_M(\vec{q}),\vec{q})\cdot\Lambda(\alpha^{\pm}_{|\delta|M(K)}(\vec{m});f_{p(\overset{\circ}{m},Y)})_{\overset{\circ}{m}}$
and from the explicit form of the Lorentz boost in the proof of Lemma I in section II
$(\omega_M(\vec{q}),\vec{q})\cdot\Lambda(\alpha^{\epsilon}_{\kappa}(m)) = \frac{1}{\kappa}(\omega_M(\vec{q})\omega_\kappa(\vec{m})-\epsilon\vec{q}\vec{m},\kappa\vec{q}-\frac{\epsilon\vec{m}}{\kappa+\omega_\kappa(\vec{m})}(\omega_M(\vec{q})\omega_\kappa(\vec{m})-\epsilon\vec{m}\vec{q}+\kappa\omega_M(\vec{q}))$.
Since for fixed $m\in\mathcal{O}^{\pm}_{\delta}$ $\rho^{\cdots}_{\cdots}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m$ is from $S(I\!R_3)$ in $\vec{q}$ it follows from the last two equations via the chain rule that $\rho^{\cdots}_{\cdots}$ is infinitely often continuous differentiable in all six variables $q^i, m^i, i\in\{1,2,3\}$ and together with all its derivatives polynomial bounded; that is $\rho^{\cdots}_{\cdots}\in\mathcal{O}_M(I\!R_6)$. Moreover for fixed $\vec{q}\in I\!R_3$ it is from $S(I\!R_3)$ in $\vec{m}$.
Finally from the HRH-condition (ii), $\eta-$isometry and the equations (48), (49) it follows by means of the orthonormality relation of u^{σ}_{M} for all $(\alpha,r)\in SU(2)^{\mathcal{I}}_{\mathcal{F}}\times SU(2)_m$ (with $SU(2)_m$ the stability group of m specified in section II):
$4\omega_M(\vec{q})^2\rho^{J,M}_{\hat{I}^3I^3,\hat{\epsilon}\epsilon}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m =$
$\sum\limits_{\hat{\sigma},\sigma=-s}^{+s}\sum\limits_{j,\ell=-I}^{+I}\rho^{J,M}_{j\ell,\hat{\sigma}\sigma}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m D^I(r)_{j\hat{I}^3}\overline{D^I_{\ell I^3}(r)}\times$
$\times u^{\hat{\epsilon}}_{M}(\vec{q})^c_{\nu'}S^{1/2}(\alpha^{-1})_{\nu'}{}^{\mu'}u^{\hat{\sigma}}_{M}(\vec{q})_{\mu'}u^{\epsilon}_{M}(\vec{q})_{\nu}\overline{S^{1/2}(\alpha^{-1})}_{\nu}{}^{\mu}u^{\sigma}_{M}(\vec{q})^c_{\mu}$,
where summation over all double indices is understood.
Integration with the Haar measures of the two groups and the orthogonality relations of the irreducible representations $D^I(r)$ respectively the irreducible components $D^{1/2}(\alpha)$ of $S^{1/2}(\alpha) = D^{1/2}(\alpha)\oplus D^{1/2}(\alpha)$ (for $\alpha\in SU(2)_m$) together with the orthonormality relations for u^{σ}_{M} lead to:
$\rho^{J,M}_{\hat{I}^3I^3,\hat{\epsilon}\epsilon}(\ldots)_m = \delta_{\hat{I}^3I^3}\delta_{\hat{\epsilon}\epsilon}\frac{1}{4(I+1)}\sum\limits_{\sigma=-s}^{+s}\sum\limits_{j=-I}^{+I}\rho^{J,M}_{jj,\sigma\sigma}(\ldots)_m =: \delta_{\hat{I}^3I^3}\delta_{\hat{\epsilon}\epsilon}\rho^{J}_{M}(\ldots)_m$ □
Inserting in equation (44) of Lemma II the special $\mathcal{O}_M(I\!R_4)-$functions

$$\tilde{g}_{t,a}(q) := \frac{q^0+\omega_M(\vec{q})}{2\omega_M(\vec{q})}e^{-i(q^0-\omega_M(\vec{q}))t}\hat{f}_a(\vec{q})^c$$

respectively

$$\tilde{g}^{\sigma}_{t,\lambda,a}(q) := \frac{1}{2\omega_M(\vec{q})}e^{-i(q^0-\omega_M(\vec{q}))t}\hat{f}_a(\vec{q})^c u^{\sigma}_{M}(\vec{q})^c_{\lambda}$$

with $\hat{f}_a(q)\in S(I\!R_3), \lambda=\mu$ or ν and summing over ν and μ we obtain the two-point functions of the quasi-particle excitations $B^{\cdots}_{\cdots}(\hat{f}_a,t)^{\dagger}_m$ for $\dagger\in\{+,*\}$:

$$
\begin{aligned}
&(B^{(I,Y,s)}_{M,I^3,\sigma}(\hat f_a,t)^{\dagger}_m\Psi_0(m)|\eta_m B^{(\tilde I,\tilde Y,\tilde s)}_{\tilde M,\tilde I^3,\tilde\sigma}(\hat f_b,t)^{\dagger}_m\Psi_0(m))_m\\
&=(\Psi_0(m)|B^{(I,Y,s)}_{M,I^3,\sigma}(\hat f_a,t)_m B^{(\tilde I,\tilde Y,\tilde s)}_{\tilde M,\tilde I^3,\tilde\sigma}(\hat f_b,t)^{+}_m\Psi_0(m))_m\\
&=\delta_{I\tilde I}\delta_{I^3\tilde I^3}\delta_{Y\tilde Y}\delta_{s\tilde s}\delta_{\sigma\tilde\sigma}\delta_{M\tilde M}\int_{I\!R_3}\frac{d\vec q}{2\omega_M(\vec q)}\hat f_a(\vec q)^c\hat f_b(\vec q)\rho^{(I,Y,s)}_M(\omega_M(\vec q),\vec q;f_{p(m,Y)})_m .
\end{aligned}
\tag{50}
$$

The independence of these relations from $\dagger\in\{+,*\}$ is a trivial consequence of $\eta_m^2=id_{\mathcal H(m)}$ and $\eta_m\Psi_0(m)=\Psi_0(m)$.

A further restriction on the spectral function $\rho_{\dots}^{\dots}$ derives from the presupposed existence of an one-particle multiplet $I^3\in\{-I,-I+1,...,I\},\sigma\in\{-s,s\}$ in the sector with quantum numbers (M,s^P,I,Y) of the so called physical Hilbert space; that means in the direct integral

$$\int_{\oplus\mathcal O^\pm_\delta} d\mu^\pm_\delta(m)N(m)/N_0(m)\text{ with }N_0(m):=N(m)\cap\mathcal H_0(m)$$

and $N(m)$ a maximal η_m-semi-normed subspace of $\mathcal H(m)$ containing the time independent states $B^{(I,Y,s^P)}_{M,I^3,\sigma}(\hat f,t)^+_m\Psi_0(m)$. Given such an one-particle multiplet, *which in addition is composed of a quark and antiquark for* $s=0$ *respectively three (anti-)quarks for* $s=1/2(1\bar/2)$, there must exist for every precompact region $G\subset I\!R_3$ an operator $\phi^{(I,Y,s^P)}_{M,\dots}(f_{p(m,Y)})^+_m$ which means in view of equation (39) some $\tilde f_k(k_1,...,k_{2(s+1)})=\tilde f_M(k_1,...,k_{2(s+1)},k)$ such that
$\int_{\mathcal O^\pm_\delta}d\mu^\pm_\delta(m)<B^{(I,Y,s^P)}_{M,I^3,\sigma}(\hat f,0)^+_m\Psi_0(m),B^{(I,Y,s^P)}_{M,I^3,\sigma}(\hat f,0)^+_m\Psi_0(m)>_m=$
$\int_{\mathcal O^\pm_\delta}d\mu^\pm_\delta(m)\int_{I\!R_3}\frac{d\vec q}{2\omega_M(\vec q)}\hat f(\vec q)^c\hat f(\vec q)\rho^{(I,Y,s^P)}_M(\omega_M(\vec q),\vec q;f_{p(m,Y)})_m>0$
for all $\hat f\in S(I\!R_3)$ with supp.$\hat f\subseteq G$. Since due to Lemma II (i) the integrations can be interchanged there must exist for every G some $\tilde f_M$ such that

$$\int_{\mathcal O^\pm_\delta}d\mu^\pm_\delta(m)\rho^{(I,Y,s^P)}_M(\omega_M(\vec q),\vec q,f_{p(m,Y)})_m>0 \tag{51}$$

for almost all $\vec q\in G$.
But then the method of Ref.[15] can be used to construct by means of Lorentz transformations (via taking linear combinations and avaraging over Lorentz transformations with a sufficiently strong decreasing measure $\mu(\Lambda(\alpha))$) *a* $\phi^{(I,Y,s^P)}_{M,\dots}(f_{p(m,Y)})^+_m$ respectively $\tilde f_M$ such that *the inequality* (51) *holds for all* $\vec q\in I\!R_3$. An obvious consequence is that the direct integral of the equivalence classes generated by the states $B^{(I,Y,s^P)}_{M,I^3,\sigma}(\hat f,t)^+_m\Psi_0(m)$ with $\hat f(\vec q):=h(\vec q)[\int_{\mathcal O^\pm_\delta}d\mu^\pm_\delta(m)\rho^{(I,Y,s^P)}_M(\omega(\vec q),\vec q;f_{p(m,Y)})_m]^{-1/2}$ and $h\in S(I\!R_3)$ of compact support span pre-Hilbert spaces the completions of which are unitary equivalent to $\mathcal L^2(I\!R_3,d\vec q/2\omega_M(\vec q))$. For the construction of the corresponding Fock-space of asymptotic collision states the following positivity property of $\rho_{\dots}^{\dots}$, which is stronger than (51) for all $\vec q\in I\!R_3$, is needed.

Positivity Condition: For every one-particle multiplet with quantum numbers (M, I, Y, s^P) there exists a $\tilde{f}_M$ such that

$$\forall_{(q,m)\in \mathcal{Q}_M^+ \times \mathcal{O}_\delta^\pm} \rho_M^{(I,Y,s^P)}(\omega_M(\vec{q}), \vec{q}; f_{p(m,Y)})_m > 0$$

$\div$

It should be pointed out that this additional condition has its origin in the direct integral structure of the physical Hilbert space enforced by the flavor symmetry and/or color confining representation of the translation group and *not* in the indefinite metric, since in case the Lorentz orbit $\mathcal{O}_\delta^\pm$ contains only one point $m = 0$ with $d\mu_\delta^\pm(m) = \delta^4(m)dm$ it holds true independently from η_0 being trivial or not.

Since except from replacing commutators by anticommutators and symmetrization by antisymmetrization both cases $s = 0$ and $s = 1/2$ run in the same way, we restrict the further procedures chiefly to the spin zero case and drop spin indices. Finally in short hand notation the channel fixing triple of indices (I, Y, s) is abbreviated by J and it is assumed $M(J) = M(\hat{J}) \Leftrightarrow J = \hat{J}$. Thus $\phi_{I^3}^J(\cdots)_m$ stands for $\phi_{M,I^3}^{(I,Y,s)}(\cdots)_m$, $B_{I^3}^J(\hat{f}, t)_m$ for $B_{M,I^3}^{(I,Y,s)}(\hat{f}, t)_m$ and $\delta_{J\tilde{J}}$ for $\delta_{I\hat{I}}\delta_{Y\tilde{Y}}\delta_{s\tilde{s}}$.

V. Asymptotic Quasi-Particle States

In this section we construct via the methods of Ref.[9,10] and especially Ref.[13,14] in each Hilbert space $(\mathcal{H}(m), (\cdot,\cdot)_m)$ asymptotic outgoing respectively incoming quasi-particle states which span subspaces with *non-negative* η_m- *"norm"*. For the sake of notational shortness we introduce some abbreviations.

$$\begin{aligned} &\Omega_{[J]_L}(\vec{q}_1, ..., \vec{q}_L) = \Omega_{J_1,...,J_L}(\vec{q}_1, ..., \vec{q}_L) := \\ &[\int_{O_\delta^\pm} d\mu_\delta^\pm(\hat{m}) \Pi_{j=1}^L \rho_{M_j}^{J_j}(\omega_{M_j}(\vec{q}_j), \vec{q}_j; f_{p(\hat{m},Y_j)})_{\hat{m}}]^{-1/2} \end{aligned} \tag{52}$$

Due to Lemma II and the positivity condition $\Omega_{[J]_L}$ is from $\mathcal{C}^\infty(I\!R_{3L})$, Lorentz-invariant [i.e. $\Omega_{[J]_L}(\Lambda(\overrightarrow{\alpha})q_1, ..., \Lambda(\overrightarrow{\alpha})q_L) = \Omega_{[J]_L}(\vec{q}_1, ..., \vec{q}_L)$], strictly positive and totally symmetric, i.e. for all π from the permutation group S_L we have $\Omega_{J_{\pi(1)},...,J_{\pi(L)}}(\vec{q}_{\pi(1)}, ..., \vec{q}_{\pi(L)}) = \Omega_{J_1,...,J_L}(\vec{q}_1, ..., \vec{q}_L)$. For arbitrary $\hat{f}_{a_j} \in S(I\!R_3)$ with *compact support* we define:

$$\begin{aligned} &\tilde{g}_{t;[a]_L}^{[J]_L}(q_1, ..., q_L) = \tilde{g}_{t;a_1,...,a_L}^{J_1,...,J_L}(q_1, ..., q_L) := \\ &\Omega_{[J]_L}(\vec{q}_1, ..., \vec{q}_L) \Pi_{j=1}^L \frac{q_j^0 + \omega_{M_j}(\vec{q}_j)}{2\omega_{M_j}(\vec{q}_j)} e^{-i(q_j^0 - \omega_{M_j}(\vec{q}_j))t} \hat{f}_{a_j}(\vec{q}_j)^c \end{aligned} \tag{53}$$

$\tilde{g}_{t;a_L}^{[J]_L}$ is from $\mathcal{O}_M(I\!R_{4L})$, of compact support in $(\vec{q}_1, ..., \vec{q}_L)$ and again totally symmetric.

Then in generalization of the quasi-particle excitations $B^J_{I^3}(\Omega_J \hat{f}_a, t)^+_m = \phi^J_{I^3}(g^J_{t;a}, p(m,Y))^+_m$ $L-$fold product operators are defined by:

$$
\begin{aligned}
&B^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j}; t)^\dagger_m := \\
&\int_{\mathbb{R}_{4L}} dx\ g^{[J]_L}_{t;[a]_L}(x_1, ..., x_L)^c \Pi^L_{j=1} \phi^{J_j}_{I^3_j}(x_j, p(m, Y_j))^\dagger_m
\end{aligned}
\tag{54}
$$

$g^{[J]_L}_{t;[a]_L}$ is the inverse Fourier-transform of $\tilde{g}^{[J]_L}_{t;[a]_L}$ and $\dagger \in \{+, *\}$.
Via developing the vacuum expectation value of the $L-$fold field product in equation (54) into truncated ones it follows in literally the same way as in Ref.[9,10,12] respectively Ref.[13,14] by means of the cluster theorems of section III, the equations (42), the boundedness of η_m and $\eta_m \Psi_0(m) = \Psi_0(m)$ that the strong Haag-Ruelle-limits

$$
\begin{aligned}
&\Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j})^{\mathrm{out/in}}_m := \underset{t\to +/-\infty}{s-\lim} B^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j}; t)^+_m \Psi_0(m) \\
&= \eta_m \underset{t\to +/-\infty}{s-\lim} B^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j}; t)^*_m \Psi_0(m)
\end{aligned}
\tag{55}
$$

exist in the color neutral closed subspace $(\overline{\mathcal{D}_0(m)}, (\cdot|\cdot)_m)$ of $(\mathcal{H}(m), (\cdot|\cdot)_m)$ and their matrix elements $< \cdot, \cdot >_m = (\cdot|\eta \cdot)_m$ are given by:

$$
\begin{aligned}
&< \Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j})^{\mathrm{out/in}}_m, \Psi^{[\hat{J}]_K}_{[\hat{I}^3]_K}(\Omega_{[\hat{J}]_K} \otimes^K_{k=1} \hat{f}_{b_k})^{\mathrm{out/in}}_m >_m = \\
&\lim_{t\to +/-\infty} (B^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j}; t)^+_m \Psi_0(m) | B^{[\hat{J}]_K}_{[\hat{I}^3]_K}(\Omega_{[\hat{J}]_K} \otimes^K_{k=1} \hat{f}_{b_k}; t)^*_m \Psi_0(m))_m = \\
&\delta_{KL} \sum_{\pi \in S_L} \int_{\mathbb{R}_{4L} \times \mathbb{R}_{4L}} dy\ dx\ g^{[J]_L}_{0;[a]_L}(y_1, ..., y_L) g^{[\hat{J}]_L}_{0;[b]_L}(x_1, ..., x_L)^c \times \\
&\times \Pi^L_{j=1}(\phi^{J_j}_{I^3_j}(y_j, p(m, Y_j))^+_m \Psi_0(m) | \eta_m \phi^{\hat{J}_{\pi(j)}}_{\hat{I}^3_{\pi(j)}}(x_{\pi(j)}, p(m, \hat{Y}_{\pi(j)}))^+_m \Psi_0(m))_m
\end{aligned}
\tag{56}
$$

The limits in (55) and (56) are aproached like $|t|^{-1/2}$ and even faster than $|t|^{-N}$ for arbitrary natural N if the set $\{\hat{f}_{a_j} \in S(\mathbb{R}_3) | j = 1, ..., L\}$ is nonoverlapping in velocity space[13,14]; that means if for all $\vec{q}_i \in supp.\hat{f}_{a_i}$ one has (even if $M_i = M_j$)$\vec{q}_i \omega_{M_i}(\vec{q}_i)^{-1} \neq \vec{q}_j \omega_{M_j}(\vec{q}_j)^{-1}$ for $i \neq j$. Moreover the equations (55) and (56) remain true if the tensor products $\otimes^L_{j=1} \hat{f}_{a_j}$ are replaced by an arbitrary function $\hat{h} \in S(\mathbb{R}_{3L})$ with compact support and the limits are uniform in $(\vec{y}_1, ..., \vec{y}_L) \in \mathbb{R}_{3L}$ for all space-translated wave-functions $\hat{f}_{a,\vec{y}}(\vec{q}) := e^{-i\vec{q}\vec{y}} \hat{f}_a(\vec{q})$.
Inserting the equations (52), (53) and (44) of Lemma II into (56) we end up with:

$$
\begin{aligned}
&< \Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes^L_{j=1} \hat{f}_{a_j})^{\mathrm{out/in}}_m, \Psi^{[\hat{J}]_K}_{[\hat{I}^3]_K}(\Omega_{[\hat{J}]_K} \otimes^K_{k=1} \hat{f}_{b_k})^{\mathrm{out/in}}_m >_m = \\
&\delta_{KL} \sum_{\pi \in S_L} (\Pi^L_{j=1} \delta_{M_j \hat{M}_{\pi(j)}} \delta_{I_j \hat{I}_{\pi(j)}} \delta_{I^3_j \hat{I}^3_{\pi(j)}} \delta_{Y_j \hat{Y}_{\pi(j)}}) \times \\
&\times \int_{\mathbb{R}_{3L}} d\vec{q} [\Pi^L_{k=1} \frac{1}{2\omega_{M_k}(\vec{q}_k)} \hat{f}_{a_k}(\vec{q}_k)^c \hat{f}_{b_{\pi(k)}}(\vec{q}_k) \rho^{(I_k, Y_k, 0)}_{M_k}(\omega_{M_k}(\vec{q}_k), \vec{q}_k; f_{p(m, Y_k)})_m] \times \\
&\times [\int_{\mathcal{O}^\pm_\delta} d\mu^\pm_\delta(\hat{m}) \Pi^L_{\ell=1} \rho^{(I_\ell, Y_\ell, 0)}_{M_\ell}(\omega_{M_\ell}(\vec{q}_\ell), \vec{q}_\ell; f_{p(\hat{m}, Y_\ell)})_{\hat{m}}]^{-1}
\end{aligned}
\tag{57}
$$

It should be pointed out in passing, that integrating both sides over the Lorentz-orbit $\mathcal{O}_\delta^{\pm}$ (,which will be done in the last step to the physical Hilbert space(s) below,) then due to Lemma II the $\mu_\delta^{\pm}-$and $\vec{q}-$integrations on the right hand side can be interchanged and we obtain just the scalar product of the asymptotic $L-$particle collision states of a Wightman theory.
If one or more Fermions with quantum numbers $J_j = (I_j, Y_j, s_j, M_j)$ are involved then in the equations (53)-(57) $\hat{f}_{a_j}$ has to be replaced by $\hat{f}_{a_j} u_{M_j}^{\sigma_j}$, $(\mathrm{q}_j^0 - \omega_{M_j}(\vec{q}))$ deleted in (53) and a factor $(-1)^{N_F(\pi)}$ to be added in equation (56) respectively $(-1)^{N_F(\pi)} \delta_{s_j \hat{s}_{\pi(j)}} (2\omega_{M_k}(\vec{q}_k))^{-1}$ on the right hand side of equations (57). $N_F(\pi)$ is the number of interchanges of Fermi-fields in the developement of the vacuum expectation values into truncated ones and therefore also the number of interchanges of Fermion wave functions in the product $\hat{f}_{b_{\pi(1)}}(\vec{q}_1) \cdot ... \cdot \hat{f}_{b_{\pi(L)}}(\vec{q}_L)$ compared to the original order $\hat{f}_{b_1}(\vec{q}_1) \cdot ... \cdot \hat{f}_{b_L}(\vec{q}_L)$ in equation (57). On the right hand side of the equations (54) and (56) $g_{t;[a]_L}^{[J]_L}$ inherited from the $u_M^\sigma(\vec{q})_\lambda$ two further $L-$tuples $[\sigma]_L$ and $[\lambda]_L([\lambda']_L)$ of indices; $\phi_{I_j^3}^{J_j}(\phi_{I_{\pi(j)}^3}^{J_{\pi(j)}})$ has to be replaced by $\phi_{I_j^3,\lambda_j}^{J_j}(\phi_{I_{\pi(j)}^3, \lambda'_{\pi(j)}}^{J_{\pi(j)}})$ and summation over all double indices $\lambda_j(\lambda'_j)$ to be performed. In the following deletion in case $s = 0$ is indicated by an additional bracket; for instance (u_M^σ), $(\bar{\otimes}\mathbb{C}_{2(2s+1)})$ etc.
In any case the equation (57) together with the Positivity Condition on the two-point functions of the quasi-particle excitations imply for ex$\in$ {out,in} and $L \geq 1$:

$$< \Psi_{[I^3]_L}^{[J]_L}(\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j}(u_{M_j}^{\sigma_j}))_m^{\mathrm{ex}}, \Psi_{[I^3]_L}^{[J]}(\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j}(u_{M_j}^{\sigma_j}))_m^{\mathrm{ex}} >_m > 0 \qquad (58)$$

This can be most easily deduced from the observation that the right hand side of the equation (57) (as well as its analogen for $s = 1/2$) is just the (Hilbert-)scalar product of a state with wave function $\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j} e_{I_j^3}^{I_j,Y_j}(u_{M_j}^{\sigma_j})$ in the Fock-space over the $\mathcal{L}^2-$spaces $\mathcal{L}^2(\mathbb{R}_3, d\vec{q}[2\omega_M(\vec{q})]^{-1}\rho_M^{(I,Y,s)}(\omega_M(\vec{q}), \vec{q}; f_{p(m,Y)})_m \bar{\otimes}\mathbb{C}_{2I+1}(\bar{\otimes}\mathbb{C}_{2(2s+1)})$. Here for fixed $(I, Y) \in \{(1/2, 1), (1, 0), (0, 0), (1/2, -1)\}$ $e_{I^3}^{I,Y}$, $I^3 = I, I-1, ..., -I$ denotes the canonical base in $\mathbb{C}_{2I+1}$ equiped with the scalar product $(z, w)_{\mathbb{C}} := z^+ w$. Therefore the asymptotic incoming respectively outgoing quasi-particle states together with $\Psi_0(m)$ span two subspaces $D_{\mathrm{ex}}(m)$ of $\mathcal{H}(m)$ such that on their closures $H_{\mathrm{ex}}(m) := \overline{D_{\mathrm{ex}}(m)}$ (with respect to the Hilbert norm) $||\Psi_m^{\mathrm{ex}}||_{\eta m} := \sqrt{< \Psi_m^{\mathrm{ex}}, \Psi_m^{\mathrm{ex}} >_m}$ defines a semi-norm. Both spaces have besides $\Psi_0(m)$ at least all one-quasi-particle states in common: $\Psi_{I^3}^J(\Omega_J \otimes \hat{f}_a(u_M^\sigma))_m^{\mathrm{out}} = \Psi_{I^3}^J(\Omega_J \otimes \hat{f}_a(u_M^\sigma))_m^{\mathrm{in}}$. In order to make the structure of $D_{ex}(m)$ and the positive definitness of $< \cdot, \cdot >_m$ on it more precise we first observe that by means of the canonical base $e_{I^3}^{IY}$ and $g_{t;[a]_L[I^3]_L([\sigma]_L)}^{[J]_L[\hat{I}^3]_L([\lambda]_L)} := g_{t;[a]_L([\sigma]_L)}^{[J]_L([\lambda]_L)} \Pi_{j=1}^L e_{I_j^3 \hat{I}_j^3}^{I_j,Y_j}$ the product operators (54) can be rewritten as:

$$\begin{aligned} & B_{[I^3]}^{[J]_L}(\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j}(u_{M_j}^{\sigma_j}); t)_m^\dagger = \\ & \textstyle\sum_{[\hat{I}^3]_L}(\sum_{[\lambda]_L}) \int_{\mathbb{R}_{4L}} dx \; g_{t;[a]_L[I^3]_L([\sigma]_L)}^{[J]_L[\hat{I}^3]_L([\lambda]_L)}(x)^c \Pi_{j=1}^L \phi_{\hat{I}_j^3,(\lambda_j)}^{J_j}(x_j; p(m, Y_j))_m^\dagger \\ & := B^{[J]_L}(\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j}(u_{M_j}^{\sigma_j}) e_{I_j^3}^{I_j,Y_j}; t)_m^\dagger \end{aligned} \qquad (59)$$

Now consider what will be called the Fock-symmetrized tensor products:

$$h^{[J]_L}_{[a]_L[I^3]_L([\sigma]_L)} := 1/L! \sum_{\pi\in S_L} \Omega_{[J]_{\pi L}} \otimes_{j=1}^{L} \hat{f}_{a_{\pi(j)}}((-1)^{N_F(\pi)} u^{\sigma_{\pi(j)}}_{M_{\pi(j)}}) e^{I_{\pi(j)},Y_{\pi(j)}}_{I^3_{\pi(j)}} \quad \text{with } [J]_{\pi L} := (J_{\pi(1)}, J_{\pi(2)}, ..., J_{\pi(L)}). \tag{60}$$

Then it follows from the equation (55) and (59) (since strong convergence implies weak convergence):

$$\begin{aligned}\Psi(\textstyle\sum_{[a]_L[I^3]_L([\sigma]_L)} \alpha_{[a]_L[I^3]_L([\sigma]_L)} h^{[J]_L}_{[a]_L[I^3]_L([\sigma]_L)})^{\text{ex}}_m = \\ \textstyle\sum_{[a]_L[I^3]_L([\sigma]_L)} \alpha_{[a]_L[I^3]_L([\sigma]_L)} \Psi(h^{[J]_L}_{[a]_L[I^3]_L([\sigma]_L)})^{\text{ex}}_m =\end{aligned} \tag{61}$$

and clearly $\Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j}))^{\text{ex}}_m = \Psi(\Omega_{[J]_L} \otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j} e^{I_j,Y_j}_{I^3_j})^{\text{ex}}_m$. Therefore on the one hand the assignment $\mathcal{I}^{\text{ex}}_{m,[J]_L} : h^{\cdots}_{\cdots} \to \Psi(h^{\cdots}_{\cdots})^{\text{ex}}_m$ defines an isomorphism from the vector space

$$\mathcal{K}^{[J]_L} := LH\{h^{[J]_L}_{[a]_L[I^3]([\sigma]_L)} | \hat{f}_{a_j} \in S_c(I\!R_3), I^3_j \in \{I_j, I_j - 1, ..., -I_j\}, (\sigma_j \in \{1/2, -1/2\})\} \tag{62}$$

onto the subspace $D^{[J]_L}_{ex}(m) := \{\Psi(h)^{\text{ex}}_m | h \in \mathcal{K}^{[J]_L}\}$, where $S_c(I\!R_n)$ is the subspace of functions from $S(I\!R_n)$ with compact support. On the other hand, if $\underline{h}$ denotes the equivalence class generated by $h \in \mathcal{K}^{[J]_L}$ in the Fock-symmetrized $L-$fold tensor product $\mathcal{L}^{2\otimes}_{[J]_L}$ of the $\mathcal{L}^2-$spaces spezified before, then the natural assigenement $\mathcal{I}^{[J]_L}_{\mathcal{L}^2} : h \to \underline{h}$ defines an isomorphism from $\mathcal{K}^{[J]_L}$ onto a subspace of $\mathcal{L}^{2\otimes}_{[J]_L}$ and the equation (57) implies that the composition $\mathcal{I}^{[J]_L}_{\mathcal{L}^2} \circ (\mathcal{I}^{\text{ex}}_{m,[J]_L})^{-1}$ is an isometric isomorphism from $(D^{[J]_L}_{ex}(m), < \cdot, \cdot >_m)$ onto a subspace of $\mathcal{L}^{2\otimes}_{[J]_L}$. Hence $< \cdot, \cdot >_m$ is positive definite and $|| \cdot ||_{\eta m}$ a norm on $D^{[J]_L}_{ex}(m)$.

Obviously $D^{[J]_L}_{ex}(m) = D^{[\hat{J}]_L}_{ex}(m)$ if and only if $\exists_{\pi\in S_L} [\hat{J}]_L = [J]_{\pi L}$ and in the opposite case $D^{[J]_L}_{ex}(m) \cap D^{[\hat{J}]_L}_{ex}(m) = \{\mathcal{O}_m\}$. The same holds true for the spaces $\mathcal{K}^{[J]_L}$ respectively $\mathcal{L}^{2\otimes}_{[J]_L}$. Hence two $L-$tuples $[J]_L$ and $[\hat{J}]_L$ of triples $J_j = (I_j, Y_j, s_j)$ respectively $\hat{J}_j = (\hat{I}_j, \hat{Y}_j, \hat{s}_j)$ are said to be *substantial different* if $\forall_{\pi\in S_L} [\hat{J}]_L \neq [J]_{\pi L}$. If $\triangle^L_{psd}$ denotes the set of all *pairwise substantial different* $L-$tuples and $D^{[J]_0}_{ex}(m) := \{d\Psi_0(m) | d \in \mathbb{C}\}$ the vacuum subspace then $D_{ex}(m)$ is the algebraic direct sum $D_{ex}(m) = \dot{+}_{L\in I\!N_0} (\dot{+}_{[J]_L\in\triangle^L_{psd}} D^{[J]_L}_{ex}(m))$ and a $|| \cdot ||_{\eta_m}-$dense subspace of the η_m-orthogonal direct sum

$$D^{\oplus}_{ex}(m) = \oplus_{L\in I\!N_0}(\oplus_{[J]_L\in\triangle^L_{psd}} D^{[J]_L}_{ex}(m)). \tag{63}$$

Both are isometrical isomorphic to corresponding subspaces of the Fock-space $\mathcal{F} = \oplus_{L\in I\!N_0}(\oplus_{[J]_L\in\triangle^L_{psd}} \mathcal{L}^{2\otimes}_{[J]_L})$ with $\mathcal{L}^{2\otimes}_{[J]_0} := (\mathbb{C}, (\cdot, \cdot)_{\mathbb{C}})$.

Apart from a domain problem due to the unboundedness of the ratation free Lorentz-operators the covariance properties of the asymptotic quasi-particle states are obtained in the same way as in Wightman theories. Since translations, rotations and internal symmetry transformations are represented by unitary operators on $\overline{\mathcal{D}_0(m)} \supseteq H_{ex}(m)$, the

Haag-Ruelle limits are in their domain. As well they are caught into the domain of the *closure* of the rotation free Lorentz operators $\mathcal{L}(\alpha)_m$ if the following equation holds, from which at the same time the transformation properties of the asymptotic quasi-particle states can immediately be read off for all $(\alpha, y, v, (r,t)) \in (SL(2,\mathbb{C})\times)\mathbb{R}_4) \times (U(1)^{\mathcal{Q}}_{\mathcal{C}} \times SU(2)^{\mathcal{U}}_{\mathcal{C}})) \times (SU(2)^{\mathcal{I}}_{\mathcal{F}} \times U(1)^{\mathcal{Y}}_{\mathcal{F}})$:

$$\begin{aligned}
&\underset{t\to+/-\infty}{s\text{-}\lim}\,\mathcal{U}(\alpha,y)_m I_m(v,(r,\tau))B^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L}\otimes^L_{j=1}\hat{f}_{a_j};t)^+_m\Psi_0(m)=\\
&(\Pi^L_{k=1}D^{I_k}(r)_{\hat{I}^3_k I^3_k}e^{iY_k\tau})\Psi^{[J]_L}_{[\hat{I}^3]_L}(\Omega_{[J]_L}\otimes^L_{j=1}(\hat{f}_{a_j})_{(\Lambda(\alpha),y)})^{\text{out/in}}_{\Lambda(\alpha)m}\\
&\text{with } (\hat{f}_{a_j})_{(\Lambda(\alpha),y)}(\vec{q}):=e^{i(\omega_{M_j}(\vec{q})y^0-\vec{q}\vec{y})}\hat{f}_{a_j}(\Lambda(\overrightarrow{\alpha^{-1}})q)
\end{aligned}$$

For Fermions $\hat{f}_{a_j}$ has to be replaced by $\hat{f}_{a_j}u^{\sigma_j}_{M_j}$ on the left hand side respectively $\hat{f}_{a_j}S^{(n,n)}(\bar{\alpha})u^{\sigma_j}_{M_j}$ on the right. By means of the equations (52)-(54), the transformation properties (29) respectively (40) of the quasi-particle excitations and the Lorentz invariance of $\Omega_{[J]_L}$ it follows that this condition is equivalent with the following one (both for all m):

$$\begin{aligned}
&s-\lim_{t\to+/-\infty}\int_{R_{4L}}dx\int_{R_{4L}}dq\,\Omega_{[J]_L}(\vec{q}_1,...,\vec{q}_L)[\Pi^L_{j=1}e^{-i(x_k-y)q_k}\hat{f}_{a_j}(\Lambda(\overrightarrow{\alpha^{-1}})q_j)\times\\
&\phi^{J_j}_{I^3_j}(x_j,p(m,Y_j))^+_m]\Psi_0(m)\{\Pi^L_{\ell=1}\frac{q^0_\ell+\omega_{M_\ell}(\vec{q}_\ell)}{2\omega_{M_\ell}(\vec{q}_\ell)}e^{i(q^0_\ell-\omega_{M_\ell}(\vec{q}_\ell))t}\\
&-\Pi^L_{k=1}\frac{(\Lambda(\alpha^{-1})q_k)^0+\omega_{M_k}(\overrightarrow{\Lambda(\alpha^{-1})q_k})}{2\omega_{M_k}(\overrightarrow{\Lambda(\alpha^{-1})q_k})}e^{i((\Lambda(\alpha^{-1})q_k)^0-\omega_{M_k}(\overrightarrow{\Lambda(\alpha^{-1})q_k}))t}\}=0
\end{aligned}$$

Since it holds trivially for all rotations and translation, it must only be proved for the one-parameter group of special Lorentz transformations in the $(q^0,q^1)-$plane, and this proof can be copied step by step from the proof of the covariance of the asymptotic creation and anihilation operators in Wightman theories given on page 190 in Ref.[14].
Denoting in the following the closure of $\mathcal{U}(\alpha,y)_m$ like that of $\mathcal{L}(\alpha)_m$ by the same symbol, then obiously $\mathcal{U}(\alpha,y)_m$ maps $D_{ex}(m)$ onto $D_{ex}(\Lambda(\alpha)m)$ and therefore into the domain of $\mathcal{U}(\beta,y)_{\Lambda(\alpha)m}$ for all $(\beta,y)\in SL(2,\mathbb{C})\times)\mathbb{R}_4$.
The results obtained so far are collected in the following theorem:

Theorem I: For any collection of local and color neutral composite field operators $\phi^J_{M,\nu,I^3}(f_{p(m,Y)})_m$, $J=(I,Y,s^P)$ satisfying the Haag-Ruelle-Hepp- and Positivity Condition the strong Haag-Ruelle-limits

$$\Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L}\otimes^L_{j=1}\hat{f}_{a_j}(u^{\sigma_j}_{M_j}))^{\text{out/in}}_m := s-\lim_{t\to+/-\infty}B^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L}\otimes^L_{j=1}\hat{f}_{a_j}(u^{\sigma_j}_{M_j});t)^+_m\Psi_0(m)$$

exist for all $\hat{f}_a\in S(\mathbb{R}_3)$ with compact support and span together with $\Psi_0(m)$ two subspaces $D_{\text{out/in}}(m)\subseteq\overline{\mathcal{D}_0(m)}$ with the properties:

- (i): $||\Psi^{\text{ex}}_m||_{\eta_m} := \sqrt{<\Psi^{\text{ex}}_m,\Psi^{\text{ex}}_m>_m}$ *defines a norm on $D_{ex}(m)$ and a semi-norm on the Hilbert closure $H_{\text{ex}}(m)=\overline{D_{\text{ex}}(m)}$.*

(ii): *The $\eta-$isometric symmetry operators* $\mathcal{U}(\alpha,y)_m I_m(v,(r,\tau))$ *map* $D_{\rm ex}(m)$ *onto* $D_{\rm ex}(\Lambda(\alpha)m)$ (= $D_{\rm ex}(m)$ for $\alpha\in SU(2)_m$) *and act on* $D_{\rm ex}(m)$ *like*:

$$\begin{array}{l}\mathcal{U}(\alpha,y)_m I_m(v,(r,\tau))\Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j}))^{\rm ex}_m = \\ (\Pi_{k=1}^L D^{I_k}(r)_{\hat{I}^3_k I^3_k} e^{iY_k\tau})\Psi^{[J]_L}_{[\hat{I}^3]_L}(\Omega_{[J]_L}\otimes_{j=1}^L (\hat{f}_{a_j}(S^{(n,n)}(\bar{\alpha})u^{\sigma_j}_{M_j}))_{(\Lambda(\alpha),y)})^{\rm ex}_{\Lambda(\alpha)m}\end{array} \quad (64)$$

$\div$

A further essential for the last step to the physical Hilbert space is the independence of the asymptotic quasi-particle states the choice of the quasi-particle excitation operators $\phi^J(f_{p(m,Y)})_m$, that is in the cases hat hand the independence from the choice of the functions $f_{p(m,Y)}(x_1,...,x_{2(s+1)}) = f_M(x_1,...,x_{2(s+1},p(m,Y))$ resulting from the solution of condition (31).

Corollary I: *Let* $\phi^J_{...}(f_{p(m,Y)})^+_m$ *and* $\phi^J_{...}(h_{p(m,Y)})^+_m$ *be quasi-particle excitation operators with* $\phi^J_{...}(f_{p(m,Y)})^+_m\Psi_0(m) = \phi^J_{...}(h_{p(m,Y})^+_m\Psi_0(m)$ *and* $\Psi^{[J]_L}_{...}(\Omega^f_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j})|f)^{\rm ex}_m$ *respectively* $\Psi^{[J]_L}_{...}(\Omega^h_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j})|h)^{\rm ex}_m$ *be the asymptotic quasi-particle states constructed from them. Then for all* $L\in I\!N$: $\Psi^{[J]_L}_{...}(\Omega^h_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j})|h)^{\rm ex}_m = \Psi^{[J]_L}_{...}(\Omega^f_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j})|f)^{\rm ex}_m$

$\div$

Proof: From $\phi^J_{...}(f_{p(m,Y})^+_m\Psi_0(m) = \phi^J_{...}(h_{p(m,Y)})^+_m\Psi_0(m)$ and Lemma II it follows $\rho^J_{...}(\omega_M(\vec{q}),\vec{q};f_{p(m,Y)})_m = \rho^J_{...}(\omega_M(\vec{q}),\vec{q};h_{p(m,Y)})_m$ and therefore $\Omega^h_{[J]_L} = \Omega^f_{[J]_L} = \Omega_{[J]_L}$. In the same way as in the proof of the Haag-Ruelle limits (via developing the vacuum expectation values into truncated ones and applying the cluster theorems and the equations (42) it follows that the Hilbert scalar product $(\cdots|\cdots)_m$ of incoming and outgoing states is given by the right hand side of equation (56) with η_m deleted in the two-point functions. But then the Corollary I follows by calculating the Hilbert norm:

$$||\Psi^{[J]_L}_{...}(\Omega_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j})|f)^{\rm ex}_m - \Psi^{[J]_L}_{...}(\Omega_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j})|h)^{\rm ex}_m||_{\mathcal{H}(m)} = 0 \qquad \Box$$

Finally we introduce in every $\mathcal{H}(m)$ an operator $S(m)$ from which the scattering matrix will be constructed in the next section. Due to the scalar product (56) and its Hilbert analogon the equations

$$\begin{array}{l}S(m)\Psi_0(m) := \Psi_0(m) \\ S(m)\Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j}))^{\rm out}_m := \Psi^{[J]_L}_{[I^3]_L}(\Omega_{[J]_L}\otimes_{j=1}^L \hat{f}_{a_j}(u^{\sigma_j}_{M_j}))^{\rm in}_m\end{array} \quad (65)$$

define (via continuous extension) a bounded operator with domain $H_{\rm out}(m)$ and range $H_{\rm in}(m)$ which is isometric with respect to both $(\cdot|\cdot)_m$ and $<\cdot,\cdot>_m$. Supplementing this definition by $S(m)\Psi(m) := \Psi(m)$ for all $\Psi(m)\in H_{\rm out}(m)^\perp$ we obtain a unitary and η_m-isometric operator in $\mathcal{H}(m)$ which mapp $H_{\rm out}(m)$ onto $H_{\rm in}(m)$.

This leads us directly to the problem of asymptotic completeness. In non-gauge Wightman theories, which means $\eta_m = id_{\mathcal{H}(m)}, \mathcal{H}(m) = H(m) \supseteq H_{ex}(m)$, the asymptotic completeness $H_{out}(m) = H_{in}(m) = \mathcal{H}(m)$ has to be put in as an independent axiom[12,14]. For the existence of an *unitary* $S-$matrix however it would be sufficient to postulate only $H_{out}(m) = H_{in}(m) \subseteq \mathcal{H}(m)$. The axiom of asymptotic completeness is obviously equivalent with the demand that $H_{out}(m)$ and $H_{in}(m)$ are both maximal in $\mathcal{H}(m)$, that means: If $\mathcal{H}'(m)$ is a subspace with $H_{ex}(m) \subseteq \mathcal{H}'(m)$ then $H_{ex}(m) = \mathcal{H}'(m)$. This equivalent version of the asymptotic completeness can easily be carried over to the gauge-theory at hand by means of the following Lemma which says that $||\cdot||_{\eta m}$ is a norm even on $H_{ex}(m)$.

Lemma III: If $\underline{H}_{ex}(m) = \overline{D_{ex}(m)}^{\eta_m}$ denotes the completion of the pre-Hilbert spaces $(D_{ex}(m), <\cdot,\cdot>_m)$ with respect to the norm $||\cdot||_{\eta m}$, then $H_{ex}(m) \subseteq \underline{H}_{ex}(m)$, that means $H_{ex}(m) \cap \mathcal{H}_0(m) = \{O\}$

$\div$

Proof: Since $||\Psi^{\text{ex}}_m||_{\eta m} \leq ||\Psi^{\text{ex}}_m||_m$ every Cauchy sequence in $(D_{ex}(m), ||\cdot||_m)$ is a Cauchy sequence in $(D_{ex}(m), ||\cdot||_{\eta m})$. Hence for every $\Psi^{\text{ex}}_m \in H_{ex}(m)$ there exists a Cauchy sequence $(\Psi^{\text{ex}}_{m,n})_{n\in I\!N}$ in $(D_{ex}(m), ||\cdot||_m)$ and a $\underline{\Psi}^{\text{ex}}_m \in \underline{H}_{ex}(m)$ with $\lim_{n\to\infty} ||\Psi^{\text{ex}}_m - \Psi^{\text{ex}}_{m,n}||_m = 0$ and $\lim_{n\to\infty} ||\underline{\Psi}^{\text{ex}}_m - \Psi^{\text{ex}}_{m,n}||_{\eta m} = 0$. Since $||\cdot||_{\eta m}$ is a semi-norm on $H_{ex}(m)$ there exists for every $\epsilon > 0$ a $N(\epsilon) \in I\!N$ such that for all $n \geq N(\epsilon)$ $||\underline{\Psi}^{\text{ex}}_m - \Psi^{\text{ex}}_m||_{\eta m} \leq ||\underline{\Psi}^{\text{ex}}_m - \Psi^{\text{ex}}_{m,n}||_{\eta m} + ||\Psi^{\text{ex}}_m - \Psi^{\text{ex}}_{m,n}||_m < \epsilon$ and therefore $\Psi^{\text{ex}}_m = \underline{\Psi}^{\text{ex}}_m \in \underline{H}_{ex}(m)$. Since $(\underline{H}_{ex}(m), ||\cdot||_{\eta m})$ is a Banach space(s) it follows $||\Psi^{\text{ex}}_m||_{\eta m} = 0 \Longleftrightarrow \Psi^{\text{ex}}_m = \mathcal{O}_m$. $\square$

In the gauge-theory at hand with $\eta_m \neq id_{\mathcal{H}(m)}$ due to the Zorn-Lemma the set of all subspaces $\{\underline{H}^{\text{ex}}_\alpha(m) \subset \mathcal{H}(m) | \alpha \in \mathcal{I}\}$ with $\underline{H}_{ex}(m) \subseteq \underline{H}^{\text{ex}}_\alpha(m)$ and $<\cdot,\cdot>_m$ defining a norm on $\underline{H}^{\text{ex}}_\alpha(m)$ for all $\alpha \in \mathcal{I}$ contains a maximal one $\underline{H}^{\text{ex}}_{\text{max}}(m)$ which is closed with respect to the Hilbert norm[30,31]. But then Lemma III implies:

Corollary II: $\underline{H}_{ex}(m)$ maximal $\Rightarrow H_{ex}(m) = \underline{H}_{ex}(m)$.

Now it is obvious what the axiom of asymptotic completeness should be in the present case (in order to have an unitary $S-$matrix).
Asymptotic Completeness: $\underline{H}_{out}(m)$ and $\underline{H}_{in}(m)$ are both maximal and equal:

$$\forall_{m\in\mathcal{O}^\pm_\delta} \underline{H}_{out}(m) = \underline{H}^{\text{out}}_{max}(m) = \underline{H}^{\text{in}}_{max}(m) = \underline{H}_{in}(m) \tag{66}$$

$\div$

This condition will be sufficient for the equality of the Hilbert spaces of outgoing and incoming (real) particle states to be constructed in the next section and also a necessary one for $\mu^\pm_\delta-$almost all $m \in \mathcal{O}^\pm_\delta$. We do not claim their equality to the maximal subspace $H(m)$. Since the tripel $(\mathcal{H}(m), (\cdot|\cdot)_m, \eta_m)$ is a Krein space all three are related to each other via the fundamental decomposition by a homeomorphism[30,31] and hopefully by a η_m-isometry. However the detailed investigation of this problem is bejond the scope of this paper. Of course a sufficient condition which together with the maximality of $\underline{H}_{out/in}(m)$ implies the equality $H(m) = H_{out}(m) = H_{in}(m)$ would be, that the time-dependent (for$L \geq 2$) states created by the L-fold products of the quasi-particle excitations (54) from the vacuum $\Psi_0(m)$ are all from $H(m)$ for $|t|$ larger than some fixed finite $t_0 > 0$.

VI. Asymptotic Collision States

In this section $\underline{H}_{out}(m)$ and $\underline{H}_{in}(m)$ are assumed to be maximal but not necessarily equal unless explicitely stated. To obtain the asymptotic Hilbert space(s) of incoming and outgoing (real) particle collision states from the quasi-particle spaces $H_{ex}(m)$ two further steps have to be performed: I.) Lifting the group of η-isometric isomorphisms (of Theorem I) $\mathcal{U}(\alpha, y)_m I_m(v,(r,\tau)) : D_{ex}(m) \to D_{ex}(\Lambda(\alpha)m)$ to groups of unitary and for $\alpha \in SU(2)_m$ unitary equivalent isomorphisms

$$\underline{\mathcal{U}}(\alpha, y)^{\mathrm{ex}}_m \underline{I}^{\mathrm{ex}}_m(v,(r,\tau)) : \underline{H}_{ex}(m) \to \underline{H}_{ex}(\Lambda(\alpha)m).$$

II.) Gluing the unitary isomorphisms of step I.) together to unitary equivalent unitary representations $\mathcal{U}_{ex}(\alpha, y) I_{ex}(v,(r,\tau))$ on a subspace $\mathcal{H}_{ex}$ (called the physical Hilbert space $\mathcal{H}_{phys}$ if $\underline{H}_{out}(m) = \underline{H}_{in}(m)$) of the direct integrals $\int_{\oplus \mathcal{O}^{\pm}_{\delta}} d\mu^{\pm}_{\delta}(m) \underline{H}_{ex}(m)$.

Ad I.): Since the symmetry operators (of Theorem I) $\mathcal{U}(\alpha, y)^{\mathrm{ex}}_m I^{\mathrm{ex}}_m(v,(r,\tau)) : D_{ex}(m) \to D_{ex}(\Lambda(\alpha)m)$ are $\eta-$isometric isomorphisms they have a unique unitary extention $\underline{\mathcal{U}}(\alpha, y)^{\mathrm{ex}}_m \underline{I}^{\mathrm{ex}}_m(v,(r,\tau)) : \underline{H}_{ex}(m) \to \underline{H}_{ex}(\Lambda(\alpha)m)$ with $\underline{\mathcal{U}}(\alpha, y)^{\mathrm{ex}}_m$ satisfying the group relation (16) and therefore define for $\alpha \in SU(2)_m$ and each ex=out respectively ex=in a unitary representation of the corresponding direct product (sub-)group on $\underline{H}_{ex}(m)$. Similarly the restriction $(S(m) \lceil D_{out}(m))$ being an η_m-isometric isomorphism from $D_{out}(m)$ onto $D_{in}(m)$ has a unique unitary extension $\underline{S}(m) : \underline{H}_{out}(m) \to \underline{H}_{in}(m)$. Moreover the equation (64) and (65) imply the commutation relations:

$$\underline{S}(\Lambda(\alpha)m) \underline{\mathcal{U}}(\alpha, y)^{\mathrm{out}}_m \underline{I}^{\mathrm{out}}_m(v,(r,\tau) = \underline{\mathcal{U}}(\alpha, y)^{\mathrm{in}}_m \underline{I}^{\mathrm{in}}_m(v,(r,\tau)) \underline{S}(m) \tag{67}$$

Hence $\underline{\mathcal{U}}(\alpha, y)^{\mathrm{in}}_m \underline{I}^{\mathrm{in}}_m(v,(r,\tau))$ and $\underline{\mathcal{U}}(\alpha, y)^{\mathrm{out}}_m \underline{I}^{\mathrm{out}}_m(v,(r,\tau))$ are unitary equivalent for $\alpha \in SU(2)_m$.

Ad II.): The last step consists in a straight forward application of the theory of direct integrals of Hilbert spaces as it can be found in Part II, chap.1 - chap.2 of Ref.[32]. Consider the vector spaces of so called vector fields defined by the Cartesian products $X_{m \in \mathcal{O}^{\pm}_{\delta}} \underline{H}_{ex}(m) = \{\{\Psi^{\mathrm{ex}}_m\}_{m \in \mathcal{O}^{\pm}_{\delta}} | \Psi^{\mathrm{ex}}_m \in \underline{H}_{ex}(m)\}, \;\; ex \in \{in, out\}$ endowed with the natural pointwise linear operations. Taking in the equations (60)-(62) for $\hat{f}_a \in S_c(\mathbb{R}_3)$ the special products $f_{(n_0,n_1,n_2,n_3)}(\vec{q}) := g_{n_0}(\vec{q}) \Pi^3_{i=1} H_{n_i}(q_i), (n_0, n_1, n_2, n_3) \in \mathbb{N}_4$ of Hermite-functions and the $\mathcal{C}^{\infty}-$functions $g_n \in S_c(\mathbb{R}_3)$ given by

$$g_n(\vec{q}) := \begin{cases} 1 & \text{for } |\vec{q}| \leq n \\ exp.(1 - \frac{n^2}{n^2 - \vec{q}^2}) & \text{for } n \leq |\vec{q}| \leq \sqrt{2}n \\ 0 & \text{for } |\vec{q}| \geq \sqrt{2}n \end{cases} \tag{68}$$

one can construct via linear combinations with rational numbers and tensor products a sequence $(z_n)_{n\in\mathbb{N}}$ of functions from $\cup_{L\in\mathbb{N}}\cup_{[J]_L\in\triangle^L_{psd}}\mathcal{K}^{[J]_L}$ such that the corresponding sequences $(\Psi(z_n)^{\rm ex}_m)_{n\in\mathbb{N}_0}$ from $D_{ex}(m)$ with $\Psi(z_0)^{\rm ex}_m := \Psi_0(m)$ are total in $\underline{H}_{ex}(m)$ for every $m\in\mathcal{O}^\pm_\delta$. Moreover Lemma II and the scalar products (57) imply that for all $k,n\in\mathbb{N}_0$ the functions $\mathcal{O}^\pm_\delta\to\mathbb{C}, m\to<\Psi(z_k)^{\rm ex}_m,\Psi(z_n)^{\rm ex}_m>_m$ are bounded $\mathcal{C}^\infty-$functions (in $\vec{m}$) and therefore $\mu^\pm_\delta$-measurable. Hence $(\Psi_{ex}(z_n))_{n\in\mathbb{N}_0}$ with $\Psi_{ex}(z_n) := \{\Psi(z_n)^{\rm ex}_m\}_{m\in\mathcal{O}^\pm_\delta}$ is a *fundamental sequence* of $\mu^\pm_\delta-$measurable vector fields generating a measurable field structure on the $\underline{H}_{ex}(m)'$s. It should be remarked that the sequence $(z_n)_{n\in\mathbb{N}_0}$ generating the fundamental sequence is independent of m.

An arbitrary vector field $\Phi_{ex}\in X_{m\in\mathcal{O}^\pm_\delta}\underline{H}_{ex}(m)$ is called *measurable* if the functions $\mathcal{O}^\pm_\delta\to\mathbb{C},\ m\to<\Phi^{\rm ex}_m,\Psi(hn)^{\rm ex}_m>_m$ are $\mu^\pm_\delta-$measurable for all $\mathbb{N}_0$ and called *square integrable* if it is measurable and $\int_{\mathcal{O}^\pm_\delta}d\mu^\pm_\delta(\vec{m})<\Phi^{\rm ex}_m,\Phi^{\rm ex}_m>_m<+\infty$

Obviously the set of all measurable vector fields is a subspace containing the subspace $X_{m\in\mathcal{O}^\pm_\delta}D_{ex}(m)$. The set of all (strictly speaking equivalence classes of $\mu^\pm_\delta-$almost equal and) square integrable vector fields endowed with (the natural pointwise linear operations and) the scalar product

$$<\Psi_{ex},\Phi_{ex}>:=\int_{\mathcal{O}^\pm_\delta}d\mu^\pm_\delta(\vec{m})<\Psi^{\rm ex}_m,\Phi^{\rm ex}_m>_m \tag{69}$$

is a separable Hilbert space[32] which is called the *direct integral of the Hilbert spaces* $\underline{H}_{ex}(m)$ and denoted by $H_{ex}:=\int_{\oplus\mathcal{O}^\pm_\delta}d\mu^\pm_\delta(\vec{m})\underline{H}_{ex}(m)$. In accordance with this its elements are denoted by $\Psi_{ex}:=\int_{\oplus\mathcal{O}^\pm_\delta}d\mu^\pm_\delta(\vec{m})\Psi^{\rm ex}_m$.

All members of the fundamental sequence with the exception of the vacuum vector field $\Psi_{ex}(z_0)=\{\Psi_0(m)\}_{m\in\mathcal{O}^\pm_\delta}$ as well as all elements of the subspaces $D^{[J]_L}_{ex}:=X_{m\in\mathcal{O}^\pm_\delta}D^{[J]_L}_{ex}(m)$ with $L\geq 1$ and therefore all elements of the subspace $(X_{m\in\mathcal{O}^\pm_\delta}D_{ex}(m))\backslash(X_{m\in\mathcal{O}^\pm_\delta}D^{[J]_0}_{ex}(m))$ are square integrable since by means of Lemma II, the Lemma of Fatou and Fubini's theorem the $\vec{q}-$ and $\vec{m}-$integrations can be interchanged when both sides of equation (57) are integrated over $\mathcal{O}^\pm_\delta$ and we obtain for all $\Psi_{ex}(h^{[J]_L}_{[a]_L[I^3]_L([\sigma]_L)})\in D^{[J]_L}_{ex}$ with $L\geq 1$ and $h^{\cdots}_{\cdots}$ defined in equation (60):

$$\begin{aligned}&<\Psi_{ex}(h^{[J]_L}_{[a]_L[I^3]_L([\sigma]_L)}),\ \ \Psi_{ex}(h^{[\hat{J}]_K}_{[b]_K[\hat{I}^3]_K([\hat{\sigma}]_K)})>=\\ &\delta_{KL}\sum_{\pi\in S_L}((-1)^{N_F(\pi)})\Pi^L_{j=1}[\delta_{J_j\hat{J}_{\pi(j)}}\delta_{I^3_j\hat{I}^3_{\pi(j)}}(\delta_{\sigma_j\hat{\sigma}_{\pi(j)}})\delta_{M(J_j)M(\hat{J}_{\pi(j)})}\times\\ &\times\int_{\mathbb{R}_3}d\vec{q}/2\omega_{M(J_j)}(\vec{q})\hat{f}_{a_j}(\vec{q})f_{b_{\pi(j)}}(\vec{q})]\end{aligned}\tag{70}$$

Hence by the same reasoning as in section V it follows that the closure $\overline{D^{[J]_L}_{ex}}$ of $D^{[J]_L}_{ex}$ is isometrically isomorphic to the $L-$fold Fock-symmetrized tensor product $\mathcal{L}^{2\otimes}_{[J]_L}(\nu)$ of the $\mathcal{L}^2-$spaces $\mathcal{L}^2_J(\nu):=\mathcal{L}^2(\mathbb{R}_3,d\nu_J(\vec{q})\bar{\otimes}\mathbb{C}_{2(I+1)}(\bar{\otimes}\mathbb{C}_{2(2s+1)})$ with $d\nu_J(\vec{q}):=d\vec{q}/2\omega_{M(J)}(\vec{q})$ describing free relativistic single particle multiplets with quantum numbers $J=(I,Y,s)$ and mass $M(J)$. Since for $L\neq K$ and/or pairwise substantial different $L-$tuples $[J]_L$ and

$[\hat{J}]_L$ from $\triangle^L_{psd}$ these subspaces are pairwise orthogonal to each other, their orthogonal direct sums

$$\mathcal{H}^o_{out/in} := \oplus_{L\in I\!N}(\oplus_{[J]_L\in\triangle^L_{psd}}\overline{D^{[J]_L}_{out/in}}) \tag{71}$$

are isometrically isomorphic (and therefore unitary equivalent) to the vacuum amputated Fock space:

$$\mathcal{F}^o = \oplus_{L\in I\!N}(\oplus_{[J]_L\in\triangle_{psd}}\mathcal{L}^{2\otimes}_{[J]_L}(\nu))$$

of free relativistic many particle multiplets.
It must be pointed out that $\mathcal{H}_{ex}$ is a *proper subspace* of the direct integral H_{ex} since for any $\Psi_{ex}(h^{[J]_L}_{\dots}) = \int_{\oplus\mathcal{O}^\pm_\delta} d\mu^\pm_\delta(\vec{m})\Psi(h^{[J]_L}_{\dots})^{\text{ex}}_m$ with $h^{[J]_L}_{\dots} \in \mathcal{K}^{[j]_L}$ and any bounded, $\mu^\pm_\delta-$measurable and on some set of finite measure non constant function $g : \mathcal{O}^\pm_\delta \rightarrow \mathbb{C}$ the vector field $(g\Psi_{ex})(h^{[J]_L}_{\dots}) := \{g(m)\Psi(h^{[J]_L}_{\dots})^{\text{ex}}_m\}_{m\in\mathcal{O}^\pm_\delta}$ is obviously from H_{ex} [32] but not in $\mathcal{H}^o_{ex}$.
For the remaining derivation of the symmetry representations the short hand notations $G := (SL(2,\mathbb{C})\times)I\!R_4)\times(U^{\mathcal{U}}_{\mathcal{C}}(1)\times SU^{\mathcal{U}}_{\mathcal{C}}(2))\times(U(1)^{\mathcal{Y}}_{\mathcal{F}}\times SU(2)^{\mathcal{I}}_{\mathcal{F}})$ for the underlying symmetry group with elements (α, y, w) and $W(\alpha,y,w)^{\text{ex}}_m := \mathcal{U}(\alpha,y)^{\text{ex}}_m I^{\text{ex}}_m(w)$ for the corresponding symmetry operators are introduced.
Besides the vector space $X_{m\in\mathcal{O}^\pm_\delta}\underline{H}_{ex}(m)$ endowed with the fundamental sequence $(\Psi_{ex}(z_n))_{n\in I\!N_0}, \Psi_{ex}(z_n) = \{\Psi(z_n)^{\text{ex}}_m\}_{m\in\mathcal{O}^\pm_\delta}$ the vector spaces $X_{m\in\mathcal{O}^\pm_\delta}\underline{H}_{ex}(\Lambda(\alpha)m)$ endowed with the fundamental sequences $(\Psi^{\Lambda(\alpha)}_{ex}(z_n))_{n\in I\!N_0}$ with $\Psi^{\Lambda(\alpha)}_{ex}(z_n) := \{\Psi(z_n)^{\text{ex}}_{\Lambda(\alpha)m}\}_{m\in\mathcal{O}^\pm_\delta}$ define for every fixed Lorentz transformation $\Lambda(\alpha)$ a measurable field structure over $\mathcal{O}^\pm_\delta$.
Then the unitary operators $\underline{W}(\alpha,y,w)^{\text{ex}}_m : \underline{H}_{ex}(m) \rightarrow \underline{H}_{ex}(\Lambda(\alpha)m)$ of step I.) define for every fixed $(\alpha,y,w) \in G$ via the mapping $m \rightarrow \underline{W}(\alpha,y,w)^{\text{ex}}_m$ a so called field of unitary mappings over $\mathcal{O}^\pm_\delta$. Moreover for all $k, j \in I\!N_0$ the mapping

$$m \rightarrow < W(\alpha,y,w)^{\text{ex}}_m\Psi(z_k)^{\text{ex}}_m, \Psi(z_j)^{\text{ex}}_{\Lambda(\alpha)m} >_{\Lambda(\alpha)m} =$$
$$< \Psi(z_k)^{\text{ex}}_m, (W(\alpha,y,w)^{\text{ex}}_m)^+\Psi(z_j)^{\text{ex}}_{\Lambda(\alpha)m} >_m$$

is $\mu^\pm_\delta-$measurable because $\Psi(z_j)^{\text{ex}}_{\Lambda(\alpha)m}$ is from $D_{ex}(\Lambda(\alpha)m$ for all $j \in I\!N_0$ and $(W(\alpha,y,w)^{\text{ex}}_m)^+$ $= (W(\alpha,y,w)^{\text{ex}}_m)^{-1} = W((\alpha,y,w)^{-1})^{\text{ex}}_{\Lambda(\alpha)m}$ maps $D_{ex}(\Lambda(\alpha)m)$ onto $D_{ex}(m)$ all elements of which are measurable. Then Proposition 1 of Part II, Chapter 2 in Ref.[32] implies, that for every measurable vector field $\Psi_{ex} = \{\Psi^{\text{ex}}_m\}_{m\in\mathcal{O}^\pm_\delta}$ the vector field $W_{ex}(\alpha,y,w)\Psi_{ex} :=$ $\{\underline{W}(\alpha,y,w)^{\text{ex}}_m\}_{m\in\mathcal{O}^\pm_\delta}$ is measurable, in which case the field of unitary mappings $m \rightarrow$ $\underline{W}(\alpha,y,w)^{\text{ex}}_m$ is called measurable. For every $\Psi_{ex} \in H_{ex}$ obviously $W_{ex}(\alpha,y,w)\Psi_{ex}$ is from H_{ex} with $< W_{ex}(\alpha,y,w)\Psi_{ex}, W_{ex}(\alpha,y,w)\Psi_{ex} >=< \Psi_{ex},\Psi_{ex} >$ due to the unitarity of $\underline{W}(\alpha,y,w)^{\text{ex}}_m$ for every $m \in \mathcal{O}^\pm_\delta$. The mappings (of equivalence classes) $W_{ex}(\alpha,y,w)$: $H_{ex} \rightarrow H_{ex}, \Psi_{ex} = \int_{\oplus\mathcal{O}^\pm_\delta} d\mu^\pm_\delta(\vec{m})\Psi^{\text{ex}}_m \rightarrow \int_{\oplus\mathcal{O}^\pm_\delta} d\mu^\pm_\delta(\vec{m})\underline{W}(\alpha,y,w)^{\text{ex}}_m\Psi^{\text{ex}}_m$ inherit from the $\underline{W}(\alpha,y,w)^{\text{ex}}_m$'s besides unitarity the *(Poincaré) composition law* $W_{ex}(\alpha,y,w)W_{ex}(\hat{\alpha},\hat{y},\hat{w}) =$ $W_{ex}(\alpha\hat{\alpha}, y + \alpha\hat{y}, w\hat{w})$ and the further property of being decomposable operators [Part II, Chap. 2, Sections 3 and 5 in Ref.[32]].
In other words the mappings $W_{out/in} : G \rightarrow Aut.H_{out/in}, (\alpha,y,w) \rightarrow W_{out/in}(\alpha,y,w)$ are unitary representations of the symmetry group G. Furthermore Theorem I, part(ii) im-

plies every subspace $\overline{D_{ex}^{[J]_L}}, L \geq 1$ and therefore also their orthogonal direct sums $\mathcal{H}_{ex}^{\circ}$ to be invariant subspaces for W_{ex}. Hence the restrictions $\mathcal{W}_{ex}(\alpha, y, w) := W_{ex}(\alpha, y, w)\lceil\mathcal{H}_{ex}^{\circ}$ define unitary representations $\mathcal{H}_{ex} : G \rightarrow Aut.\mathcal{H}_{ex}^{\circ}$ of G on $\mathcal{H}_{ex}^{\circ}$ which due to the equations (64) and (70) are unitary equivalent to the representation $\mathcal{W}_{\mathcal{F}^{\circ}} : G \rightarrow Aut.\mathcal{F}^{\circ}$ on the Fock space $\mathcal{F}^{\circ}$ formed by the direct sum of tensor products of apart from parity doubling in case $s \in \{1/2, \overline{1/2}\}$ irreducible unitary representations of G in the single particle subspaces $\mathcal{L}_J^2(\nu)$ of $\mathcal{F}^{\circ}$.

Apart from the existence of translational invariant vacuum states on the level of asymptotic limits we have now arrived at exactly the same situation familiar from classical Wightman theories[10−14].

However for every equivalence class of complex functions $h : \mathbb{R}_4 \rightarrow \mathbb{C}$ from $\mathcal{L}^2(\mathcal{O}_\delta^{\pm}, d\mu_\delta^{\pm}(\vec{m}))$ the vector field $\Psi_0(h) := \{h(m)\Psi_0(m)\}_{m\in\mathcal{O}_\delta^{\pm}}$ is from $H_{out} \cap H_{in}$, the closed subspace $H^o \subset H_{out} \cap H_{in}$ of all these vector fields $\Psi_0(h)$ isometrically isomorphic to $\mathcal{L}^2(\mathcal{O}_\delta^{\pm}, d\mu_\delta^{\pm}(\vec{m}))$ and due to the invariance of the measure $\mu_\delta^{\pm}$ an invariant subspace for the representation W_{ex}. Since the restrictions $\mathcal{W}_0(\alpha, y, w) = W_{out}(\alpha, y, w)\lceil H^0 = W_{in}(\alpha, y, w)\lceil H^o$ map the vector field $\Psi_0(h) = \{h(m)\Psi_0(m)\}$ onto $\Psi_0(h_{\Lambda(\alpha)}) = \{h(\Lambda(\alpha)^{-1}m)\Psi_0(m)\}_{m\in\mathcal{O}_\delta^{\pm}}$ all states from H^o are invariant under translations as well as internal symmetry transformations and transform under Lorentz transformations like scalar particles of mass $|\delta|M(K)$ (with $M(K)$ the Kaon mass). This "vacuum-representation" $\mathcal{W}_0 : G \rightarrow Aut.H^0$ is certainly unitary equivalent to the "scalar-representation" $\mathcal{W}_S : G \rightarrow Aut.\mathcal{L}^2(\mathcal{O}_\delta^{\pm}.d\mu_\delta^{\pm}(\vec{m}))$ defined by $(\mathcal{W}_S(\alpha, y, w)h)(m) := h(\Lambda(\alpha)^{-1}m)$. Adding these representations to the former ones via direct sums $\mathcal{H}_{ex} := H^o \oplus \mathcal{H}_{ex}^o, \mathcal{U}_{ex} := \mathcal{W}_0 \oplus \mathcal{W}_{ex}$ respectively $\mathcal{F} := \mathcal{L}^2(\mathcal{O}_\delta^{\pm}, d\mu_\delta^{\pm}(\vec{m})) \oplus \mathcal{F}^o$ and $\mathcal{U}_{\mathcal{F}} := \mathcal{W}_0 \oplus \mathcal{W}_{\mathcal{F}^0}$ then the pairs $(\mathcal{H}_{out/in}, \mathcal{U}_{out/in})$ are unitary equivalent to the familiar Fock-representation of free relativistic many particle systems with the one dimensional "vacuum representation" $(\mathbb{C}, id_{\mathbb{C}})$ replaced by the infinite dimensional "scalar representation" $(\mathcal{L}^2(\mathcal{O}_\delta^{\pm}, d\mu_\delta^{\pm}(\vec{m})), \mathcal{W}_S)$.

It remains to investigate the scattering operator. From the definition of the operators $S(m)$ in equation (65) respectively $\underline{S}(m)$ in step I.) it follows that the mapping $m \rightarrow \underline{S}(m)$ defines a measurable field of unitary mappings because for all $\Psi_m^{\text{out}} \in \underline{H}_{out}(m)$ and all $j \in \mathbb{N}_0$ the mapping $m \rightarrow < \underline{S}(m)\Phi_m^{\text{out}}, \Psi(z_j)_m^{\text{in}} > = < \Phi_m^{\text{in}}, \Psi(z_j)_m^{\text{in}} >$ is $\mu_\delta^{\pm}-$measurable. Then the unitarity of the $\underline{S}(m)$'s implies that for every $\Psi_{out} = \{\Psi_m^{\text{out}}\}_{m\in\mathcal{O}_\delta^{\pm}} \in H_{out}$ the vector field $S\Psi_{out} := \{\underline{S}(m)\Psi_m^{\text{out}}\}_{m\in\mathcal{O}_\delta^{\pm}} = \{\Psi_m^{\text{in}}\}_{m\in\mathcal{O}_\delta^{\pm}} =: \Psi_{in}$ is from H_{in} and the mapping $S : H_{out} \rightarrow H_{in}, \Psi_{out} \rightarrow S\Psi_{out} := \Psi_{in} = \int_{\oplus\mathcal{O}_\delta^{\pm}} d\mu_\delta^{\pm}(\vec{m})\underline{S}(m)\Psi_m^{\text{out}}$ is unitary and decomposable. Because it maps $\mathcal{H}_{out}^o$ onto $\mathcal{H}_{in}^o$ and every $\Psi_0 \in H^o$ onto itself the restriction $\mathcal{S} := S\lceil\mathcal{H}_{out}$ is the direct sum $\mathcal{S} = id_{H^0} \oplus \mathcal{S}_0$ with $\mathcal{S}_0$ the restriction of S to $\mathcal{H}_{out}^o$. Moreover from the scalar product (69) and the commutation relations (67) of step I.) we obtain via a simple calculation the commutation relations $SW_{out}(\alpha, y, w) = W_{in}(\alpha, y, w)S$ and therefore $\mathcal{S}\mathcal{U}_{out} = \mathcal{U}_{in}\mathcal{S}$.

If asymptotic completeness $\underline{H}_{out}(m) = \underline{H}_{in}(m)$ holds for almost all $m \in \mathcal{O}_\delta^{\pm}$ which implies $\mathcal{H}_{out} = \mathcal{H}_{in} =: \mathcal{H}_{phys}$ and $\mathcal{U}_{out} = \mathcal{U}_{in} =: U$ then $\mathcal{S}$ is an unitary operator in $\mathcal{H}_{phys}$ which commutes with the representation of the symmetry group $US = SU$ and is called the *scattering operator*. Summing up our final result is:

Theorem II: For any given collection of color-singlet single-particle flavor multiplets satisfying the Haag-Ruelle-Hepp- and positivity condition on the two-point function of the corresponding excitation operators there exist:

(i): *Two Hilbert spaces $\mathcal{H}_{out/in}$ being the direct sums of asymptotic outgoing respectively incoming multi-particle collision states and an infinite dimensional space H° of translational invariant (vacuum-) states.*

(ii): *Isometric isomorphism from $\mathcal{H}_{out/in}$ onto the Fock space $\mathcal{F}$ over the $\mathcal{L}^2_{(I,Y,S)}$ spaces $\mathcal{L}^2(\mathbb{R}_3, d\vec{q}/2\omega_{M(I,Y,s)}(\vec{q})) \otimes \mathbb{C}_{2I+1}(\otimes\mathbb{C}_{2(2s+1)})$ and its (classical) one dimensional vacuum subspace $(\mathbb{C}, (\cdot,\cdot)_{\mathbb{C}})$ replaced by the infinite dimensional space $\mathcal{L}^2(\mathcal{O}^{\pm}_{\delta}, d\mu^{\pm}_{\delta}(\vec{q}))$.*

(iii): *On $\mathcal{H}_{out/in}$ (standard) unitary representations $\mathcal{U}_{out/in}$ of the direct product group built from the $SL(2,\mathbb{C})\times)\mathbb{R}_4$ Poincaré group, the $SU(2)^{\mathcal{I}}_{\mathcal{F}} \times U(1)^{\mathcal{Y}}_{\mathcal{F}}$ flavor group and the $SU(2)^{\mathcal{U}}_{\mathcal{C}} \times U(1)^{\mathcal{Q}}_{\mathcal{C}}$ color group which are unitary equivalent to the direct sum of tensor products of apart from parity doubling for $s \neq 0$ irreducible unitary representations in $\mathcal{L}^2_{(I,Y,s)}$ and the "scalar representation of (orbit-)mass $|\delta|M(K)$" in $\mathcal{L}^2(\mathcal{O}^{\pm}_{\delta}, d\mu^{\pm}_{\delta}(\vec{q}))$.*

(iv): *An unitary operator $\mathcal{S} : \mathcal{H}_{out} \to \mathcal{H}_{in}$ with the properties $\mathcal{S}\mathcal{U}_{out} = \mathcal{U}_{in}\mathcal{S}$ and $\mathcal{S}\lceil H^{\circ} = id_{H^{\circ}}$.*

If asymptotic completeness $\underline{H}_{out}(m) = \underline{H}_{in}(m)$ holds for almost all $m \in \mathcal{O}^{\pm}_{\delta}$ then $\mathcal{H}_{out} = \mathcal{H}_{in} =: \mathcal{H}_{phys}$ (called the strong physical Hilbert space) and $\mathcal{U}_{out} = \mathcal{U}_{in} =: \mathcal{U}$ commuting with $\mathcal{S}$ (called the scattering operator).

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Remark: All results remain true if the Haag-Ruelle-Hepp-condition on the spectrum is replaced by the weaker continuity condition near the mass shells of Herbst[16].

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