SOME REMARKS ON EXTREME INFINITE DOUBLY STOCHASTIC MATRICES

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ABSTRACT. The main results of this article are contained in sections 2 and 3. In section 2 we show that for non-trivial infinite systems of marginals $\mathcal{I}$ there are always continuum many extreme matrices that are doubly stochastic with respect to $\mathcal{I}$ (cf. Theorem 3). Corollary 1 contains the corresponding result for the substochastic case. In section 3 we show that the entries of an extreme doubly stochastic matrix are always in the closure of the additive subgroup of $\mathbb{R}$ that is generated by the respective marginals (cf. Theorem 4). Theorem 5 deals with the corresponding substochastic case.

1. PRELIMINARIES

Doubly stochastic matrices, systems of marginals. Given sets I_1, I_2, R denote by $\mathcal{M}_R(I_1, I_2) := R^{I_1 \times I_2}$ the set of all *matrices* with coefficients in R whose rows and columns are indexed by I_1 and I_2 respectively. Both rows and columns will be called *lines*.

To any matrix $P \in \mathcal{M}_{\mathbb{R}}(I_1, I_2)$ we assign a weighted bipartite graph $G(P) = (V, E, w)$. V consists of all $(i, 0)$ and $(j, 1)$ such that $i \in I_1$ and $j \in I_2$ respectively. Following traditional notation we write x_i instead of $(i, 0)$ and y_j instead of $(j, 1)$. E consists of all $\{x_i, y_j\}$ such that $p_{i,j} \neq 0$. w assigns $p_{i,j}$ to each edge $\{x_i, y_j\}$. The *support* of P is defined as the set of all indices (i, j) where $p_{i,j} \neq 0$. It is denoted by $\mathrm{supp}(P)$.

A $\leq$-*system of marginals* is a 4-tupel $\mathcal{I} = (I_1, I_2, r, s)$ where I_1, I_2 are sets and $r = (r_i)_{i \in I_1}$, $s = (s_j)_{j \in I_2}$ are families of non-negative real numbers. An index $i \in I_1$ is called *non-trivial* iff $r_i > 0$ and analogously for $j \in I_2$. The sets of non-trivial indices are denoted by $I_{1,nt}$ and $I_{2,nt}$ respectively. $\mathcal{I}$ will be called finite if both $I_{1,nt}$ and $I_{2,nt}$ are finite, and infinite otherwise. The $\leq$-system of marginals $\mathcal{I}$ is a *system of marginals* iff $\sum_{i \in I_1} r_i = \sum_{j \in I_2} s_j$. Following GRZĄŚLEWICZ (cf. [3]) we will call a matrix $P \in \mathcal{M}_{\mathbb{R}_0^+}(I_1, I_2)$ *doubly stochastic* with respect to $\mathcal{I}$ iff $\sum_{l \in I_2} p_{i,l} = r_i$ for all $i \in I_1$ and $\sum_{k \in I_1} p_{k,j} = s_j$ for all $j \in I_2$, even though the word "stochastic" is strictly justified only for probability measures. Note that in this case only countably many elements in each line can be non-zero. The set of all matrices which are doubly stochastic with respect to $\mathcal{I}$ is denoted by $\mathcal{D}(\mathcal{I})$. Similarly one defines the set $\mathcal{D}(\leq \mathcal{I})$ of all matrices which are *doubly substochastic* with respect to $\mathcal{I}$, the only difference being that the sum over each line is required to be less than or equal to the corresponding r_i or s_j. The sets $\mathcal{D}(\mathcal{I})$ and $\mathcal{D}(\leq \mathcal{I})$ are convex and we write $\mathrm{E}(\mathcal{D}(\mathcal{I}))$ and $\mathrm{E}(\mathcal{D}(\leq \mathcal{I}))$ for their sets of extreme points.

Date: 23rd May 1997.

If $P \in \mathcal{M}_{\mathbb{R}_0^+}(I_1, I_2)$ for given sets I_1, I_2, then define $\mathcal{I}(P) := (I_1, I_2, r, s)$ where $r_i := \sum_{\nu \in I_2} p_{i,\nu}$ for $i \in I_1$ and $s_j := \sum_{\mu \in I_1} p_{\mu,j}$ for $j \in I_2$. Note that $\mathcal{I}(P)$ is a system of marginals if and only if all the r_i and s_j are finite. In this case $P \in \mathcal{D}(\mathcal{I}(P))$.

Given a $\leq$-system of marginals $\mathcal{I} := (I_1, I_2, r, s)$ let $\mathcal{G} = \mathcal{G}(\mathcal{I})$ be the additive subgroup of $\mathbb{R}$ generated by the union of the r_i and s_j. Furthermore if $P \in \mathcal{D}(\leq \mathcal{I})$ and $G = G(P)$, then denote by $V_< = V_<(P)$ the set of all vertices such that the corresponding marginal value is unattained, i.e.

$$V_< := \{x_i \in V(G) : \sum_{j \in I_2} p_{i,j} < r_i\} \dot\cup \{y_j \in V(G) : \sum_{i \in I_1} p_{i,j} < s_j\}. \tag{1}$$

Graphs, rooted trees, ϵ-flows, ϵ-bitrees and ϵ-paths. A *weighted graph* will be denoted $G = (V, E, w)$ where $V = V(G)$ is the set of *vertices*, $E = E(G)$ is the set of *edges* and $w : E \longrightarrow \mathbb{R}$ is some *weight function*. A *wood* is a graph without circles and is called a *tree* iff it is connected. A tree is called a *path* iff all its vertices are contained in at most two of its edges.

A *weighted rooted tree* is a 4-tuple $T = (V, E, q, w)$ where (V, E, w) is a weighted graph and q is a specified vertex called the *root* of T. The nth *level* $L_n = L_n(q) = L_n(q, T)$ is the set of all vertices that have distance precisely n to the root. For $v \in L_n$ let $N^\pm(v) = N_T^\pm(v) := \{u \in L_{n\pm 1} : \{u, v\} \in E\}$. Finally let $E_n = E_n(q) = E_n(q, T)$ be the set of edges between L_{n-1} and L_n.

For a rooted tree and $v \in V \setminus \{q\}$, $N^-(v)$ consists of precisely one element which we will call v^-. Moreover $N^+(u)$ and $N^+(v)$ are disjoint for all distinct vertices $u, v \in V$.

Given a rooted tree $T = (V, E, q)$ and $\epsilon > 0$ a map $f : E \longrightarrow \mathbb{R}_0^+$ is called an *ϵ-flow* iff $\sum_{u \in N^+(q)} f\{q, u\} = \epsilon$ and $\sum_{u \in N^+(v)} f\{v, u\} = f\{v^-, v\}$ for all $v \in V \setminus \{q\}$.

An easy induction shows that for an ϵ-flow f on (V, E, q), $\sum_{e \in E_n} f(e) = \epsilon$ holds for all n. Hence $\sum_{e \in E} f(e)$ is infinite and so is the set V.

A *weighted bitree* $B = (G, T_1, T_2)$ is a weighted tree $G = (V, E, w)$ together with rooted trees $T_1 = (V_{T_1}, E_{T_1}, q_1)$, $T_2 = (V_{T_2}, E_{T_2}, q_2)$ such that G consists of all the vertices and edges of T_1 and T_2 plus the edge $\{q_1, q_2\}$. Moreover for $\epsilon > 0$, B is called an *ϵ-bitree* iff there are ϵ-flows f_1, f_2 on T_1, T_2 respectively such that $f_1 \leq w\restriction_{E_{T_1}}$, $f_2 \leq w\restriction_{E_{T_2}}$ and $\epsilon \leq w\{q_1, q_2\}$. We say that a weighted graph $G = (V, E, w)$ *has an ϵ-bitree* iff there is an ϵ-bitree (G', T_1', T_2') such that G' is a subgraph of G.

Given a weighted graph $G = (V, E, w)$ and $\epsilon > 0$, we say that G *has an infinite ϵ-path* iff there is a path W of infinite length in G such that $w(e) \geq \epsilon$ for all edges e belonging to W.

Characterizations. Using a slightly different notation GRZĄŚLEWICZ proved in [3, cf. Theorems 1 and 2] graph-theoretic characterizations of the sets $\mathrm{E}(\mathcal{D}(\mathcal{I}))$ and $\mathrm{E}(\mathcal{D}(\leq \mathcal{I}))$ which we will state as Theorem 1 and Theorem 2 for the convenience of subsequent reference.

Theorem 1. *Let $\mathcal{I} := (I_1, I_2, r, s)$ be a system of marginals such that both I_1 and I_2 are countable. A matrix $P \in \mathcal{D}(\mathcal{I})$ is extreme if and only if $G(P)$ is a wood and has no ϵ-bitree for any $\epsilon > 0$.* $\square$

Theorem 2. *Let $\mathcal{I} := (I_1, I_2, r, s)$ be a $\leq$-system of marginals such that both I_1 and I_2 are countable. A matrix $P \in \mathcal{D}(\leq \mathcal{I})$ is extreme if and only if the following three conditions hold.*

- **(i):** *$G(P)$ is a wood and has no ϵ-bitree for any $\epsilon > 0$.*
- **(ii):** *$\#(V(H) \cap V_<) \leq 1$ for all connected components H of $G(P)$.*
- **(iii):** *If H is a connected component of $G(P)$ and there is $\epsilon > 0$ such that H has an infinite ϵ-path, then $V(H) \cap V_< = \emptyset$.* □

2. Cardinality of $\mathrm{E}(\mathcal{D}(\mathcal{I}))$

It is well-known that for finite systems of marginals there can be only finitely many extreme matrices (in [5, cf. Theorem 4.1] Jurkat and Ryser presented a procedure that gives all extreme matrices in finitely many steps, each step producing only finitely many matrices). In the following Theorem 3 we will show that if the system of marginals $\mathcal{I}$ is no longer finite (and has as least two rows and two columns), there are continuum many extreme matrices. In the proof we will use that $\mathrm{E}(\mathcal{D}(\mathcal{I}))$ contains at least *two* distinct elements. In [3, p. 529] Grząślewicz gave an explicit example of a matrix $P \in \mathcal{D}(\mathcal{I})$. One can show that this matrix is actually in $\mathrm{E}(\mathcal{D}(\mathcal{I}))$ and by a slight modification one can get another element of $\mathrm{E}(\mathcal{D}(\mathcal{I}))$ that is distinct from P.

Theorem 3. *Let $\mathcal{I} := (I_1, I_2, r, s)$ be an infinite system of marginals such that both I_1 and I_2 are countable and contain at least two non-trivial elements. Then $\#\mathrm{E}(\mathcal{D}(\mathcal{I})) = 2^\omega$, where ω is the smallest infinite cardinal.*

Proof. Without loss of generality assume that I_2 is infinite and that both I_1 and I_2 contain only non-trivial elements. Since the cardinality of $\mathbb{R}^{I_1 \times I_2}$ is 2^ω it suffices to show that there are at least 2^ω extreme matrices.

First assume $\sum_{i \in I_1} r_i = \sum_{j \in I_2} s_j < \infty$. Fix an arbitrary $i_0 \in I_1$ and a $j_0 \in I_2$ such that $s_{j_0} = \max\{s_j : j \in I_2\}$. By using the finiteness of $\sum_{j \in I_2} s_j$ together with the infiniteness of I_2 and the maximality of s_{j_0}, it is not hard to show that there is a finite (possibly empty) set $T_f \subseteq I_2 \setminus \{j_0\}$ that satisfies

$$r_{i_0} - s_{j_0} < \sum_{j \in T_f} s_j < r_{i_0}. \tag{2}$$

Now let $\delta := \frac{1}{2}(r_{i_0} - \sum_{j \in T_f} s_j)$ and choose an infinite subset T_ω of $(I_2 \setminus \{j_0\}) \setminus T_f$ such that $\sum_{j \in T_\omega} s_j \leq \delta$. Then the set $\mathcal{T} := \{T_f \dot{\cup} T : T \subseteq T_\omega\}$ has cardinality 2^ω. Our goal is to define a matrix $P^T \in \mathrm{E}(\mathcal{D}(\mathcal{I}))$ for every $T \in \mathcal{T}$ such that the map $T \mapsto P^T$ is one-to-one. We proceed by defining new systems of marginals $\mathcal{I}^T := (I_1^T, I_2, r^T, s^T)$ for all $T \in \mathcal{T}$. Here $I_1^T := I_1 \setminus \{i_0\}$, $r^T := r\rceil_{I_1^T}$ and $s^T = (s_j^T)_{j \in I_2}$, where

$$s_j^T := \begin{cases} 0 & \text{for } j \in T, \\ s_j & \text{for } j \in I_2 \setminus (T \cup \{j_0\}), \\ s_{j_0} - r_{i_0} + \sum_{\nu \in T} s_\nu & \text{for } j = j_0. \end{cases} \tag{3}$$

Now choose $Q \in \mathrm{E}(\mathcal{D}(\mathcal{I}^T))$ and define $P^T = (p^T_{i,j})_{(i,j)\in I_1\times I_2}$ by letting $P^T\restriction_{I_1^T\times I_2} := Q$ and

$$p^T_{i_0,j} := \begin{cases} s_j & \text{for } j \in T, \\ r_{i_0} - \sum_{\nu\in T} s_\nu & \text{for } j = j_0, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Clearly $P^T \in \mathrm{E}(\mathcal{D}(\mathcal{I}))$ for all and $T \in \mathcal{T}$ and the map $T \mapsto P^T$ is one-to-one. Thus the proof of the case $\sum_{i\in I_1} r_i = \sum_{j\in I_2} s_j < \infty$ is complete.

In the case $\sum_{i\in I_1} r_i = \sum_{j\in I_2} s_j = \infty$, both I_1 and I_2 are infinite. By decomposing the set I_1 into infinitely many infinite sets $I_{1,k}$, $k \in \omega$, such that $\sum_{i\in I_{1,k}} r_i = \infty$ for all k and by using an analogous procedure on the set I_2 one arrives at infinitely many systems of marginals $\mathcal{I}_k := (I_{1,k}, I_{2,k}, r^k, s^k)$, $k \in \omega$. Here r^k, s^k are the restrictions of r, s to $I_{1,k}$, $I_{2,k}$ respectively. For each $k \in \omega$ choose two distinct matrices $P^{(k)}, Q^{(k)} \in \mathrm{E}(\mathcal{D}(\mathcal{I}_k))$. Given any function $f : \omega \longrightarrow \{0,1\}$ let $P^f = (p^f_{i,j})_{(i,j)\in I_1\times I_2}$ be defined by

$$p^f_{i,j} := \begin{cases} p^{(k)}_{i,j} & \text{for } (i,j) \in I_{1,k}\times I_{2,k} \text{ and } f(k) = 0, \\ q^{(k)}_{i,j} & \text{for } (i,j) \in I_{1,k}\times I_{2,k} \text{ and } f(k) = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Then clearly $P^f \in \mathrm{E}(\mathcal{D}(\mathcal{I}))$ and the map $f \mapsto P^f$ is one-to-one. This completes the proof of the case $\sum_{i\in I_1} r_i = \sum_{j\in I_2} s_j = \infty$ and thus the proof the theorem. □

Corollary 1. *Let $\mathcal{I} := (I_1, I_2, r, s)$ be an infinite system of marginals such that both I_1 and I_2 are countable and contain at least two non-trivial elements. Then $\#\mathrm{E}(\mathcal{D}(\leq \mathcal{I})) = 2^\omega$.* □

Remark 1. By generalizing the argument used in the proof of the case $\sum_{i\in I_1} r_i = \sum_{j\in I_2} s_j = \infty$, one can easily extend the above theorem and its corollary to the case of a general infinite cardinal α instead of ω. For this case one has to assume $\#I_{1,nt} = \#I_{2,nt}$, though, since otherwise one finds the set $\mathcal{D}(\mathcal{I})$ empty.

3. Extreme matrices and their relation to $\mathcal{G}(\mathcal{I})$

The abovementioned procedure by Jurkat and Ryser shows that if $\mathcal{I}$ is a finite system of marginals and $P \in \mathrm{E}(\mathcal{D}(\mathcal{I}))$, then all the entries of P are actually in $\mathcal{G}(\mathcal{I})$. In Theorem 4 we show that this result carries over to the infinite case if one substitutes $\mathcal{G}(\mathcal{I})$ by its closure. This also provides a new proof of Birkhoff's Theorem (cf. [2] or [1, Theorem II.2.3] for a more recent presentation) and its generalizations by Isbell and Kendall (see [4] and [6] respectively) as will be noted in Remark 3. Example 1 shows that in general it is not true that the entries of an extreme infinite doubly stochastic matrix are in $\mathcal{G}(\mathcal{I})$. We start by stating without proof a very easy lemma that will be used in the proof of Theorem 4.

Lemma 1. *Given a countable set S, a positive real number α and a family of positive real numbers $\mathcal{A} = (a_i)_{i\in S}$ such that $\beta := \sum_{i\in S} a_i > \alpha$, there is a finite set $S' \subseteq S$ and a family of positive real numbers $\mathcal{A}' = (a'_i)_{i\in S'}$ such that*

$$\underset{i\in S'}{\forall}\ a'_i < a_i \quad \text{and} \quad \sum_{i\in S'} a'_i = \alpha. \tag{6}$$ □

Theorem 4. *Let $\mathcal{I} := (I_1, I_2, r, s)$ be an infinite system of marginals such that both I_1 and I_2 are countable. If $\mathcal{G} := \mathcal{G}(\mathcal{I})$, then*

$$P \in \mathrm{E}(\mathcal{D}(\mathcal{I})) \quad \Rightarrow \quad \{p_{i,j} : (i,j) \in I_1 \times I_2\} \subseteq \overline{\mathcal{G}}. \tag{7}$$

Proof. Since any closed additive subgroup that is strictly contained in $\mathbb{R}$ is equal to $t\mathbb{Z}$ for some $t \in \mathbb{R}_0^+$, one can assume $\mathcal{G} = \overline{\mathcal{G}} = t\mathbb{Z}$ for some positive t. Consider $P \in \mathcal{D}(\mathcal{I})$ such that $p_{i_0,j_0} \notin \mathcal{G}$ for some $(i_0, j_0) \in I_1 \times I_2$. We will show that $P \notin \mathrm{E}(\mathcal{D}(\mathcal{I}))$ by using Theorem 1. Assume that $G = G(P) = (V, E, w)$ is a wood. We have to show that there is $\epsilon > 0$ such that G has an ϵ-bitree. For any $p \in \mathbb{R}$ define $d(p) := \mathrm{dist}(p, \mathcal{G})$. Then clearly $d(p) = d(-p)$ for all $p \in \mathbb{R}$, and $d(\sum_{i=1}^{\infty} p_i) \leq \sum_{i=1}^{\infty} d(p_i)$ for all sequences $(p_i)_{i \in \mathbb{N}}$ such that $\sum_{i=1}^{\infty} p_i \in \mathbb{R}$. These two properties of d imply that

$$\underset{(i,j) \in I_1 \times I_2}{\forall} \sum_{\mu \in I_1 \setminus \{i\}} d(p_{\mu,j}) \geq d(p_{i,j}) \quad \text{and} \quad \sum_{\nu \in I_2 \setminus \{j\}} d(p_{i,\nu}) \geq d(p_{i,j}). \tag{8}$$

We are now ready to construct the ϵ-bitree. To this end let $\epsilon := \frac{1}{2} d(p_{i_0,j_0})$, $q_1 := x_{i_0}$, $q_2 := y_{j_0}$ and consider the bitree $B' = (G', T_1', T_2')$ where for $\alpha = 1, 2$ $T_\alpha' = (V_{T_\alpha'}, E_{T_\alpha'}, q_\alpha)$ is the connected component of q_α in the graph $G(P) - \{\{q_1, q_2\}\}$. We are now going to construct the ϵ-bitree $B = (G, T_1, T_2)$ such that for $\alpha = 1, 2$ $T_\alpha = (V_{T_\alpha}, E_{T_\alpha}, q_\alpha)$ is a subgraph of T_α'. We deal with T_1 first. Simultaniously with T_1 we construct an ϵ-flow f_1 on T_1. We proceed by a recursive definition of sets $L_n'(q_1) \subseteq L_n(q_1)$ and sets $E_n'(q_1) \subseteq E_n(q_1)$. To use Lemma 1 for the first step put $S = \{j \in I_2 \setminus \{j_0\} : p_{i_0,j} \notin \mathcal{G}\}$, $\mathcal{A} = (d(p_{i_0,j}))_{j \in S}$, $\beta = \sum_{j \in S} di(p_{i_0,j})$, $\alpha = \epsilon$ (note that $\beta \geq 2\alpha > \alpha$ by (8)). Lemma 1 provides us with a finite set $S' \subseteq S$ and a family of positive real numbers $\mathcal{A}' = (a_i')_{i \in S'}$ that satisfy (6). Let $N(q_1) := \{y_j : j \in S'\}$, $L_0'(q_1) := \{q_1\}$, $L_1'(q_1) := N(q_1)$, $E_1'(q_1) := \{\{q_1, y_j\} \in E_1(q_1) : y_j \in N(q_1)\}$ and $f_1\{q_1, y_j\} := a_j'$ for all $y_j \in N(q_1)$. It then follows that $\sum_{y_j \in N(q_1)} f_1\{q_1, y_j\} = \epsilon$ and $f_1\{q_1, y_j\} < d(p_{i_0,j}) \leq p_{i_0,j} = w\{q_1, y_j\}$ for all $y_j \in N(q_1)$.

Now consider the case that $n \geq 1$ is an odd number and suppose that the sets $L_m'(q_1) \subseteq L_m(q_1)$ and $E_m'(q_1) \subseteq E_m(q_1)$ are defined for $m \leq n$ such that for $m \neq 0$ and all $v \in L_m'(q_1)$, $v^- \in L_{m-1}'(q_1)$. Furthermore assume that for $m \leq n$, $f_1 \restriction_{E_m'(q_1)}$ is defined such that $0 < f_1 \restriction_{E_m'(q_1)} < (d \circ w) \restriction_{E_m'(q_1)} \leq \min\{t, w \restriction_{E_m'(q_1)}\}$. Suppose $y_{j_n} \in L_n'(q_1)$ and $x_{j_n} := y_{j_n}^-$. Then $0 < f_1\{x_{i_n}, y_{j_n}\} < d(p_{i_n,j_n}) \leq \min\{t, w\{x_{i_n}, y_{j_n}\}\}$. For another application of Lemma 1 put $S = \{i \in I_1 \setminus \{i_n\} : p_{i,j_n} \notin \mathcal{G}\}$, $\mathcal{A} = (d(p_{i,j_n}))_{i \in S}$, $\beta = \sum_{i \in S} d(p_{i,j_n})$, $\alpha = f_1\{x_{i_n}, y_{j_n}\}$ and note that $\alpha = f_1\{x_{i_n}, y_{j_n}\} < d(p_{i_n,j_n}) \leq \beta$ as is implied again by (8). As in the first step Lemma 1 provides us with a finite set $S' \subseteq S$ and a family of positive real numbers $\mathcal{A}' = (a_i')_{i \in S'}$ that satisfy (6). Let $N(y_{j_n}) := \{x_i : i \in S'\}$ and $f_1\{x_i, y_{j_n}\} := a_i'$ for all $x_i \in N(y_{j_n})$. Since T_1 is a tree, the sets $N(y)$, $y \in L_n'(q_1)$ are all disjoint. Finally, if one lets $L_{n+1}'(q_1) := \dot{\bigcup}_{y \in L_n'(q_1)} N(y)$ and $E_{n+1}'(q_1) := \{\{x, y\} \in E_{n+1} : x \in L_{n+1}'(q_1), y \in L_n'(q_1)\}$, then $\sum_{x \in N(y)} f_1\{x, y\} = f_1\{y, y^-\}$ for all $y \in L_n'(q_1)$.

The definitions of $L_{n+1}'(q_1)$ and $E_{n+1}'(q_1)$ for the case that $n \geq 1$ is even are done precisely analogously. Then V_{T_1} and E_{T_1} are defined as the unions of the $L_n'(q_1)$

and $E'_n(q_1)$ respectively. As was already checked during the construction, f_1 has all the properties of an ϵ-flow on T_1.

T_2 and f_2 are constructed in exactly the same manner as T_1 and f_1. Finally let $\hat{V} := V_{T_1} \dot{\cup} V_{T_2}$, $\hat{E} := E_{T_1} \dot{\cup} E_{T_2} \dot{\cup} \{\{q_1, q_2\}\}$ and $G = (\hat{V}, \hat{E}, w\restriction_{\hat{E}})$. Then $B = (G, T_1, T_2)$ is an ϵ-bitree and the proof of (7) is complete. $\square$

Example 1. Let $\mathcal{I} := \mathcal{I}(P)$ where

$$P := \begin{pmatrix} 1-a_1 & a_2 & & & & & \\ a_2 & & 1-a_3 & & & & \\ & 1-a_3 & & a_4 & & & \\ & & a_4 & & 1-a_5 & & \\ & & & 1-a_5 & & & \\ & & & & & & \ddots \end{pmatrix} \tag{9}$$

and

$$a_n := (10^n\sqrt{2} - \lfloor 10^n\sqrt{2} \rfloor) \cdot 10^{-n} \tag{10}$$

for all $n \in \mathbb{N}$ ($\sqrt{2}$ is used for definiteness, any other element from $\mathbb{R}^+ \setminus \mathbb{Q}$ will do). a_n is then irrational and positive for all $n \in \mathbb{N}$. On the other hand $-a_n + a_{n+1} \in \mathbb{Q}$ for all $n \in \mathbb{N}$ and hence $\mathcal{G}(\mathcal{I}) \subseteq \mathbb{Q}$. It is easily checked that $P \in \mathrm{E}(\mathcal{D}(\mathcal{I}))$, though.

Example 1 also shows that in general it is not true that an extreme matrix P is determined in $\mathcal{D}(\mathcal{I})$ by its support in the sense that

$$(\tilde{P} \in \mathcal{D}(\mathcal{I}) \text{ and } \mathrm{supp}(\tilde{P}) \subseteq \mathrm{supp}(P)) \quad \Rightarrow \quad \tilde{P} = P. \tag{11}$$

Indeed, if one lets

$$Q := \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & & -\frac{1}{2} & \\ & -\frac{1}{2} & & \\ & & & \ddots \end{pmatrix}, \tag{12}$$

and P as in Example 1, then P and $P + Q$ have the same support and $P + Q \in \mathcal{D}(\mathcal{I})$. On the other hand (11) holds for an extreme matrix $P \in \mathcal{D}(\mathcal{I})$ if $\mathcal{I} = (I_1, I_2, r, s)$ is a system of marginals such that $\sum_{i \in I_1} r_i = \sum_{j \in I_2} s_j < \infty$ as was shown in [3, Proposition 1], even though the hypothesis of the finite sums was forgotten. If the system of marginals $\mathcal{I}$ is such that (11) holds true (e.g. if $\sum_{i \in I_1} r_i = \sum_{j \in I_2} s_j < \infty$), then the set of extreme points of $\mathcal{D}(\mathcal{I})$ is the same as the set of *exposed points* of $\mathcal{D}(\mathcal{I})$, i.e. $\mathrm{E}(\mathcal{D}(\mathcal{I})) = \mathrm{Exp}(\mathcal{D}(\mathcal{I}))$ (cf. [3, Proposition 2]). The question if $\mathrm{E}(\mathcal{D}(\mathcal{I})) = \mathrm{Exp}(\mathcal{D}(\mathcal{I}))$ in general seems to be difficult. It is an open problem if the matrix P from Example 1 is exposed. One way to deal with the general case might be to modify the definition of an exposed point of $\mathcal{D}(\mathcal{I})$ by admitting only continuous linear functionals with respect to an appropriate topology on $\mathcal{D}(\mathcal{I})$, but currently no final results on this issue can be given.

By using Theorem 2 one finds that Theorem 4 has an analogue for the substochastic case which will be stated as Theorem 5. Though slightly more involved, the proof is similar to the proof of Theorem 4 and will be omitted.

Theorem 5. *Let $\mathcal{I} := (I_1, I_2, r, s)$ be an infinite $\leq$-system of marginals such that both I_1 and I_2 are countable. If $\mathcal{G} := \mathcal{G}(\mathcal{I})$, then*

$$(13)\quad P \in \mathrm{E}(\mathcal{D}(\leq \mathcal{I})) \quad \Rightarrow \quad \{p_{i,j} : (i,j) \in I_1 \times I_2\} \subseteq \overline{\mathcal{G}}. \qquad \square$$

Remark 2. One can easily extend Theorem 4 and Theorem 5 to the case that the sets I_1 and I_2 are of arbitrary cardinality. The key to reducing the general case to the countable case is to note that for any doubly (sub-)stochastic matrix P the graph $G(P)$ decomposes into connected components which have countably many vertices.

Remark 3. The classical results that for systems of marginals (I_1, I_2, r, s) such that $r_i = s_j = 1$ for all $i \in I_1$, $j \in I_2$, the set of extreme doubly stochastic matrices consists of all permutation matrices and the set of extreme doubly substochastic matrices consists of all subpermutation matrices, are immediate consequences of Theorems 4 and 5 respectively.

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