NET SPACES IN CATEGORICAL TOPOLOGY

Josef Šlapal

Abstract. In the paper we introduce the notion of $\mathscr{S}$-nets, $\mathscr{S}$ a non-empty construct, as a generalization of the usual nets. $\mathscr{S}$-net spaces and continuous maps between them are then defined in a natural way. For $\mathscr{S}$-net spaces we introduce a number of axioms and study the categories obtained. We find some topological and cartesian closed categories of $\mathscr{S}$-net spaces and describe their relations to certain known topological categories.

0. Introduction

Cartesian closed topological categories belong to the most important objects studied in categorical topology. The usefulness of these categories for applications to many classical branches of mathematics is well known. And the interest in cartesian closed categories became even stronger after discovering the important role that they play in computer science. When looking for new cartesian closed topological categories categorical topologists usually start from the category of topological spaces and generalize or modify the considered axioms of topology. These axioms depend on the chosen access to topology (closure operations, neighborhood structures, systems of open sets, etc.). In this connection the most often used access is that one which views topologies as convergence structures and uses filters for describing the convergence. This filter access gave arise to a rich hierarchy of useful categories and an intensive study is being performed in the area at present.

In this paper we will also view topologies as convergence structures, but we shall use nets instead of filters for expressing the convergence. This access is closely related to that one using filters and some connections between certain types of categories of convergence filter spaces and those of convergence net spaces are described in [14]. Of course, from the classical point of view filters are more suitable for investigations in topology than nets which have, essentially, an order-theoretic nature. However, in our considerations we will not deal with the usual nets, but

1980 *Mathematics Subject Classification* (1985 *Revision*). AMS classification: 54A05, 54A20, 54B30, 18B30, 18D15.

Key words and phrases. Net, subnet, net space, convergence space, limit space, topological category, cartesian closed category, topological universe, diagonality axiom, condition of iterated limits, preclosure space, closure space, pseudo-topological space, topological space, R_0-space, T_0-space, T_1-space.

This paper was finished during the author's stay at the Free University of Berlin as a fellow of the Konferenz der deutschen Akademien der Wissenschaften (Stipendienprogram gefördert durch die Volkswagenstiftung) from October 1994 to April 1995. .

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

with a certain type of generalized nets which have lost their order-theoretic nature in the benefit of a category-theoretic one. Our use of this generalized nets, called $\mathscr{S}$-nets (where $\mathscr{S}$ is a construct), could be hardly duplicated by using filters.

For $\mathscr{S}$-net spaces introduced we will consider a number of axioms including, at the first place, the axioms that are extensions of the usual net axioms. By this way we obtain a number of full subcategories of the category of $\mathscr{S}$-net spaces (with extended continuous maps as morphisms) which are then studied. We will especially deal with the question of cartesian closedness of these subcategories and with their relations to some known topological categories (subcategories and supercategories of Top). We discover a number of cartesian closed topological categories of $\mathscr{S}$-net spaces some of which are trivial, some are isomorphic to certain well known categories, but many are new and useful for applications to various branches of mathematics. We also receive some known results (see e.g. [5] and [7]) as trivial consequences of certain more general facts.

1. Basic definitions

Throughout the paper, $\mathscr{S}$ will denote a non-empty construct (i.e. concrete category over *Set*) whose objects have non-empty underlying sets. As usual, we will not distinguish notationally between $\mathscr{S}$-objects, resp $\mathscr{S}$-morphisms, and their underlying sets, resp. maps. If X, Y are sets and $y \in Y$ is an element, then the constant map $f : X \to Y$ with $f(x) = y$ for each $x \in X$ will be called the *y-constant map*.

1.1. Definitions.

(1) Let X be a set. By an *$\mathscr{S}$-net* in X we understand any pair (S, f) where $S \in \mathscr{S}$ is an object and $f : S \to X$ is a map. The class of all $\mathscr{S}$-nets in X will be denoted by $\langle X \rangle_{\mathscr{S}}$.

(2) If $(S, f), (T, g) \in \langle X \rangle_{\mathscr{S}}$, then (S, f) is called a subnet of (T, g) if there is an $\mathscr{S}$-morphism $\varphi : S \to T$ such that $f = g \circ \varphi$.

1.2. Examples.

1. Let *Dir* be the category of directed sets whose morphisms are the cofinal maps, i.e. maps $\varphi : (S_1, \leq_1) \to (S_2, \leq_2)$ having the property that for any $s_2 \in S_2$ there is $s_1 \in S_1$ such that $s_2 \leq_2 \varphi(s)$ whenever $s \in S_1, s_1 \leq_1 s$. If $\mathscr{S} = \mathit{Dir}$, then the notions of an $\mathscr{S}$-net and a subnet (of an $\mathscr{S}$-net) coincide with the usual notions of a net and a subnet (see e.g. [7]), respectively.
2. Let $\alpha > 0$ be an ordinal and let $\underline{\alpha}$ be the category whose only object is α and whose morphisms are isotone injections of α into itself. Clearly, $\underline{\alpha}$ is a subcategory of *Dir*. For $\mathscr{S} = \underline{\alpha}$ the notions of an $\mathscr{S}$-net and a subnet coincide with the notions of a sequence of type α and a subsequence, respectively. Sequences of type ω are usually called briefly sequences.
3. Let Set^+ be the category of non-empty sets. If $\mathscr{S} = Set^+$ and $(S, f), (T, g) \in \langle X \rangle_{\mathscr{S}}$, then, clearly, (S, f) is a subnet of (T, g) iff $f(S) \subseteq g(T)$.

1.3. Denotation. If X is a set and $\pi : \langle X \rangle_{\mathscr{S}} \to \exp X$ a map, then we will usually write $(S, f) \xrightarrow{\pi} x$ instead of $x \in \pi(S, f)$, and $(S, f) \stackrel{\pi}{\nrightarrow} x$ instead of $x \in X - \pi(S, f)$.

The set $\pi(S,f)$ can be viewed as the set of all limits of the net (S,f) w.r.t. π and therefore we will sometimes say that (S,f) converges to x (w.r.t. π) if $(S,f) \xrightarrow{\pi} x$.

1.4. Definition.

(1) A pair (X,π) where X is a set and $\pi : \langle X\rangle_{\mathscr{S}} \to \exp X$ a map is called an *$\mathscr{S}$-net space* provided that $(S,f) \xrightarrow{\pi} x$ whenever f is the x-constant map.

(2) An $\mathscr{S}$-net space (X,π) is called a *convergence $\mathscr{S}$-net space* provided that from $(S,f) \xrightarrow{\pi} x$ it follows that $(T,g) \xrightarrow{\pi} x$ for each subnet (T,g) of (S,f).

(3) A convergence $\mathscr{S}$-net space (X,π) is called a *limit $\mathscr{S}$-net space* provided that, whenever $(S,f) \stackrel{\pi}{\not\to} x$, there is a subnet (T,g) of (S,f) such that $(U,h) \stackrel{\pi}{\not\to} x$ for any subnet (U,h) of (T,g).

1.5. Examples.

1. Let $\mathscr{S} = Dir$ or $\mathscr{S} = \underline{\alpha}$ where $\alpha > 0$ is an ordinal. Let $(X,\mathscr{O})$ be a topological space (given by the system $\mathscr{O}$ of open sets). Let (X,π) be the $\mathscr{S}$-net space associated with $(X,\mathscr{O})$, i.e., for any $(S,f) \in \langle X\rangle_{\mathscr{S}}, (S,f) \xrightarrow{\pi} x$ iff for each $A \in \mathscr{O}$ with $x \in A$ there exists $s_A \in S$ such that $f(s) \in A$ for every $s \in S, s \geq s_A$. Then, obviously, (X,π) is a limit $\mathscr{S}$-net space.
2. The $\mathscr{B}$-convergence structures studied in [15] for special subcategories $\mathscr{B}$ of Dir are nothing else than the maps $\pi : \langle X\rangle_{\mathscr{B}} \to \exp X$ for which (X,π) are convergence $\mathscr{B}$-net spaces.
3. Let $\mathscr{S} = \underline{\omega}$. Then for convergence $\mathscr{S}$-net spaces (X,π) the maps π are precisely the multivalued convergences on X from [10]. The convergence or limit $\mathscr{S}$-net spaces (X,π) fulfilling card $\pi(S,f) \leq 1$ for each $(S,f) \in \langle X\rangle_{\mathscr{S}}$ are known as the (Fréchet) $\mathscr{L}$-spaces or $\mathscr{L}^*$-spaces, respectively (see e.g. [4]).

1.6. Definition. Let $(X,\pi),(Y,\rho)$ be a pair of $\mathscr{S}$-net spaces. A map $F : X \to Y$ is called a *continuous map* of (X,π) into (Y,ρ) provided that $(S, F\circ f) \xrightarrow{\rho} F(x)$ whenever $(S,f) \xrightarrow{\pi} x$. We denote by $Net_{\mathscr{S}}$ the category of $\mathscr{S}$-net spaces and continuous maps, and by $Conv_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ its full subcategories of convergence $\mathscr{S}$-net spaces and limit $\mathscr{S}$-net spaces respectively. The set of all continuous maps of (X,π) into (Y,ρ) will be denoted by $C((X,\pi),(Y,\rho))$.

1.7. Examples.

1. Let $(X,\pi) \in Net_{\mathscr{S}}$ be an object and let $\pi' : \langle X\rangle_{\mathit{Set}^+} \to \exp X$ be the map given by $(S,f) \xrightarrow{\pi'} x$ iff f is the x-constant map or there exists $(S,f) \in \langle X\rangle_{\mathscr{S}}$ such that $(S,f) \xrightarrow{\pi} x$. Then the assignation $(X,\pi) \mapsto (X,\pi')$ determines a concrete embedding of $Net_{\mathscr{S}}$ into Net_{Set^+}.
2. For $\mathscr{S} = Dir$ the category $Lim_{\mathscr{S}}$ is studied in [5] where its objects are called L*-spaces.

2. Properties of $Net_{\mathscr{S}}$, $Conv_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$

The category $Net_{\mathscr{S}}$ is not fibre-small in general (it is fibre-small whenever $\mathscr{S}$ is small). Therefore we do not suppose fibre-smallness when working with initially structured or topological categories. More precisely, by an *initially structured category* we understand a construct which is closed under formation of initial objects for all mono-sources and in which each constant map between its objects is a morphism. An initially structured category which is initially complete (i.e. closed under formation of initial objects for all sources) is said to be a *topological category*. Also the notion of a *topological universe* is meant here in the generalized sense obtained by omitting the requirement of fibre-smallness (cf. [1]). Thus, a topological universe is a cartesian closed topological category $\mathscr{K}$ with the property that every $\mathscr{K}$-object C has a one-point extension, i.e. a $\mathscr{K}$-object $\overline{C} = C \cup \{\infty_C\}$ where $\infty_C \notin C$ such that C is a subobject of $\overline{C}$ and for any $\mathscr{K}$-object A, any subobject B of A and any $\mathscr{K}$-morphism $\varphi : B \to C$ the map $\overline{\varphi} : A \to C$, given by $\overline{\varphi}(a) = \varphi(a)$ whenever $a \in B$ and $\overline{\varphi}(a) = \infty_C$ whenever $A \notin B$, is a $\mathscr{K}$-morphism. Let us remark here that by a subobject of an object we always understand an initial object with regard to an inclusion map.

A topological universe is said to be *strong* provided that its quotient maps are productive (cf. [12]). Throughout the paper, products in $\mathscr{S}$ are meant to be products over non-empty index sets.

2.1. Theorem. *$Net_{\mathscr{S}}$ is a strong topological universe.*

Proof. Clearly, each constant map between objects of $Net_{\mathscr{S}}$ is continuous. Let $F_i : X \to (X_i, \pi_i), i \in I$, be a source in $Net_{\mathscr{S}}$ and let $\pi : \langle X\rangle_{\mathscr{S}} \to \exp X$ be given as follows: $(S, f) \xrightarrow{\pi} x$ iff $(S, F_i \circ f) \xrightarrow{\pi_i} F_i(x)$ for each $i \in I$. Clearly, $(X, \pi) \in Net_{\mathscr{S}}$. Let $(Y, \rho) \in Net_{\mathscr{S}}$ be an object and $G : Y \to X$ a map such that $F_i \circ G : (Y, \rho) \to (X_i, \pi_i)$ is continuous for each $i \in I$. Let $(T, g) \in \langle Y\rangle_{\mathscr{S}}, (T, g) \xrightarrow{\rho} y$. Then $(T, F_i \circ G \circ g) \xrightarrow{\pi_i} F_i \circ G(y)$ for each $i \in I$. Hence $(T, G \circ g) \xrightarrow{\pi} G(y)$, i.e. $G : (Y, \rho) \to (X, \pi)$ is a continuous map. Therefore (X, π) is the initial object of the source $F_i, i \in I$, and it is unique because $Net_{\mathscr{S}}$ is transportable. Consequently, $Net_{\mathscr{S}}$ is initially complete. We have shown that $Net_{\mathscr{S}}$ is a topological category.

Let $(X, \pi), (Y, \rho) \in Net_{\mathscr{S}}$ be objects. Let $Z = C((X, \pi), (Y, \rho))$ and let $\sigma : \langle Z\rangle_{\mathscr{S}} \to \exp Z$ be given by $z \in \sigma(S, g)$ iff from $(S, f) \xrightarrow{\pi} x$ it follows that $(S, g^f) \xrightarrow{\rho} z(x)$ where $g^f : S \to Y$ is defined by $g^f(s) = g(s)(f(s))$. Then $(Z, \sigma) \in Net_{\mathscr{S}}$ and we will show that (Z, σ) is a function space of (X, π) and (Y, ρ) in $Net_{\mathscr{S}}$. To this end, let $e : (X, \pi) \times (Z, \sigma) \to (Y, \rho)$ be the evaluation map, i.e. the map with $e(x, z) = z(x)$ whenever $(x, z) \in X \times Z$. Put $(V, \tau) = (X, \pi) \times (Z, \sigma)$ and let $(S, f) \in \langle V\rangle_{\mathscr{S}}, (S, f) \xrightarrow{\tau} (x, z)$. Then $(S, pr_X \circ f) \xrightarrow{\pi} x$ and $(S, pr_Z \circ f) \xrightarrow{\sigma} z$. Hence $(S, h^p) \xrightarrow{\rho} z(x)$ where $h = pr_Z \circ f$ and $p = pr_X \circ f$. We have $h^p(s) = h(s)(p(s)) = e(p(s), h(s)) = e(pr_X(f(s)), pr_Z(f(s))) = e(f(s))$ for every $s \in S$. Therefore $h^p = e \circ f$ which yields $(S, e \circ f) \xrightarrow{\rho} e(x, z)$. Consequently, $e : (X, \pi) \times (Z, \sigma) \to (Y, \rho)$ is continuous. Now, let $F : (X, \pi) \times (W, \upsilon) \to (Y, \rho)$ be a continuous map and let $F^* : W \to Z$ be the map given by $F^*(w)(x) = F(x, w)$. Let $(S, g) \in \langle W\rangle_{\mathscr{S}}, (S, g) \xrightarrow{\upsilon} w$, and $(S, f) \in \langle X\rangle_{\mathscr{S}}, (S, f) \xrightarrow{\pi} x$. Then $(S, (f, g))$ converges to (x, w) in the space $(X, \pi) \times (W, \upsilon)$. Hence $(S, F \circ (f, g)) \xrightarrow{\rho} F(x, w)$. This results in $(S, F^* \circ g) \xrightarrow{\sigma} F^*(w)$ because $F((f, g)(s)) = F^*(g(s))(f(s)) = (F^* \circ g)^f(s)$ whenever $s \in S$. Consequently, $F^* : (W, \upsilon) \to (Z, \sigma)$ is a continuous map. Since clearly $e \circ (id_X \times F^*) = F$, we have shown that (Z, σ) is a function space of (X, π) and (Y, ρ) in $Net_{\mathscr{S}}$. Therefore $Net_{\mathscr{S}}$ is cartesian closed.

Let $(X,\pi) \in Net_{\mathscr{S}}$ be an object and ∞_X a point with $\infty_X \notin X$. Let $(X \cup \{\infty_X\}, \pi') \in Net_{\mathscr{S}}$ be defined as follows: if $(S,f) \in \langle X \cup \{\infty_X\}\rangle_{\mathscr{S}}$, then $\pi'(S,f) = \pi(S,f) \cup \{\infty_X\}$ whenever $\infty_X \notin f(S)$ and $\pi'(S,f) = X \cup \{\infty_X\}$ whenever $\infty_X \in f(S)$. Clearly, (X,π) is a subspace (i.e. a subobject) of $(X \cup \{\infty_X\}, \pi')$. Let $(Y,\rho) \in Net_{\mathscr{S}}$, let (Z,σ) be a subspace of (Y,ρ) and let $F : (Z,\sigma) \to (X,\pi)$ be a continuous map. Let $\overline{F} : Y \to X \cup \{\infty_X\}$ be defined by $\overline{F}(y) = F(y)$ whenever $y \in Z$ and $\overline{F}(y) = \infty_X$ whenever $y \in Y - Z$. It can easily be seen that $\overline{F} : (Y,\rho) \to (X \cup \{\infty_X\}, \pi')$ is a continuous map. Consequently, $Net_{\mathscr{S}}$ is a topological universe. We will show that this universe is strong.
Let $(X,\pi) \in Net_{\mathscr{S}}$ be an object, Y a set and $F : X \to Y$ a surjection. Evidently, an object $(Y,\rho) \in Net_{\mathscr{S}}$ is the quotient object of (X,π) with regard to F if and only if the following condition is fulfilled: $(S,g) \xrightarrow{\rho} y$ iff there is $(S,f) \in \langle X\rangle_{\mathscr{S}}, (S,f) \xrightarrow{\pi} x$, such that $F \circ f = g$ and $F(x) = y$. Let $F_i : (X_i,\pi_i) \to (Y_i,\rho_i), i \in I$, be a family of quotient maps in $Net_{\mathscr{S}}$. Let $(X,\pi) = \prod_{i\in I}(X_i,\pi_i)$, $(Y,\rho) = \prod_{i\in I}(Y_i,\rho_i)$ be the (cartesian) products and let $p_i : (X,\pi) \to (X_i,\pi_i)$, $q_i : (Y,\rho) \to (Y_i,\rho_i), i \in I$, be the corresponding projections. Finally, let $F : (X,\pi) \to (Y,\rho)$ be the unique continuous map with $q_i \circ F = F_i \circ p_i$ for all $i \in I$. Obviously, F is a surjection. Let $(S,g) \in \langle Y\rangle_{\mathscr{S}}, (S,g) \xrightarrow{\rho} y$. Then $(S, q_i \circ g) \xrightarrow{\rho_i} q_i(y)$ for each $i \in I$. Thus, for each $i \in I$ there exists $(S,f_i) \in \langle X_i\rangle_{\mathscr{S}}$ such that $(S,f_i) \xrightarrow{\pi_i} x_i$, $F_i \circ f_i = q_i \circ g$ and $F_i(x_i) = q_i(y)$. Let $f : S \to X$ be the map with $p_i \circ f = f_i$ for each $i \in I$, and let $x \in X$ be the element with $p_i(x) = x_i$. Then $(S,f) \xrightarrow{\pi} x$ and we have $F \circ f = g$ because $q_i \circ F \circ f = F_i \circ p_i \circ f = F_i \circ f_i = q_i \circ g$ for each $i \in I$. Next, we have $F(x) = y$ because $q_i \circ F(x) = F_i \circ p_i(x) = F_i(x_i) = q_i(y)$ for each $i \in I$. Conversely, let there is $(S,f) \in \langle X\rangle_{\mathscr{S}}$, $(S,f) \xrightarrow{\pi} x$, $F \circ f = g$ and $F(x) = y$. Then $(S, p_i \circ f) \xrightarrow{\pi_i} p_i(x)$ and, for each $i \in I$, we have $F_i \circ p_i \circ f = q_i \circ F \circ f = q_i \circ g$ and $F_i(p_i(x)) = q_i(F(x)) = q_i(y)$. Consequently, $(S, q_i \circ g) \xrightarrow{\rho_i} q_i(y)$ for each $i \in I$, i.e. $(S,g) \xrightarrow{\rho} y$. We have shown that (Y,ρ) is a quotient object of (X,π) with regard to F. Hence quotient maps are productive in $Net_{\mathscr{S}}$. Thus $Net_{\mathscr{S}}$ is a strong topological universe. The proof is complete.

$Conv_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ are obviously closed under formation of initial objects in $Net_{\mathscr{S}}$ and so they are topological. Consequently, $Conv_{\mathscr{S}}$ is a bireflective subcategory of $Net_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ is a bireflective subcategory of $Conv_{\mathscr{S}}$. The bireflection of an object $(X,\pi) \in Net_{\mathscr{S}}$ in $Conv_{\mathscr{S}}$ is given by the identity map $id_X : (X,\pi) \to (X,\widehat{\pi})$ where $\widehat{\pi} : \langle X\rangle_{\mathscr{S}} \to \exp X$ is defined by $(T,g) \xrightarrow{\widehat{\pi}} x$ iff there is $(S,f) \in \langle X\rangle_{\mathscr{S}}$ with $(S,f) \xrightarrow{\pi} x$ such that (T,g) is a subnet of (S,f). The bireflection of an object $(X,\pi) \in Conv_{\mathscr{S}}$ in $Lim_{\mathscr{S}}$ is given by the identity map $id_X : (X,\pi) \to (X,\widehat{\pi})$ where $\widehat{\pi} : \langle X\rangle_{\mathscr{S}} \to \exp X$ is defined by $(T,g) \xrightarrow{\widehat{\pi}} x$ iff for each subnet (S,f) of (T,g) there is a subnet (U,h) of (S,f) with $(U,h) \xrightarrow{\pi} x$.

2.2. Theorem. *$Conv_{\mathscr{S}}$ is a bicoreflective subcategory of $Net_{\mathscr{S}}$.*

Proof. Let $(X,\pi) \in Net_{\mathscr{S}}$ and let $\pi^* : \langle X\rangle_{\mathscr{S}} \to \exp X$ be the map given by $(T,g) \xrightarrow{\pi^*} x$ iff, whenever (S,f) is a subnet of (T,g), $(S,f) \xrightarrow{\pi} x$. Clearly, $(X,\pi^*) \in Conv_{\mathscr{S}}$ and $id_X : (X,\pi^*) \to (X,\pi)$ is a bimorphism. Let $(Y,\rho) \in Conv_{\mathscr{S}}$ be an object and $F : (Y,\rho) \to (X,\pi)$ a continuous map. Let $(T,g) \in \langle Y\rangle_{\mathscr{S}}, (T,g) \xrightarrow{\rho} y$. Then each subnet (U,h) of (T,g) has the property that $(U,h) \xrightarrow{\rho} y$. Let (S,f) be a subnet of $(T, F\circ g)$. Then there is an $\mathscr{S}$-morphism $\varphi : S \to T$ with $f = F\circ g\circ\varphi$. Since $(S, g\circ\varphi)$ is a subnet of (T,g), we have $(S, g\circ\varphi) \xrightarrow{\rho} y$. Consequently, $(S,f) = (S, F\circ g\circ\varphi) \xrightarrow{\pi} F(y)$. Hence $(T, F\circ g) \xrightarrow{\pi^*} F(y)$, i.e. $F : (Y,\rho) \to (X,\pi^*)$ is continuous. Hence $id_X : (X,\pi^*) \to (X,\pi)$ is a bicoreflection of (X,π) in $Conv_{\mathscr{S}}$ and the proof is complete.

2.3. Corollary. *$Conv_{\mathscr{S}}$ is a cartesian closed category.*

2.4. Theorem. *If $\mathscr{S}$ has finite cartesian products, then $Lim_{\mathscr{S}}$ is cartesian closed.*

Proof. Let $(X,\pi),(Y,\rho) \in Lim_{\mathscr{S}}$. Let $Z = C((X,\pi),(Y,\rho))$ and let $\sigma : \langle Z\rangle_{\mathscr{S}} \to \exp Z$ be the map defined by $(T,g) \xrightarrow{\sigma} z$ iff from $(S,f) \xrightarrow{\pi} x$ it follows that $(S\times T, g^f) \xrightarrow{\rho} z(x)$ where $g^f : T\times S \to Y$ is the map given by $g^f(s,t) = g(t)(f(s))$.
Let $z \in Z$ and let $(T,g) \in \langle Z\rangle_{\mathscr{S}}$ where g is the z-constant map. Let $(S,f) \in \langle X\rangle_{\mathscr{S}}$, $(S,f) \xrightarrow{\pi} x$. Then $g^f(s,t) = g(t)(f(s)) = z(f(s))$. Hence $g^f = z\circ f\circ pr_S$. As z is continuous, $(S, z\circ f) \xrightarrow{\pi} z(x)$. Therefore $(S\times T, g^f) \xrightarrow{\rho} z(x)$, i.e. $(T,g) \xrightarrow{\sigma} z$.
Let $(T,g) \in \langle Z\rangle_{\mathscr{S}}$, $(T,g) \xrightarrow{\sigma} z$, and let $(S,f) \xrightarrow{\pi} x$. Let (U,h) be a subnet of (T,g). Then there is an $\mathscr{S}$-morphism $\varphi : U \to T$ such that $h = g\circ\varphi$. We have $(S\times T, g^f) \xrightarrow{\rho} z(x)$. As the map $\psi : S\times U \to S\times T$ given by $\psi(s,u) = (s,\varphi(u))$ is obviously an $\mathscr{S}$-morphism, $(S\times U, g^f\circ\psi)$ is a subnet of $(S\times T, g^f)$ and thus $(S\times U, g^f\circ\psi) \xrightarrow{\sigma} z(x)$. But $g^f(\psi(s,u)) = h(s,\varphi(u)) = g(\varphi(u))(f(s)) = p(u)(f(s)) = h^f(s,u)$ for any $(s,u) \in S\times U$ and consequently $(U,h) \xrightarrow{\sigma} z$.
Let $(T,g) \in \langle Z\rangle_{\mathscr{S}}$, $(T,g) \overset{\sigma}{\nrightarrow} z$. Then there exists $(S,f) \in \langle X\rangle_{\mathscr{S}}$ such that $(S,f) \xrightarrow{\pi} x$ but $(S\times T, g^f) \overset{\rho}{\nrightarrow} z(x)$. Thus, there exists a subnet (U,h) of $(S\times T, g^f)$ such that $(V,p) \overset{\rho}{\nrightarrow} z(x)$ for any subnet (V,p) of (U,h). Let $\varphi : U \to S\times T$ be the $\mathscr{S}$-morphism with $h = g^f\circ\varphi$. As $pr_T\circ\varphi : U \to T$ is an $\mathscr{S}$-morphism, $(U, g\circ pr_T\circ\varphi)$ is a subnet of (T,g). Let (W,r) be a subnet of $(U, g\circ pr_T\circ\varphi)$. Then there is an $\mathscr{S}$-morphism $\psi : W \to U$ such that $r = g\circ pr_T\circ\varphi\circ\psi$. Let $\chi : W \to S\times W$ be the map given by $\chi(w) = (pr_S(\varphi(\psi(w))), w)$. Then χ is an $\mathscr{S}$-morphism and we have $g^f(\varphi(\psi(w))) = g^f(pr_S(\varphi(\psi(w))), pr_T(\varphi(\psi(w)))) = g(pr_T(\varphi(\psi(w))))(f(pr_S(\varphi(\psi(w))))) = r(w)(f(pr_S(\chi(w)))) = r^f(pr_S(\chi(w)), w) = r^f(\chi(w))$ for every $w \in W$. Hence $(W, g^f\circ\varphi\circ\psi)$ is a subnet of $(S\times W, r^f)$. Since $(W, g^f\circ\varphi\circ\psi)$ is a subnet of (U,h), we have $(W, g^f\circ\varphi\circ\psi) \overset{\rho}{\nrightarrow} z(x)$ and consequently $(S\times W, r^f) \overset{\rho}{\nrightarrow} z(x)$. Hence $(W,r) \overset{\sigma}{\nrightarrow} z$ which means that no subnet of $(U, g\circ pr_T\circ\varphi)$ converges to z. We have shown that $(Z,\sigma) \in Lim_{\mathscr{S}}$.
Let $e : (X,\pi)\times(Z,\sigma) \to (Y,\rho)$ be the evaluation map. Denote $(V,\tau) = (X,\pi)\times(Z,\sigma)$ and let $(S,f) \in \langle X\rangle_{\mathscr{S}}$, $(S,f) \xrightarrow{\tau} (x,z)$. Then $(S, pr_X\circ f) \xrightarrow{\pi} x$ and $(S, pr_Z\circ f) \xrightarrow{\sigma} z$. Hence $(S\times S, h^g) \xrightarrow{\rho} z(x)$ where $G = pr_x\circ f$ and $h = pr_Z\circ f$. Let $\varphi : S \to S\times S$ be the map given by $\varphi(s) = (s,s)$. Then φ is an $\mathscr{S}$-morphism and consequently $(S, h^g\circ\varphi)$ is a subnet of $(S\times S, h^g)$. Thus $(S, h^g\circ\varphi) \xrightarrow{\rho} z(x)$. For any $s \in S$ we have $h^g(\varphi(s)) = h^g(s,s) = pr_Z(f(s))(pr_X(f(s))) = e(pr_X(f(s)), pr_Z(f(s))) = e(f(s))$. Therefore $(S, e\circ f) \xrightarrow{\rho} e(x,z)$ and we have shown that e is continuous.
Let $(W,\upsilon) \in Lim_{\mathscr{S}}$ and let $F : (X,\pi)\times(W,\upsilon) \to (Y,\rho)$ be a continuous map. Let $F^* : W \to Z$ be the map given by $F^*(w)(x) = F(x,w)$. Let $(S,f) \in \langle S\rangle_{\mathscr{S}}$, $(S,f) \xrightarrow{\pi} x$, and $(T,g) \in \langle W\rangle_{\mathscr{S}}, (T,g) \xrightarrow{\upsilon} w$. Then the $\mathscr{S}$-net $(S\times T, (f\circ pr_S, g\circ pr_T)) =$

$(S\times T, f\times g)$ converges to (x,w) in $(X,\pi)\times(W,\upsilon)$ and thus $(S\times T, F\circ(f\times g)) \xrightarrow{\rho} F(x,w)$. For any $(s,t)\in S\times T$ we have $F((f\times g)(s,t)) = F(f(s),g(t)) = F^*(g(t))(f(s)) = (F^*\circ g)^f(s,t)$. Hence $(S\times T,(F^*\circ g)^f) \xrightarrow{\rho} F^*(w)(x)$ which results in $(S, F^*\circ g) \xrightarrow{\sigma} F^*(w)$. Thus, $F^* : (W,\upsilon)\to(Z,\sigma)$ is a continuous map. As the equality $e\circ(id_X\times F^*) = F$ is clearly valid, we have shown that (Z,σ) is the function space of (X,π) and (Y,σ) in $Lim_{\mathscr{S}}$. The proof is complete.

2.5. Example. The cartesian closedness of $Lim_{\mathscr{S}}$ for $\mathscr{S} = Dir$, which is an immediate consequence of Theorem 2.4, is also proved in [5].

3. Diagonal $\mathscr{S}$-net spaces

In the previous paragraph it has been shown that $Conv_{\mathscr{S}}$ is a cartesian closed topological subcategory of the topological preuniverse $Net_{\mathscr{S}}$ but the function spaces in $Conv_{\mathscr{S}}$ are not inherited from $Net_{\mathscr{S}}$. The same is valid also for $Lim_{\mathscr{S}}$ provided that $\mathscr{S}$ has finite cartesian products. In this paragraph we will find such cartesian closed topological subcategories of $Conv_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ whose function spaces are inherited from $Net_{\mathscr{S}}$. These subcategories will be received as intersection of $Conv_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ with certain full subcategory of $Net_{\mathscr{S}}$. This subcategory itself will be topological and cartesian closed with function spaces inherited from $Net_{\mathscr{S}}$ and, moreover, its function spaces will have the point-wise structure.

3.1. Definition. An $\mathscr{S}$-net space (X,π) is called *diagonal* provided that it satisfies the following condition (called the *diagonality axiom*):
Let $\{(S,f_i); i\in S\}$ and $\{(S,g_j); j\in S\}$ be families of $\mathscr{S}$-nets in X such that $f_i(j) = g_j(i)$ whenever $i,j\in S$ and let $(S,f_i)\xrightarrow{\pi} x_i$ whenever $i\in S$ and $(S,g_j)\xrightarrow{\pi} y_j$ whenever $j\in S$. Let $f,g,h : S\to X$ be the maps given by $f(i)=x_i$, $g(j)=y_j$ and $h(i)=f_i(i)$. Then from $(S,f)\xrightarrow{\pi} x$ and $(S,g)\xrightarrow{\pi} x$ it follows that $(S,h)\xrightarrow{\pi} x$.

We denote by $DNet_{\mathscr{S}}$ the full subcategory of $Net_{\mathscr{S}}$ whose objects are precisely the diagonal $\mathscr{S}$-net spaces.

3.2. Examples.

1. If $\mathscr{S} = \underline{\omega}$, then the diagonality of $\mathscr{S}$-net spaces coincides with the diagonality defined in [13].
2. Let $\mathscr{S} = \underline{1}$. Then $DNet_{\mathscr{S}}$ is isomorphic with the category of quasi-discrete topological spaces [3] (i.e. topological spaces with completely additive closures).

3.3. Lemma. *Let $(X,\pi)\in Net_{\mathscr{S}}$ and $(Y,\sigma)\in DNet_{\mathscr{S}}$ be objects, let $Z = C((X,\pi),(Y,\sigma))$, and let (Z,σ) be the function space of (X,π) and (Y,ρ) in $Net_{\mathscr{S}}$. Then the space (Z,σ) has point-wise structure, i.e., whenever $(S,g)\in\langle Z\rangle_{\mathscr{S}}$, $(S,g)\xrightarrow{\sigma} z$ iff $(S,g^x)\xrightarrow{\rho} z(x)$ for each $x\in X$ where $g^x : S\to Y$ is the map given by $g^x(i) = g(i)(x)$.*

Proof. Recall that $(S,g)\xrightarrow{\sigma} z$ iff $(S,f)\xrightarrow{\pi} x$ implies $(S,g^f)\to z(x)$ where $g^f : S\to Y$ is the map given by $g^f(i) = g(i)(f(i))$ - see the proof of Theorem 2.1. Let $\tau : \langle Z\rangle_{\mathscr{S}}\to \exp Z$ be the map given by $(S,g)\xrightarrow{\tau} z$ iff $(S,g^x)\xrightarrow{\rho} z(x)$ for each $x\in X$. Let $(S,g)\in\langle Z\rangle_{\mathscr{S}}$, $(S,g)\xrightarrow{\tau} z$, and let $(S,f)\in\langle X\rangle_{\mathscr{S}}, (S,f)\xrightarrow{\pi} x$. Then for each $i\in S$ we have $(S,g(i)\circ f)\xrightarrow{\rho} g(i)(x)$. Put $u_i = g(i)\circ f$ for each

$i \in S$ and $v_j = g^{f(j)}$ for each $j \in S$. Then $u_i(j) = g(i)(f(j)) = g^{f(j)}(i) = v_j(i)$ whenever $i, j \in S$. Obviously, $(S, u_i) \xrightarrow{\rho} g(i)(x)$ for each $i \in S$ and $(S, v_j) \xrightarrow{\rho} z(f(j))$ for each $j \in S$. Further, we have $(S, g^x) \xrightarrow{\rho} z(x)$ and $(S, z \circ f) \xrightarrow{\rho} z(x)$. As $g^f(i) = g(i)(f(i)) = u_i(i)$ for each $i \in S$, we have $(S, g^f) \xrightarrow{\rho} z(x)$ and consequently $(S, g) \xrightarrow{\rho} z$. Thus the implication $(s, g) \xrightarrow{\tau} z \Rightarrow (S, g) \xrightarrow{\sigma} z$ is valid. The inverse implication is obvious: For any $x \in X$ we have $(S, f) \xrightarrow{\pi} x$ whenever $f : S \to X$ is the x-constant map, thus from $(S, g) \xrightarrow{\sigma} z$ it follows that $(S, g^f) \xrightarrow{\rho} z(x)$ where $g^f(i) = g(i)(f(i)) = g(i)(x) = g^x(i)$. Hence $(S, g) \xrightarrow{\tau} z$. Therefore $\sigma = \tau$ which proves the statement.

Let us denote $DConv_{\mathscr{S}} = DNet_{\mathscr{S}} \cap Conv_{\mathscr{S}}$ and $DLim_{\mathscr{S}} = DNet_{\mathscr{S}} \cap Lim_{\mathscr{S}}$. Then we have

3.4. Theorem. *$DNet_{\mathscr{S}}$, $DConv_{\mathscr{S}}$ and $DLim_{\mathscr{S}}$ are cartesian closed topological categories. Their function spaces are inherited from $Net_{\mathscr{S}}$ and have point-wise structure.*

Proof. Let $F_k : X \to (X_k, \pi_k), k \in K$, be a source in $DNet_{\mathscr{S}}$ and let (X, π) be its initial object in $Net_{\mathscr{S}}$, i.e. $(S, f) \xrightarrow{\pi} x$ iff $(S, F_k \circ f) \xrightarrow{\pi_k} F_k(x)$ for each $k \in K$ - see the proof of Theorem 2.1. Let $\{(S, f_i); i \in S\}$ and $\{(S, g_j); j \in S\}$ be families of $\mathscr{S}$-nets in X such that $f_i(j) = g_j(i)$ whenever $i, j \in S$ and let $(S, f_i) \xrightarrow{\pi} x_i$ whenever $i \in S$ and $(S, g_j) \xrightarrow{\pi} y_j$ whenever $j \in S$. Further, let $f, g, h : S \to X$ be the maps given by $f(i) = x_i$, $g(j) = y_j$ and $h(i) = f_i(i)$, and suppose that $(S, f) \xrightarrow{\pi} x$, $(S, g) \xrightarrow{\pi} x$. Then, for each $k \in K$, $\{(S, F_k \circ f_i); i \in S\}$ and $\{(S, F_k \circ g_j), j \in S\}$ are families of $\mathscr{S}$-nets in X_k such that $(F_k \circ f_i)(j) = (F_k \circ g_j)(i)$ whenever $i, j \in S$, $(S, F_k \circ f_i) \xrightarrow{\pi_k} F_k(x_i)$ whenever $i \in S$ and $(S, F_k \circ g_j) \xrightarrow{\pi_k} F_k(y_j)$ whenever $j \in S$. Next, for each $k \in K$, the map $F_k \circ f : S \to X_k$ fulfils $(F_k \circ f)(i) = F_k(x_i)$ for each $i \in S$, the map $F_k \circ g : S \to X_k$ fulfils $(F_k \circ g)(j) = F_k(y_j)$ for each $j \in S$, and the map $F_k \circ h : S \to X_k$ fulfils $(F_k \circ h)(i) = F_k(f_i(i))$ for each $i \in S$. As $(S, F_k \circ g) \xrightarrow{\pi_k} F_k(x)$ and $(S, F_k \circ g) \xrightarrow{\pi_k} F_k(x)$ for each $k \in K$, $(S, F_k \circ h) \xrightarrow{\pi_k} F_k(x)$ for each $k \in K$ too. Consequently, $(S, h) \xrightarrow{\pi} x$. We have shown that $DNet_{\mathscr{S}}$ is closed in $Net_{\mathscr{S}}$ under formation of initial objects. As $Conv_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ are closed in $Net_{\mathscr{S}}$ under formation of initial objects, so are also $DConv_{\mathscr{S}}$ and $DLim_{\mathscr{S}}$. Hence $DNet_{\mathscr{S}}$, $DConv_{\mathscr{S}}$ and $DLim_{\mathscr{S}}$ are topological.

Let $(X, \pi), (Y, \rho) \in DNet_{\mathscr{S}}$ be objects and let (Z, σ) be the function space of (X, π) and (Y, ρ) in $Net_{\mathscr{S}}$. By the Lemma 3.3 we have $(S, p) \xrightarrow{\sigma} z$ iff $(S, p^x) \xrightarrow{\sigma} z(x)$ for each $x \in X$ where $p^x : S \to Y$ is the map defined by $g^x(s) = g(s)(x)$. Let $\{(S, f_i); i \in S\}$ and $\{(S, g_j); j \in S\}$ be families of $\mathscr{S}$-nets in Z such that $f_i(j) = g_j(i)$ whenever $i, j \in S$ and let $(S, f_i) \xrightarrow{\sigma} y_i$ whenever $i \in S$ and $(S, g_j) \xrightarrow{\sigma} z_j$ whenever $j \in S$. Let $f, g, h : S \to Z$ be the maps given by $f(i) = y_i$, $g(j) = z_j$, $h(i) = f_i(i)$, and let $(S, f) \xrightarrow{\sigma} z$, $(S, g) \xrightarrow{\sigma} z$. Then for any $x \in X$, $\{(S, f_i^x); i \in S\}$ and $\{(S, g_j^x); j \in S\}$ are families of $\mathscr{S}$-nets in Y such that $f_i^x(j) = f_i(j)(x) = g_j(i)(x) = g_j^x(i)$ whenever $i, j \in S$, $(S, f_i^x) \xrightarrow{\rho} y_i(x)$ whenever $i \in S$ and $(S, g_j^x) \xrightarrow{\rho} z_j(x)$ whenever $j \in S$. Further, for any $x \in X$ we have $f^x(i) = y_i(x)$ whenever $i \in S$, $g^x(j) = z_j(x)$ whenever $j \in S$, $h^x(i) = f_i^x(i)$ whenever $i \in S$, $(S, f^x) \xrightarrow{\rho} z(x)$ and $(S, g^x) \xrightarrow{\rho} z(x)$. Therefore $(S, h^x) \xrightarrow{\rho} z(x)$ for each $x \in X$. This results in $(S, h) \xrightarrow{\sigma} z$, i.e. $(Z, \sigma) \in DNet_{\mathscr{S}}$. Consequently, $DNet_{\mathscr{S}}$ is cartesian closed, and its function spaces

are inherited from $Net_{\mathscr{S}}$ and have point-wise structure.
Let $(X,\pi),(Y,\sigma)\in DConv_{\mathscr{S}}$ be objects and let (Z,σ) be their function space in $Net_{\mathscr{S}}$. Let $(S,p)\xrightarrow{\sigma} z$ and let $x\in X$. Let (T,q) be a subnet of (S,p). Then there is an $\mathscr{S}$-morphism $\varphi: T\to S$ with $q=p\circ\varphi$, and $(T,p^x\circ\varphi)$ is a subnet of (S,p^x). Therefore $(T,p^x\circ\varphi)\xrightarrow{\rho} z(x)$. But for every $t\in T$ we have $p^x(\varphi(t))=p(\varphi(t))(x)=q(t)(x)=q^x(t)$, i.e. $p^x\circ\varphi=q^x$. Hence $(T,q^x)\xrightarrow{\rho} z(x)$. We have shown that $(T,q)\xrightarrow{\sigma} z$. Consequently, $(Z,\sigma)\in Conv_{\mathscr{S}}$ and regarding the previous part of the proof $(Z,\sigma)\in DConv_{\mathscr{S}}$. Hence $DConv_{\mathscr{S}}$ is cartesian closed, and its function spaces are inherited from $Net_{\mathscr{S}}$ and have point-wise structure.
Finally, let $(X,\pi),(Y,\sigma)\in DLim_{\mathscr{S}}$ be objects and let (Z,σ) be their function space in $Net_{\mathscr{S}}$. Let $(S,f)\in\langle Z\rangle_{\mathscr{S}}$ has the property that each its subnet has a subnet converging to z (w.r.t. σ). Let $x\in X$ and let (T,g) be a subnet of (S,f^x). Then there is an $\mathscr{S}$-morphism $\varphi: T\to S$ with $g=f^x\circ\varphi$ and we have $(f^x\circ\varphi)(t)=f(\varphi(t))(x)=(f\circ\varphi)^x(t)$ for each $t\in T$, i.e. $g=(f\circ\varphi)^x$. As $(T,f\circ\varphi)$ is a subnet of (S,f), there is a subnet (U,g) of $(T,f\circ\varphi)$ such that $(U,g)\xrightarrow{\sigma} z$. Hence $(U,g^x)\xrightarrow{\rho} z(x)$. Let $\psi: U\to T$ be the $\mathscr{S}$-morphism with $g=f\circ\varphi\circ\psi$. As $g^x(u)=g(u)(x)=f(\varphi(\psi(u)))(x)=(f\circ\varphi)^x\psi(u)=g(\psi(u))$ for each $u\in U$, (U,g^x) is a subnet of (T,g). Consequently, $(S,f^x)\xrightarrow{\rho} z(x)$ and hence $(S,f)\xrightarrow{\sigma} z$. Thus, with regard to the previous part of the proof, $(Z,\sigma)\in DLim_{\mathscr{S}}$. Therefore $DLim_{\mathscr{S}}$ is cartesian closed, and its function spaces are inherited from $Net_{\mathscr{S}}$ and have point-wise structure. The statement is proved.

4. Relations of $Net_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ to some supercategories of *Top*

We denote by *Pcl* the category of preclosure spaces (i.e. pairs (X,u) where X is a set and $u:\exp X\to\exp X$ is a map fulfilling $u\phi=\phi$, $A\subseteq X\Rightarrow A\subseteq uA$ and $A\subseteq B\subseteq X\Rightarrow uA\subseteq uB$) and continuous maps (i.e. maps $F:(X,u)\to(Y,v)$ with $f(uA)\subseteq vf(A)$ whenever $A\subseteq X$). Further, we denote by *Clo* and *Wet* the full subcategories of *Pcl* whose objects are precisely the closure spaces (i.e. the preclosure spaces (X,u) fulfilling $u(A\cup B)=uA\cup uB$ whenever $A,B\subseteq X$) and the weakly topological spaces (i.e. the preclosure spaces (X,u) fulfilling $uuA=uA$ whenever $A\subseteq X$) respectively. Finally, we denote by *Top* the category of topological spaces, i.e. $Top=Clo\cap Wet$.

It is well known that for $\mathscr{S}=Dir$ the category *Top* can be fully concretely embedded into $Lim_{\mathscr{S}}$ - see [7]. It is also known that for $\mathscr{S}=\underline{\omega}$ the full subcategory of $Lim_{\mathscr{S}}$ given by the limit $\mathscr{S}$-net spaces in which each sequence converges to at most one point can be fully concretely embedded into *Top* - see [4]. We will find some next subcategories of $Net_{\mathscr{S}}$ and $Lim_{\mathscr{S}}$ that can be, for special categories $\mathscr{S}$, fully concretely embedded into *Pcl*, *Clo*, *Wet* or *Top*.

For any $\mathscr{S}$-net space (X,π) let $u_\pi:\exp X\to\exp X$ be the map given by $A\subseteq X\Rightarrow u_\pi A=\{x\in X;\text{ the exists }(S,f)\in\langle X\rangle_{\mathscr{S}}\text{ such that }f(S)\subseteq A\text{ and }(S,f)\xrightarrow{\pi} x\}$. The following assertion is evident:

4.1. Proposition. *The assignation $\pi\mapsto u_\pi$ determines a concrete functor from $Net_{\mathscr{S}}$ into Pcl.*

The concrete functor from 4.1, and also every its restriction, will be denoted by $\mathscr{F}_{\mathscr{S}}$.

4.2. Example. For $\mathscr{S} = \underline{1}$ the functor $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $\mathit{Net}_{\mathscr{S}}$ into *Pcl*. Namely, in this case, $\mathscr{F}_{\mathscr{S}}$ is an isomorphism of $\mathit{Net}_{\mathscr{S}}$ onto the full subcategory of *Clo* whose objects are precisely the quasi-discrete closure spaces [3], i.e. the closure spaces with completely additive closures.

4.3. Definition. The category $\mathscr{S}$ is called

(1) an *additively hereditary category* if for any $\mathscr{S}$-object S and any pair of sets T, U with $S = T \cup U$ there is a subobject of S whose underlying set is T or U,

(2) a *fork category* if for any pair of $\mathscr{S}$-objects T, U and any pair of elements $t \in T, u \in U$ there exists an $\mathscr{S}$-object S such that both the t-constant map of S into T and the u-constant map of S into U are $\mathscr{S}$-morphisms.

4.4. Example.

1. If $\mathscr{S} = \mathit{Dir}$, then $\mathscr{S}$ is additively hereditary.
2. If in $\mathscr{S}$ all constant maps are $\mathscr{S}$-morphisms, then $\mathscr{S}$ is a fork category.
3. If $\mathscr{S}$ is initially complete, then $\mathscr{S}$ is both additively hereditary and fork category.

4.5. Proposition. *If $\mathscr{S}$ is additively hereditary, then $\mathscr{F}_{\mathscr{S}}$ is a concrete functor from $\mathit{Net}_{\mathscr{S}}$ into Clo.*

Proof. Let $(X, \pi) \in \mathit{Net}_{\mathscr{S}}$, $A, B \subseteq X$ and $x \in u_\pi(A \cup B)$. Then there is $(S, f) \in \langle X\rangle_{\mathscr{S}}$ with $(S, f) \xrightarrow{\pi} x$ such that $f(S) \subseteq A \cup B$. Put $C = f(S) \cap A$ and $D = f(S) \cap B$. Then $f(S) = C \cup D$, i.e. $S = f^{-1}(C) \cup f^{-1}(D)$. Thus, there is a subobject T of S whose underlying set is $f^{-1}(C)$ or $f^{-1}(D)$. Without loss of generality we can suppose that $f^{-1}(C)$ is the underlying set of T. Then $(f^{-1}(C), f \restriction f^{-1}(C))$ is a subnet of (S, f). Hence $(f^{-1}, f \restriction f^{-1}(C)) \xrightarrow{\pi} x$ and, since $(f \restriction f^{-1}(C))(f^{-1}(C)) = C \subseteq A$, we have $x \in u_\pi A \subseteq u_\pi A \cup u_\pi B$. Therefore $u_\pi(A \cup B) \subseteq u_\pi A \cup u_\pi B$. As the converse inclusion is obvious, $(X, u_\pi) \in \mathit{Clo}$. Now the statement follows from 4.1.

4.6. Definition. We denote by $\widetilde{\mathit{Net}}_{\mathscr{S}}$ and $\widetilde{\mathit{Lim}}_{\mathscr{S}}$ the full subcategories of $\mathit{Net}_{\mathscr{S}}$ and $\mathit{Lim}_{\mathscr{S}}$, respectively, given by those their objects (X, π) for which the following condition is satisfied:
If $(S, f) \xrightarrow{\pi} x$ and $(T_s, g_s) \xrightarrow{\pi} f(s)$ for each $s \in S$, then there is $(U, h) \in \langle X\rangle_{\mathscr{S}}$ such that $(U, h) \xrightarrow{\pi} x$ and $h(U) \subseteq \bigcup\limits_{s \in S} g_s(T_s)$.

4.7. Proposition. *$\mathscr{F}_{\mathscr{S}}$ is a concrete functor from $\widetilde{\mathit{Net}}_{\mathscr{S}}$ into Wet.*

Proof. Let $(X, \pi) \in \widetilde{\mathit{Net}}_{\mathscr{S}}$, $A \subseteq X$ and $x \in u_\pi u_\pi A$. Then there is a net $(S, f) \in \langle X\rangle_{\mathscr{S}}$ with $(S, f) \xrightarrow{\pi} x$ such that $f(S) \subseteq u_\pi A$. Further, for each $s \in S$ there is a net $(T_s, g_s) \in \langle X\rangle_{\mathscr{S}}$ with $(T_s, g_s) \xrightarrow{\pi} f(s)$ such that $g_s(T_s) \subseteq A$. Thus, there is a net $(U, h) \in \langle X\rangle_{\mathscr{S}}$ with $(U, h) \xrightarrow{\pi} x$ and $h(U) \subseteq \bigcup\limits_{s \in S} g_s(T_s) \subseteq A$. Therefore $x \in u_\pi A$ and we have proved the inclusion $u_\pi u_\pi A \subseteq u_\pi A$. As the converse inclusion is obvious, $(X, u_\pi) \in \mathit{Wet}$ and the statement follows from 4.1.

4.8. Corollary. *If $\mathscr{S}$ is additively hereditary, then $\mathscr{F}_{\mathscr{S}}$ is a concrete functor from $\widetilde{\mathit{Net}}_{\mathscr{S}}$ into Top.*

4.9. Lemma. *Let $\mathscr{S}$ be a fork category, X a set and $(S,f),(T,g) \in \langle X\rangle_{\mathscr{S}}$. If $f(S) \subseteq g(T)$, then there exists a subnet of (S,f) which is also a subnet of (T,g).*

Proof. Let $x \in f(S)$ and let $s \in S, t \in T$ be arbitrary elements with $f(s) = g(t) = x$. Then there is an $\mathscr{S}$-object U such that both the s-constant map $\varphi : U \to S$ and the t-constant map $\psi : U \to T$ are $\mathscr{S}$-morphisms. Let $h : U \to X$ be the x-constant map. Then clearly (U,h) is a subnet of both (S,f) and (T,g).

4.10. Lemma. *Let $\mathscr{S}$ be a fork category and $(X,\pi) \in Lim_{\mathscr{S}}$ an object. Then for any $(S,f) \in \langle X\rangle_{\mathscr{S}}$ and any $x \in X$ we have $(S,f) \xrightarrow{\pi} x$ iff $x \in u_\pi g(T)$ for each subnet (T,g) of (S,f).*

Proof. Suppose that $(S,f) \stackrel{\pi}{\not\to} x$. Then there is a subnet (T,g) of (S,f) such that for any subnet (U,g) of (T,g) there holds $(U,h) \stackrel{\pi}{\not\to} x$. Let $(V,p) \in \langle X\rangle_{\mathscr{S}}$, $p(V) \subseteq g(T)$. By Lemma 4.9, there exists a subnet (W,q) of (V,p) which is also a subnet of (T,g). But then $(W,q) \stackrel{\pi}{\not\to} x$, hence $(V,p) \stackrel{\pi}{\not\to} x$. Therefore $x \notin u_\pi g(T)$. We have proved that from $x \in u_\pi g(T)$ for each subnet (T,g) of (S,f) it follows that $x \in u_\pi g(T)$. As the converse implication is obvious, the proof is complete.

4.11. Theorem. *If $\mathscr{S}$ is a fork category, then $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $Lim_{\mathscr{S}}$ into Pcl.*

Proof. Let $(X,\pi),(X,\rho) \in Lim_{\mathscr{S}}$ be objects, $\pi \neq \rho$. Without loss of generality we can suppose that there are $(S,f) \in \langle X\rangle_{\mathscr{S}}$ and $x \in X$ such that $(S,f) \xrightarrow{\pi} x$ but $(S,f) \stackrel{\rho}{\not\to} x$. Thus, by Lemma 4.10, $x \in u_\pi g(T)$ for each subnet (T,g) of (S,f) and there is a subnet (T_0,g_0) of (S,f) such that $x \notin u_\rho g_0(T_0)$. Hence $u_\pi \neq u_\rho$, i.e. $\mathscr{F}_{\mathscr{S}}$ is injective on objects.
Let $(X,\pi),(Y,\rho) \in Lim_{\mathscr{S}}$ and let $F : (X,u_\pi) \to (Y,u_\rho)$ be a continuous map. Let $(S,f) \xrightarrow{\pi} x$. Then, by Lemma 4.9, for each subnet (T,g) of (S,f) we have $x \in u_\pi g(T)$ and consequently $F(x) \in u_\rho F(g(T))$. But every subnet of $(S, F\circ f)$ has the form $(T, F\circ g)$ where (T,g) is a subnet of (S,f). Thus, by Lemma 4.10, $(S, F\circ f) \xrightarrow{\rho} F(x)$. Therefore $\mathscr{F}_{\mathscr{S}}$ is full. The assertion now results from 4.1.

4.12. Corollary. *If $\mathscr{S}$ is an additively hereditary and fork category, then $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $Lim_{\mathscr{S}}$ into Clo.*

Proof. It follows from 4.5 and 4.11.

4.13. Corollary.

(a) *If $\mathscr{S}$ is a fork category, then $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $\widetilde{Lim}_{\mathscr{S}}$ into Wet.*

(b) *If $\mathscr{S}$ is an additively hereditary and fork category, then $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $\widetilde{Lim}_{\mathscr{S}}$ into Top.*

Proof. It is a consequence of 4.7, 4.8 and 4.11.

4.14. Example. Let $\mathscr{S} = \underline{\omega}$. It is well known that the full subcategory of $Lim_{\mathscr{S}}$ whose objects are precisely the $\mathscr{L}^*$-spaces (see 1.5, example 3) is embeddable by $\mathscr{F}_{\mathscr{S}}$ into *Clo*. The $\mathscr{L}^*$-spaces that are objects of $\widetilde{Lim}_{\mathscr{S}}$ are called $\mathscr{S}^*$-spaces (see [6]). It is known that $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of the category of $\mathscr{S}^*$-spaces into *Top* (more precisely, $\mathscr{F}_{\mathscr{S}}$ is an isomorphism of the category of $\mathscr{S}^*$-spaces onto the category of those Fréchet topological spaces in which every sequence has at most one limit).

5. The condition of iterated limits

The condition introduced in 4.6 has several lacks. It is known that for $\mathscr{S} = \mathit{Dir}$ the functor $\mathscr{F}_{\mathscr{S}} : \widetilde{Lim}_{\mathscr{S}} \to \mathit{Top}$ is surjective on objects but it is not an isomorphism. Next, $\widetilde{Net}_{\mathscr{S}}$ and $\widetilde{Lim}_{\mathscr{S}}$ are evidently not closed under formation of finite products in $\mathit{Net}_{\mathscr{S}}$, and the same is valid for the full subcategories of $\widetilde{Net}_{\mathscr{S}}$ and $\widetilde{Lim}_{\mathscr{S}}$ given by their diagonal objects. Consequently, these subcategories are not closed under formation of function spaces in $\mathit{Net}_{\mathscr{S}}$. We will therefore replace the condition from 4.6 with a more convenient condition, the condition of iterated limits, which is an extension of the known condition introduced by J. L. Kelley for the case $\mathscr{S} = \mathit{Dir}$ (see [7]).

5.1. Definition. Let $\mathscr{S}$ have cartesian products and let (X, π) be a limit $\mathscr{S}$-net space. We say that (X, π) fulfils the *condition of iterated limits* if the following is valid: Let $S \in \mathscr{S}$ be an object and let $(T_s, g_s) \in \langle X\rangle_{\mathscr{S}}$ for each $s \in S$. Let $U = S \times \prod\limits_{s\in S} T_s$ and let $h : U \to X$ be the map given by $h(s,t) = g_s(t(s))$. Then from $(T_s, g_s) \xrightarrow{\pi} x_s$ for each $s \in S$ and $(S, f) \xrightarrow{\pi} x$ where $f : S \to X$ is given by $f(s) = x_s$ it follows that $(U, h) \xrightarrow{\pi} x$.

We denote by $\mathit{Cil}_{\mathscr{S}}$ the full subcategory of $\mathit{Lim}_{\mathscr{S}}$ whose objects are precisely the limit $\mathscr{S}$-net spaces fulfilling the condition of iterated limits. Clearly, $\mathit{Cil}_{\mathscr{S}}$ is a full subcategory of $\widetilde{Lim}_{\mathscr{S}}$ and thus $\mathscr{F}_{\mathscr{S}}$ is a concrete functor from $\mathit{Cil}_{\mathscr{S}}$ into Wet.

5.2. Definition. Let $\mathscr{S}$ have cartesian products. Then $\mathscr{S}$ is called *fine* if for each $\mathscr{S}$-object S there is a map $i \mapsto S^{(i)}$ of S into the class of all subobjects of S such that

(1) $\varphi : S \to T$ is an $\mathscr{S}$-morphism iff for any $j \in T$ there exists $i \in S$ such that $\varphi(S^{(i)}) \subseteq T^{(j)}$,
(2) $j \in S^{(i)} \Rightarrow S^{(j)} \subseteq S^{(i)}$,
(3) $(\prod\limits_{k\in K} S_k)^{(v)} \subseteq \prod\limits_{k\in K} S_k^{(v(k))}$.

5.3. Example.

1. The category Dir is fine. For any directed set $S = (S, \leq)$ and any $i \in S$ the directed subset $S^{(i)}$ of S is the set $S^{(i)} = \{s \in S; s \geq i\}$ directed by $\leq$.
2. The category Set^+ is fine. For any non-empty set S and any $i \in S$ the subset $S^{(i)}$ of S is given by $S^{(i)} = S$.

5.4. Lemma. *Let $\mathscr{S}$ be fine and let $(X, \pi) \in \mathit{Cil}_{\mathscr{S}}$ be an object. Then for any $(S, f) \in \langle X\rangle_{\mathscr{S}}$ and any $x \in X$ we have $(S, f) \xrightarrow{\pi} x$ iff $x \in u_\pi g(T)$ for each subnet (T, g) of (S, f).*

Proof. Let $x \in u_\pi g(T)$ for each subnet (T, g) of (S, f). Let (T, g) be a subnet of (S, f). Then, for each $i \in T$, $(T^{(i)}, g \restriction T^{(i)})$ is a subnet of (S, f) and hence $x \in u_\pi g(T^{(i)})$. Consequently, for each $i \in T$ there is a net $(U_i, h_i) \in \langle X\rangle_{\mathscr{S}}$ with $(U_i, h_i) \xrightarrow{\pi} x$ and $h(U_i) \subseteq g(T^{(i)})$. Put $W = T \times \prod\limits_{i\in T} U_i$ and let $p : W \to X$ be the map given by $p(i, v) = h_i(v(i))$. Now, since $(T, q) \xrightarrow{\pi} x$ whenever $q : T \to X$ is the x-constant map, we have $(W, p) \xrightarrow{\pi} x$. We will show that (W, p) is a subnet of (T, g). For each $i \in T$ let $\varphi_i : U_i \to T^{(i)}$ be an arbitrary map with the property

$h_i = g \circ \varphi_i$. Now we can define a map $\varphi : W \to T$ by putting $\varphi(i, v) = \varphi_i(v(i))$. Let $j \in T$ be an arbitrary element. Then for any $v \in \prod_{i \in T} U_i$ and any $(k, w) \in W^{(j,v)}$ we have $(k, w) \in (T \times \prod_{i \in T} U_i)^{(j,v)} \subseteq T^{(j)} \times \prod_{i \in T} U_i^{(v(i))}$. Hence $k \in T^{(j)}$ and $w(i) \in U_i^{(v(i))}$ for each $i \in T$. Consequently, $\varphi(k, w) = \varphi_k(w(k)) \in T^{(k)} \subseteq T^{(j)}$. Thus $\varphi(k, w) \in T^{(j)}$ and therefore $\varphi : W \to T$ is an $\mathscr{S}$-morphism. For any $(i, v) \in W$ we have $g(\varphi(i, v)) = g(\varphi_i(v(i))) = h_i(v(i)) = p(i, v)$, hence $g \circ \varphi = p$. We have shown that (W, p) is a subnet of (T, g). But (T, g) has been chosen arbitrarily and so it follows that $(S, f) \xrightarrow{\pi} x$. As the converse inclusion is obvious, the proof is complete.

5.5. Theorem. *If $\mathscr{S}$ is fine, then $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $Cil_{\mathscr{S}}$ into Wet.*

Proof. Using Lemma 5.4 we can prove Theorem 5.5 in an analogous way as we proved Theorem 4.11 by using Lemma 4.10.

¿From 4.5 and 5.5 we immediately get

5.6. Corollary. *If $\mathscr{S}$ is fine and additively hereditary, then $\mathscr{F}_{\mathscr{S}}$ is a full concrete embedding of $Cil_{\mathscr{S}}$ into Top.*

5.7. Example. It is well known that for $\mathscr{S} = Dir$ the functor $\mathscr{F}_{\mathscr{S}}$ is a concrete isomorphism of $Cil_{\mathscr{S}}$ onto *Top* - see [7]. Thus, if $\mathscr{S}$ is fine and additively hereditary, then $Cil_{\mathscr{S}}$ can be embedded into Cil_{Dir}.

If $\mathscr{S}$ has cartesian products, then we denote by $DCil_{\mathscr{S}}$ the full subcategory of $Cil_{\mathscr{S}}$ determined by those its objects that are diagonal.

5.8. Theorem. *Let $\mathscr{S}$ have cartesian products. Then $DCil_{\mathscr{S}}$ is a cartesian closed topological category with function spaces inherited from $Net_{\mathscr{S}}$.*

Proof. Let $F_k : X \to (X_k, \pi_k), k \in K$, be a source in $DCil_{\mathscr{S}}$ and let (X, π) be its initial object in $Net_{\mathscr{S}}$, i.e. let $(S, f) \xrightarrow{\pi} x$ iff $(S, F_k \circ f) \xrightarrow{\pi_k} F_k(x)$ for each $k \in K$ - see the proof of 2.1. Let $S \in \mathscr{S}$ be an object, let $(T_s, g_s) \in \langle X \rangle_{\mathscr{S}}$, $(T_s, g_s) \xrightarrow{\pi} x_s$ for each $s \in S$, and let $(S, f) \xrightarrow{\pi} x$ where $f : S \to X$ is the map given by $f(s) = x_s$. Further, let $U = S \times \prod_{s \in S} T_s$ and let $h : U \to X$ be the map given by $h(s, t) = g_s(t(s))$. Then for each $k \in K$ we have $(T_s, F_k \circ g_s) \in \langle X_k \rangle_{\mathscr{S}}$, $(T_s, F_k \circ g_s) \xrightarrow{\pi_k} F_k(x_s)$ for each $s \in S$, and $(S, F_k \circ f) \xrightarrow{\pi_k} F_k(x)$ where $F_k \circ f(s) = F_k(x_s)$. Next, we have $F_k \circ h : U \to X_k$, $(F_k \circ h)(s, t) = (F_k \circ g_s)(t(s))$. Therefore $(U, F_k \circ h) \xrightarrow{\pi_k} F_k(x)$ and consequently $(U, h) \xrightarrow{\pi} x$. Hence the initial object (X, π) fulfils the condition of iterated limits. Now, because $DLim_{\mathscr{S}}$ is closed under formation of initial objects in $Net_{\mathscr{S}}$, so is also $DCil_{\mathscr{S}}$. Therefore $DCil_{\mathscr{S}}$ is a topological category.
Let $(X, \pi), (Y, \rho) \in DCil_{\mathscr{S}}$ be objects and let (Z, σ) be their function space in $Net_{\mathscr{S}}$, i.e. let $Z = C((X, \pi), (Y, \rho))$ and let $(T, g) \xrightarrow{\sigma} z$ iff $(V, g^x) \xrightarrow{\rho} z(x)$ for each $x \in X$ where $g^x : V \to Y$ is the map defined by $g^x(s) = g(s)(x)$ - see 3.3. Let $S \in \mathscr{S}$ be an object, let $(T_s, g_s) \in \langle Z \rangle_{\mathscr{S}}$, $(T_s, g_s) \xrightarrow{\sigma} z_s$ for each $s \in S$, and let $(S, f) \xrightarrow{\pi} z$ where $f : S \to Z$ is the map given by $f(s) = z_s$. Further, let $U = S \times \prod_{s \in S} T_s$ and let $h : U \to Z$ be the map given by $h(s, t) = g_s(t(s))$. Then for each $x \in X$ we have $(T_s, g_s^x) \in \langle Y \rangle_{\mathscr{S}}, (T_s, g_s^x) \xrightarrow{\rho} z_s(x)$ for each $s \in S$, and $(S, f^x) \xrightarrow{\rho} z(x)$

where $f^x : S \to Y$ fulfils $f^x(s) = f(s)(x) = z_s(x)$ for each $s \in S$. Next, we have $h^x : U \to Y$ and $h^x(s,t) = h(s,t)(x) = g_s(t(s))(x) = g_s^x(t(s))$. Therefore $(U, h^x) \xrightarrow{\rho} z(x)$ and consequently $(U,h) \xrightarrow{\sigma} z$. We have shown that the function space (Z, σ) fulfils the condition of iterated limits. Because $DLim_{\mathscr{S}}$ has function spaces inherited from $Net_{\mathscr{S}}$, so has $DCil_{\mathscr{S}}$.

5.9. Example. If $\mathscr{S} = Dir$, then from 5.6 and 5.8 it follows that $\mathscr{F}_{\mathscr{S}}(DCil_{\mathscr{S}})$ is a cartesian closed topological subcategory of *Top*. But this subcategory is precisely the category of indiscrete topological spaces.

5.10. Remark. Obviously, the condition of iterated limits can be considered also for $\mathscr{S}$-net spaces or convergence $\mathscr{S}$-net spaces, respectively. Then, by 3.4 and the proof of 4.8, the full subcategory of $DNet_{\mathscr{S}}$ or $DConv_{\mathscr{S}}$, respectively, given by those its objects that fulfil the condition of iterated limits is a cartesian closed topological category with function spaces inherited from $Net_{\mathscr{S}}$.

6. $\mathscr{S}$-NET SPACES AND SEPARATION AXIOMS

In what follows we will deal with the question under which conditions on an $\mathscr{S}$-net space (X, π) the preclosure space (X, u_π) fulfils the lower separation axioms R_0, T_0 or T_1, respectively. For preclosure spaces these axioms are defined analogically as for topological spaces. It means that a preclosure space (X, u) is said to be
an R_0-*space* if $x \in u\{y\} \Rightarrow y \in u\{x\}$ whenever $x, y \in X$,
a T_0-*space* if $x \in u\{y\}, y \in u\{x\} \Rightarrow x = y$ whenever $x, y \in X$,
a T_1-*space* if $u\{x\} = \{x\}$ whenever $x \in X$.
In [8] R_0-spaces are called B^*-spaces, and in [2] T_0-spaces and T_1-spaces are called K-spaces and B-spaces, respectively. We denote by R_0-*Pcl*, T_0-*Pcl* and T_1-*Pcl* the full subcategories of *Pcl* whose objects are precisely the R_0-spaces, T_0-spaces and T_1-spaces, respectively. Clearly, a preclosure space is a T_1-space iff it is both R_0-space and T_0-space, i.e. $T_1\text{-}Pcl = R_0\text{-}Pcl \cap T_0\text{-}Pcl$.

The following assertion is obvious:

6.1. Proposition. *If (X, π) is an $\mathscr{S}$-net space, then*

(1) $(X, u_\pi) \in R_0$-*Pcl iff for any pair $x, y \in X$ and any $(S, f) \in \langle X \rangle_{\mathscr{S}}$ such that f is the x-constant map and $(S, f) \xrightarrow{\pi} y$ there exists $(T, g) \in \langle X \rangle_{\mathscr{S}}$ such that g is the y-constant map and $(T, g) \xrightarrow{\pi} x$,*
(2) $(X, u_\pi) \in T_0$-*Pcl iff for any $x, y \in X$ and any pair $(S, f), (T, g) \in \langle X \rangle_{\mathscr{S}}$ such that f is the x-constant map and g is the y-constant map from $(S, f) \xrightarrow{\pi} y$ and $(T, g) \xrightarrow{\pi} x$ it follows that $x = y$,*
(3) $(X, u_\pi) \in T_1$-*Pcl iff* $\operatorname{card} \pi(S, f) = 1$ *whenever $(S, f) \in \langle X \rangle_{\mathscr{S}}$ and f is a constant map.*

The full subcategory of $Net_{\mathscr{S}}$ whose objects are the $\mathscr{S}$-net spaces (X, π) fulfilling $(X, u_\pi) \in R_0$-*Pcl* is a bicoreflective subcategory of $Net_{\mathscr{S}}$ but, in general, it is not closed under formation of finite products in $Net_{\mathscr{S}}$ and hence is not closed under formation of function spaces in $Net_{\mathscr{S}}$. Neither the full subcategory of $Net_{\mathscr{S}}$ whose objects are the $\mathscr{S}$-net spaces (X, π) fulfilling $(X, u_\pi) \in T_0$-*Pcl*, which is closed under formation of initial objects for monosources in $Net_{\mathscr{S}}$ (and hence initially structured), is closed under formation of function spaces in $Net_{\mathscr{S}}$ in general. In each of these two categories we will therefore find a full cartesian closed subcategory with function spaces inherited from $Net_{\mathscr{S}}$.

6.2. Definition. We denote by R_0-$Net_{\mathscr{S}}$ the full subcategory of $Net_{\mathscr{S}}$ whose objects are precisely the $\mathscr{S}$-net spaces (X,π) fulfilling the following condition:
Let $\{(S,f_i); i \in S\}$ be a family of $\mathscr{S}$-nets in X such that $f_i(i) = x$ ($x \in X$ an element) for each $i \in S$. Let $(S,f_i) \xrightarrow{\pi} x_i$ for each $i \in I$ and let $(S,f) \in \langle X\rangle_{\mathscr{S}}$, $(S,f) \xrightarrow{\pi} x$. For every $j \in S$ let $g_j : S \to X$ be the map given by $g_j(i) = f_i(j)$ and let $g : S \to X$ be the map given by $g(i) = x_i$. Then $(S,g_j) \xrightarrow{\pi} f(j)$ for each $j \in S$ and $(S,g) \xrightarrow{\pi} x$.

R_0-$Net_{\mathscr{S}}$ is clearly closed under formation of initial objects in $Net_{\mathscr{S}}$, i.e. R_0-$Net_{\mathscr{S}}$ is a topological category.

As an immediate consequence of 4.1 and 6.2 we get

6.3. Corollary. *$\mathscr{F}_{\mathscr{S}}$ is a concrete functor from R_0-$Net_{\mathscr{S}}$ into R_0-Pcl.*

6.4. Lemma. *Let $(X,\pi) \in R_0$-$Net_{\mathscr{S}}$, $(S,f) \in \langle X\rangle_{\mathscr{S}}$ and $(S,f) \xrightarrow{\pi} x$.*

(1) *If $j,k \in S$ and $f_{j,k} : S \to X$ is the map given by $f_{j,k}(i) = f(j)$ whenever $i \neq j$ and $f_{j,k}(j) = f(k)$, then $(S,f_{j,k}) \xrightarrow{\pi} f(j)$.*
(2) *If $j \in S$ and $g_j : S \to X$ is the map given by $g_j(i) = f(j)$ whenever $i \neq j$ and $g_j(j) = x$, then $(S,g_j) \xrightarrow{\pi} f(j)$.*
(3) *If $g : S \to X$ is the x-constant map, then $(S,g) \xrightarrow{\pi} f(j)$ for each $j \in S$.*

Proof. Let the assumption of (1) be fulfilled and for each $i \in S$ let $f_i : S \to X$ be the map given by $f_i(m) = f(j)$ whenever $i \neq j$ and $f_j(m) = f(m)$. Then $(S,f_i) \in \langle X\rangle_{\mathscr{S}}$ and $f_i(i) = f(j)$ for each $i \in S$. Next, we have $(S,f_i) \xrightarrow{\pi} f(j)$ whenever $i \neq j$ and $(S,f_j) \xrightarrow{\pi} x$. Let $h : S \to X$ be the $f(j)$-constant map. Then $(S,h) \xrightarrow{\pi} f(j)$. Finally, let $g_k : S \to X$ be the map defined by $g_k(i) = f_i(k)$. Then, by the definition of R_0-$Net_{\mathscr{S}}$, $(S,g_k) \xrightarrow{\pi} h(k) = f(j)$. But it is evident that $g_k = f_{j,k}$.
Let the assumption of (2) be fulfilled and let $f_i, i \in I$, and h be defined in the same way as in the previous part of the proof. Let $g : S \to X$ be the map given by $g(i) = f(j)$ whenever $i \neq j$ and $g(j) = x$. Then, by the definition of R_0-$Net_{\mathscr{S}}$, $(S,g) \xrightarrow{\pi} f(j)$. But it is evident that $g = g_j$.
Finally, let the assumption of (3) be fulfilled and for each $i \in S$ let $f_i : S \to X$ be the x-constant map. Then, for each $i \in S$, $f_i(i) = x$ and $(S,f_i) \xrightarrow{\pi} x$. Let $j \in S$ and let $g_j : S \to X$ be the x-constant map. Then $g_j(i) = f_i(j)$ for each $i \in S$ and, because $(S,f) \xrightarrow{\pi} x$, by the definition of R_0-$Net_{\mathscr{S}}$ we get $(S,g_j) \xrightarrow{\pi} f(j)$. But $g_j = g$. The proof is complete.

6.5. Theorem. *R_0-$Net_{\mathscr{S}}$ is a cartesian closed topological category with function spaces inherited from $Net_{\mathscr{S}}$.*

Proof. Clearly, R_0-$Net_{\mathscr{S}}$ is closed under formation of initial objects in $Net_{\mathscr{S}}$. Let $(X,\pi),(Y,\rho) \in R_0$-$Net_{\mathscr{S}}$ and let (Z,σ) be the function space of (X,π) and (Y,ρ) in $Net_{\mathscr{S}}$. Let $\{(S,h_i); i \in S\}$ be a family of $\mathscr{S}$-nets in Z such that $h_i(i) = z$ ($z \in Z$) for each $i \in I$. Let $(S,h_i) \xrightarrow{\sigma} z_i$ for each $i \in I$ and let $(S,h) \in \langle Z\rangle_{\mathscr{S}}$, $(S,h) \xrightarrow{\sigma} z$. Let $j \in S$ and let $q_j : S \to Z$ be the map given by $q_j(i) = h_i(j)$. Let

$(S,f) \in \langle X\rangle_{\mathscr{S}}$, $(S,f) \xrightarrow{\pi} x$. For any $k \in S$ let $f_{j,k} : S \to X$ be the map defined in Lemma 6.4. Then $(S, f_{j,k}) \xrightarrow{\pi} f(j)$ and, obviously, $f_{j,k}(k) = f(j)$ for each $k \in K$. Let $g_j : S \to X$ be the map defined in Lemma 6.4. Then$(S, g_j) \xrightarrow{\pi} f(j)$. Thus, for each $i \in S$ we have $(S, h_i^p) \xrightarrow{\rho} z_i(f(j))$ where $p = f_{j,i}$, and $h_i^p(i) = z(f(j))$ for each $i \in S$. Next, we have $(S, h^{g_j}) \xrightarrow{\rho} z(f(j))$. Consequently, if $r : S \to Y$ is the map given by $r(i) = h_i^p(j)$, then $(S,r) \xrightarrow{\rho} h^{g_j}(j)$. But $r(i) = h_i^p(j) = h_i(j)(p(j)) = h_i(j)(f_{j,i}(j)) = h_i(j)(f(i)) = q_j(i)(f(i)) = q_j^f(i)$ for each $i \in S$ and $h^{g_j}(j) = h(j)(g_j(j)) = h(j)(x)$. Therefore $(S, q_j) \xrightarrow{\sigma} h(j)$. Now, let $q : S \to Z$ be the map given by $q(i) = z_i$. For any $i \in S$ let $f_i : S \to X$ be the x-constant map. Then $f_i(i) = x$ and by Lemma 6.4 we have $(S, f_i) \xrightarrow{\pi} f(i)$. Consequently, $(S, h_i^{f_i}) \xrightarrow{\rho} z_i(f(i))$ and $h_i^{f_i}(i) = z(x)$. Next, we also have $(S, h^f) \xrightarrow{\rho} z(x)$. Thus, if $g : S \to Y$ is the map given by $g(i) = z_i(f(i))$, then $(S,g) \xrightarrow{\rho} z(x)$. But $g(i) = z_i(f(i)) = q(i)(f(i)) = q^f(i)$ and therefore $(S,q) \xrightarrow{\sigma} z$. We have proved that $(Z,\sigma) \in R_0\text{-}Net_{\mathscr{S}}$. This completes the proof.

6.6. Example. If $\mathscr{S} = \underline{1}$, then $\mathscr{F}_{\mathscr{S}}$ is an isomorphism of R_0-$Net_{\mathscr{S}}$ onto the category of quasi-discrete closure R_0-spaces. In [3] closure R_0-spaces are called semi-uniformizable spaces.

6.7. Definition. We denote by T_0-$Net_{\mathscr{S}}$ the full subcategory of $Net_{\mathscr{S}}$ whose objects are precisely the $\mathscr{S}$-net spaces (X,π) fulfilling the following condition:
Let $\{(S,f_i); i \in S\}$ and $\{(S,g_j); j \in S\}$ be families of $\mathscr{S}$-nets in X such that $f_i(j) = g_j(i)$ for each $i,j \in S$ and $f_i(i) = x$ ($x \in X$ an element) for each $i \in S$. Let $(S,f_i) \xrightarrow{\pi} x_i$ for each $i \in S$ and $(S,g_j) \xrightarrow{\pi} y_j$ for each $j \in S$. Let $f,g : S \to X$ be the maps given by $f(i) = x_i$, $g(j) = y_j$, and let $(S,f) \xrightarrow{\pi} x$ and $(S,g) \xrightarrow{\pi} x$. Then f_i is x-constant for each $i \in S$ and also both f and g are x-constant.

Clearly, T_0-$Net_{\mathscr{S}}$ is closed under formation of initial objects for monosources in $Net_{\mathscr{S}}$, hence T_0-$Net_{\mathscr{S}}$ is initially structured. The following statement is an immediate consequence of 4.1 and 6.7:

6.8. Corollary. *$\mathscr{F}_{\mathscr{S}}$ is a concrete functor from T_0-$Net_{\mathscr{S}}$ into T_0-Pcl.*

6.9 Theorem. *T_0-$Net_{\mathscr{S}}$ is a cartesian closed initially structured category with function spaces inherited from $Net_{\mathscr{S}}$.*

Proof. Clearly, T_0-$Net_{\mathscr{S}}$ is closed under formation of initial objects for monosources in $Net_{\mathscr{S}}$. Let $(X,\pi),(Y,\rho) \in T_0\text{-}Net_{\mathscr{S}}$ and let (Z,σ) be the function space of (X,π) and (Y,ρ) in $Net_{\mathscr{S}}$. Let $\{(S,p_i); i \in S\}$ and $\{(S,q_j); j \in S\}$ be families of $\mathscr{S}$-nets in Z such that $p_i(j) = q_j(i)$ for each $i,j \in S$ and $p_i(i) = z$ for each $i \in S$. Let $(S,p_i) \xrightarrow{\sigma} z_i$ for each $i \in S$ and $(S,q_j) \xrightarrow{\sigma} t_j$ for each $j \in S$. Let $p,q : S \to Z$ be the maps given by $p(i) = z_i$, $q(j) = t_j$ and let $(S,p) \to z$, $(S,q) \to z$. Let $x \in X$ be an element and $f : S \to X$ the x-constant map. Then $p_i^f(j) = p_i(j)(f(j)) = q_j(i)(f(i)) = q_j^f(i)$ for each $i,j \in S$ and $p_i^f(i) = z(x)$ for each $i \in S$. Next, as $(S,f) \xrightarrow{\pi} x$, we have $(S,p_i^f) \xrightarrow{\rho} z_i(x)$ and $(S,q_j^f) \xrightarrow{\rho} t_j(x)$. Let $g,h : S \to Y$ be the maps given by $g(i) = z_i(x)$ and $h(j) = t_j(x)$. Then $(S,g) \xrightarrow{\rho} z(x)$ and $(S,h) \xrightarrow{\rho} z(x)$ because $g(i) = z_i(x) = p(i)(f(i)) = p^f(i)$ for each $i \in S$ and $h(j) = t_j(x) = q(j)(f(j)) = q^f(j)$ for each $j \in S$. Consequently, p_i^f is the $z(x)$-constant map and both g and h

are the $z(x)$-constant maps. Thus, for each $i \in S$ we have $p_i(i)(x) = p_i^f(i) = z(x)$, hence p_i is the z-constant map. Similarly, we have $p(i)(x) = z_i(x) = g(i) = z(x)$ for each $i \in S$ and $q(j)(x) = t_j(x) = h(j) = z(x)$ for each $j \in S$. Hence also p and q are the z-constant maps. This completes the proof.

6.10. Example. Let $\mathscr{S} = \underline{1}$. Then $\mathscr{F}_{\mathscr{S}}$ is an isomorphism of T_0-$Net_{\mathscr{S}}$ onto the category of quasi-discrete closure T_0-spaces. In [3] closure T_0-spaces are called feebly semi-separated spaces (and closure T_1-spaces are called semi-separated spaces).

6.11. Remark. R_0-$Net_{\mathscr{S}} \cap T_0$-$Net_{\mathscr{S}}$ is precisely the category of discrete $\mathscr{S}$-net spaces.

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Department of Mathematics, Technical University of Brno, Technická 2, 616 69 Brno, Czech Republic

E-mail: slapal@kinf.fme.vutbr.cz