Compact Traces in Weighted Sobolev Spaces

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Preprint Nr. 29 A 95

Abstract. We study trace operators in weighted Sobolev spaces $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\partial\Omega; w)$ for sufficiently regular unbounded domains $\Omega \subset I\!\!R^N$ with noncompact boundary. We show that under certain conditions on the weight functions v_0, v_1, w, this operator is compact. This result is then used to prove existence of a nontrivial solution of a semilinear elliptic equation with nonlinear boundary condition in an unbounded domain.

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1. INTRODUCTION

In the study of partial differential equations in a domain $\Omega \subset I\!R^N$ it is often useful to know that embeddings of the Sobolev space $W^{m,p}(\Omega)$ into the Lebesgue space $L^q(\Omega)$ are compact. In particular for nonlinear equations of variational form such a compactness property is often used to show that the energy functional for this equation satisfies the Palais-Smale condition. In the case when Ω is not bounded, compactness of the embedding above fails in general. So it seems to be natural to study more general function spaces, for instance weighted Sobolev spaces, where compact embeddings can be obtained for suitable weight functions (see e. g. [1], [4], [6]). Such an approach using weighted norms was used by several authors for the study of semilinear elliptic equations in $I\!R^N$ (see e. g. [2], [7], [12]). Now in the study of nonhomogeneous (or nonlinear) Neumann boundary value problems, there appears an additional difficulty since then a boundary integral appears in the energy functional. If Ω is an unbounded domain with noncompact boundary one is confronted with two problems: first, there is no general result for embeddings $W^{m,p}(\Omega; V) \hookrightarrow L^q(\Omega; w)$ if the boundary $\partial\Omega$ is not compact (here V denotes an admissible set of weight functions acting on the derivatives of functions from $W^{m,p}$), and second there is no result which guarantees that the trace operator $W^{m,p}(\Omega; V) \hookrightarrow L^q(\partial\Omega; w)$ is compact. The aim of the present paper is to fill this gap and to give an application of these embedding, resp. trace theorems to a semilinear elliptic problem with nonlinear boundary condition, which was studied before by the author in [8] with different methods.

This paper is organized as follows. In Section 2 we introduce the necessary notations and state a recent result on S.-K. Chua concerning the existence of a continuous extension operator in weighted Sobolev spaces. Then we prove a basic lemma, which is used several times in the paper. At the end of this section we state our main results (Theorems 3 and 4). In Section 3 we study traces on $(N-k)$–dimensional planes in $I\!R^N$ (Theorem 5) and obtain as a corollary (Corollary 6) a compact embedding result for fairly general domains Ω. In Section 4 the proof of Theorems 3 and 4 is given, and in Section 5 we state some concrete examples for admissible weight functions. In the last Section 6 a nonlinear elliptic boundary value problem is studied, where Theorem 4 and Corollary 6 are essentially used. For the proof of our trace theorems we use an approach similar to that of Kufner/Opic in [6] for embeddings of weighted Sobolev spaces.

2. NOTATIONS AND MAIN RESULTS

For a measurable function u and a positive measurable function w, defined in a domain

$\Omega \subset \mathbb{R}^N$, we define the weighted p-norm $(1 \leq p < \infty)$

$$\|u\|_{p,\Omega,w} = \left(\int_\Omega |u(x)|^p w(x)\, dx \right)^{1/p}$$

and denote by $L^p(\Omega; w)$ the space of all measurable functions u such that $\|u\|_{p,\Omega,w}$ is finite. Correspondingly for a submanifold Γ in $\mathbb{R}^N$, we define $L^q(\Gamma; w)$ to be the space of all measurable functions u such that $\|u\|_{q,\Gamma,w}$ is finite. The weighted Sobolev space $W^{m,p}(\Omega; v_0, \ldots, v_m)$ is defined as the set of all functions $u \in L^p(\Omega, v_0)$ such that all derivatives $D^\alpha u$ $(|\alpha| \leq m)$ belong to $L^p(\Omega, v_{|\alpha|})$. The corresponding norm is defined by

$$\|u\|_{m,p,\Omega,v_\alpha} = \left(\int_\Omega \sum_{|\alpha|=0}^{m} |D^\alpha u(x)|^p v_{|\alpha|}(x)\, dx \right)^{1/p}.$$

A domain $\Omega \subset \mathbb{R}^N$ is called an (ε, δ)-domain if for all $x, y \in \Omega$ satisfying $|x-y| < \delta$ there exists a rectifiable curve ω in Ω connecting x and y, such that

$$\ell(\omega) < \frac{|x-y|}{\varepsilon} \qquad \text{and} \qquad d(z) > \frac{\varepsilon |x-z|\,|y-z|}{|x-y|} \quad \text{for every } z \in \omega$$

hold. Here, $\ell(\omega)$ is the length of the curve ω and $d(z)$ denotes the distance of z from the boundary of Ω.

The Muckenhoupt class A_p is defined as the set of all positive functions v in $\mathbb{R}^N$ which satisfy

$$\begin{aligned} \frac{1}{|Q|} \left(\int_Q v\, dx \right)^{1/p} \left(\int_Q v^{\frac{-1}{p-1}}\, dx \right)^{(p-1)/p} &\leq C \qquad\qquad (1 < p < \infty) \\ \frac{1}{|Q|} \int_Q v\, dx &\leq C \operatorname*{ess\,inf}_{x \in Q} v(x) \qquad (p = 1) \end{aligned}$$

for all cubes Q in $\mathbb{R}^N$.

In the present paper we shall often make use of the following result of S.-K. Chua:

THEOREM 1. (Chua [3]) *Let Ω be an unbounded (ε, ∞)-domain, let $m \in \mathbb{N}$, $1 \leq p < \infty$, $v_i \in A_p$ $(i = 0, \ldots, m)$ be given. Then there exists a bounded linear extension operator*

$$\Lambda : W^{m,p}(\Omega; v_0, \ldots, v_m) \longrightarrow W^{m,p}(\mathbb{R}^N; v_0, \ldots, v_m)$$

satisfying $\Lambda f = f$ a. e. on Ω. The norm of Λ depends only on $\varepsilon, v_i, m, p, N$.

We assume in the sequel that our weight functions v_i and w are defined in all of $\mathbb{R}^N$ and belong to A_p. Since we consider traces, we also assume that w is continuous so that its trace on a submanifold is defined. The theorem above is then used to transfer our trace theorems in $I\!R^N$ to more general domains Ω. In order to avoid too much notation, we denote a function u defined in a domain Ω, its extension Λu on $I\!R^N$ and its trace γu on a submanifold Γ by the same symbol u. Our main tool in the proof of the trace theorems below is the following.

LEMMA 2. *Let D_n be an increasing sequence of bounded domains in $I\!R^N$ such that $I\!R^N = \cup_n D_n$. Let Γ be a smooth $(N-k)$-dimensional submanifold of $I\!R^N$ and denote $\Gamma_n = D_n \cap \Gamma$, $\Gamma^n = \Gamma \setminus \overline{\Gamma}_n$. Assume that*

$$W^{1,p}(D_n; v_0, v_1) \hookrightarrow L^q(\Gamma_n; w) \qquad \text{is compact for every } n, \tag{1}$$

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,I\!R^N,v_0,v_1}\le 1} \|u\|_{q,\Gamma^n,w} \; = \; 0\,. \tag{2}$$

Then the trace operator $W^{1,p}(I\!R^N; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ is compact. On the other hand, if the trace operator is compact, then (2) holds.
If

$$W^{1,p}(D_n; v_0, v_1) \hookrightarrow L^q(\Gamma_n; w) \qquad \text{is continuous for every } n, \tag{3}$$

$$\lim_{n\to\infty} \sup_{\|u\|_{1,p,I\!R^N,v_0,v_1}\le 1} \|u\|_{q,\Gamma^n,w} \; < \; \infty\,, \tag{4}$$

then the trace operator $W^{1,p}(I\!R^N; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ is continuous. If the trace operator is continuous, then (4) holds.

Proof. First of all we notice that (2) is equivalent to the statement that to every $\varepsilon > 0$ there exists an $\tilde{n}$, such that

$$\|u\|^q_{q,\Gamma,w} \le \varepsilon \|u\|^q_{1,p,I\!R^N,v_0,v_1} + \|u\|^q_{q,\Gamma_{\tilde{n}},w} \quad \text{for every} \quad u \in W^{1,p}(I\!R^N; v_0, v_1)\,. \tag{5}$$

Indeed, let (2) hold; then for every $\varepsilon > 0$ there is some $\tilde{n} \in I\!N$ such that for every $n \ge \tilde{n}$ we have

$$\|u\|^q_{q,\Gamma^n,w} \le \varepsilon \|u\|^q_{1,p,I\!R^N,v_0,v_1}\,.$$

Since

$$\|u\|^q_{q,\Gamma,w} = \|u\|^q_{q,\Gamma_n,w} + \|u\|^q_{q,\Gamma^n,w}\,,$$

we obtain (5).
On the other hand, if (5) is satisfied, for every $\varepsilon > 0$ we can choose $\tilde{n}$, such that

$$\|u\|^q_{q,\Gamma^{\tilde{n}},w} + \|u\|^q_{q,\Gamma_{\tilde{n}},w} \le \varepsilon \|u\|^q_{1,p,I\!R^N,v_0,v_1} + \|u\|^q_{q,\Gamma_{\tilde{n}},w},$$

which implies

$$\|u\|^q_{q,\Gamma^n,w} \le \varepsilon \|u\|^q_{1,p,I\!R^N,v_0,v_1} \qquad \text{for every} \quad n \ge \tilde{n}$$

and (2) follows.
Now let u_j be a bounded sequence in $W^{1,p}(\mathbb{R}^N; v_0, v_1)$, $\|u_j\|_{1,p,\mathbb{R}^N,v_0,v_1} \leq C$ and let $\varepsilon > 0$ be given. From (1) it follows that there is a subsequence which is a Cauchy sequence in $L^q(\Gamma_{\tilde{n}}; w)$. Therefore we can find n_ε such that for every $i, j \geq n_\varepsilon$

$$\begin{aligned} \|u_j - u_i\|^q_{q,\Gamma,w} &\leq \varepsilon \|u_j - u_i\|^q_{1,p,\mathbb{R}^N,v_0,v_1} + \|u_j - u_i\|^q_{q,\Gamma_{\tilde{n}},w} \\ &\leq (2^q C^q + 1)\varepsilon \,. \end{aligned}$$

Consequently, u_j contains a Cauchy sequence in $L^q(\Gamma; w)$ and the trace operator is compact.
Now let $W^{1,p}(\mathbb{R}^N; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ be compact and assume that (5) does not hold. Then there exist an $\varepsilon > 0$ and a sequence $(u_n)_n$ in $W^{1,p}(\mathbb{R}^N; v_0, v_1)$, such that

$$\|u_n\|^q_{q,\Gamma,w} > \varepsilon \|u_n\|^q_{1,p,\mathbb{R}^N,v_0,v_1} + \|u_n\|^q_{q,\Gamma^n,w} \,.$$

This shows $a_n = \|u_n\|^q_{1,p,\mathbb{R}^N,v_0,v_1} > 0$ and for $\tilde{u}_n = u_n / a_n$ we obtain

$$\|\tilde{u}_n\|^q_{q,\Gamma,w} > \varepsilon + \|\tilde{u}_n\|^q_{q,\Gamma^n,w} \,. \tag{6}$$

Since $\tilde{u}_n$ is bounded in $W^{1,p}(\mathbb{R}^N; v_0, v_1)$, there is a subsequence converging to $\tilde{u}$ in $L^q(\Gamma; w)$. Now taking the limit in (6), we get

$$\|\tilde{u}\|^q_{q,\Gamma,w} > \varepsilon + \|\tilde{u}\|^q_{q,\Gamma,w} \,,$$

which is a contradiction.
The second part of Lemma 2 can be proved in a similar way; one just has to note that (4) implies $\|u\|_{q,\Gamma^n,w} \leq C_1 \|u\|_{1,p,\mathbb{R}^N,v_0,v_1}$ and (3) means that $\|u\|_{q,\Gamma_n,w} \leq C_2 \|u\|_{1,p,D_n,v_0,v_1}$. It follows that $\|u\|_{q,\Gamma,w} \leq C \|u\|_{1,p,\mathbb{R}^N,v_0,v_1}$. □

To apply this lemma to certain classes of weight functions, we need some estimates for the weights. First of all we define our covering by the family of cubes with edge lenght n

$$D_n = \{x \in \mathbb{R}^N \mid |x| < n\} \,, \qquad D^n = \operatorname{int} \left(\mathbb{R}^N \setminus D_n\right)$$

($|\cdot|$ denotes the maximum norm in $\mathbb{R}^N$). In particular we have $\Gamma_n = \{x \in \Gamma \mid |x| < n\}$, $\Gamma^n = \{x \in \Gamma \mid |x| > n\}$. Then we assume the existence of positive, continuous functions r, b_0, b_1 defined in $\mathbb{R}^N$ and constants K_0, c_r such that for some $\tilde{n} \geq 2$ the following conditions hold

$$r(x) \leq |x|/3 \qquad \text{for every } \; x \in D^{\tilde{n}} \,, \tag{7}$$

$$c_r^{-1} \leq r(x)/r(y) \leq c_r \qquad \text{for every } \; x \in D^{\tilde{n}} \,, y \in Q(x, r(x)) \tag{8}$$

$$v_1(x) r^{-p}(x) \leq K_0 v_0(x) \qquad \text{for a. e. } \; x \in D^{\tilde{n}} \tag{9}$$

$$w(y) \leq b_0(x) \,,\; b_1(x) \leq v_1(y) \qquad \text{for every } \; x \in D^{\tilde{n}} \,, \text{ a. e. } \; y \in Q(x, r(x)) \tag{10}$$

Examples for such functions are given in Section 5. We denote by $Q(x,r(x))$ the cube in $I\!R^N$ centered at x with edge lenght $r(x)$. Condition (7) implies that if $Q(x,r(x)) \cap D^{3n} \neq \emptyset$ and $n \geq \tilde{n}$, then $|x| > 2n$. Consequently, if $y \in Q(x,r(x))$, then $|y| \geq |x| - r(x) \geq (2/3)|x| > n$, which shows that $Q(x,r(x)) \subset D^n$ if $Q(x,r(x)) \cap D^{3n} \neq \emptyset$.

Finally we define

$$\mathcal{B}_{n,k} = \sup_{x \in D^n} \frac{b_0^{1/q}(x)}{b_1^{1/p}(x)} \, r^{\frac{N-k}{q} - \frac{N}{p} + 1}(x) \, .$$

In this paper we always suppose that Ω is an unbounded, connected domain with sufficiently smooth boundary $\Gamma = \partial\Omega$ (the exact assumptions on Γ are formulated in Section 4). In particular, Ω satisfies the assumptions of Theorem 1. We define $\Omega_n = D_n \cap \Omega$, $\Gamma_n = D_n \cap \Gamma$. Then $\Omega = \bigcup \Omega_n$, $\Gamma = \bigcup \Gamma_n$ and we have the following results:

<u>THEOREM 3.</u> *Let* $1 \leq p \leq q < \infty$ *and* $\dfrac{N-1}{q} - \dfrac{N}{p} + 1 \geq 0$. *Assume that* $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Gamma_n; w)$ *is continuous for every* n *and* r, v_0, v_1, w *satisfy (7)–(10).*
If $\lim_{n\to\infty} \mathcal{B}_{n,1} < \infty$, *then there exists a continuous trace operator* $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\partial\Omega; w)$.

<u>THEOREM 4.</u> *Let* $1 \leq p \leq q < \infty$ *and* $\dfrac{N-1}{q} - \dfrac{N}{p} + 1 \geq 0$. *Assume that* $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Gamma_n; w)$ *is compact for every* n *and* r, v_0, v_1, w *satisfy (7)–(10).*
If $\lim_{n\to\infty} \mathcal{B}_{n,1} = 0$, *then the trace operator* $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\partial\Omega; w)$ *is compact.*

The proof of these theorems is given in Section 4.

<u>3. COMPACT TRACES ON PLANES</u>

The aim of this section is to prove a special case of Theorems 3 and 4, namely

<u>THEOREM 5.</u> *Let* $1 \leq p \leq q < \infty$ *and* $\dfrac{N-k}{q} - \dfrac{N}{p} + 1 \geq 0$ *and* Γ *be the intersection of the domain* Ω *with an* $(N-k)-$ *dimensional plane. Denote* $\Omega_n = D_n \cap \Omega$, $\Gamma_n = \Gamma \cap \Omega_n$, $\Gamma^n = \Gamma \setminus \overline{\Gamma}_n$ *and let* r, v_0, v_1, w *satisfy (7)–(10)*
Assume that $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Gamma_n; w)$ *is compact for every* n. *If* $\lim_{n\to\infty} \mathcal{B}_{n,k} = 0$, *then*

$W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ *is compact.*

If $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Gamma_n; w)$ *is continuous for every* n *and* $\lim_{n\to\infty} \mathcal{B}_{n,k} < \infty$*, then* $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ *is continuous.*

Proof. Without restriction we may assume that Γ has the form

$$\Gamma = \Omega \cap I\!R^{N-k} = \{x \in \Omega \mid x = (x', 0),\ x' \in I\!R^{N-k}\}\,.$$

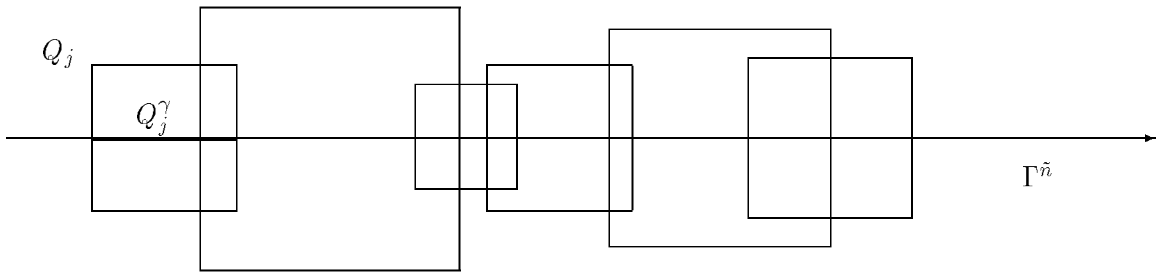
First we treat the case $\Omega = I\!R^N$, i. e. $\Gamma = I\!R^{N-k}$. Let $\Gamma_R^n = \{x \in \Gamma^n \mid |x| < R\}$, where $|x|$ denotes the maximum norm in $I\!R^{N-k}$. Since $\Gamma_R^{\tilde{n}}$ is bounded, there exists a locally finite covering with cubes

$$Q^\gamma(x_j, r(x_j)) = \{x \in I\!R^{N-k} \mid |x - x_j| < r(x_j)\} \qquad (j \in I\!N)\,,$$

such that

$$\sum_j \chi_{Q^\gamma(x_j, r(x_j))}(\eta) \le \kappa \qquad \text{for every} \quad \eta \in I\!R^{N-k}\,,$$

where χ_{Q^γ} is the characteristic function on Q^γ and the constant κ only depends on the dimension $N-k$. This result, known as Besicovitch covering lemma can be found in [5]. We denote by $Q(x_j, r(x_j))$ the corresponding cube in $I\!R^N$ with same center and edge length.



Since $y \in Q(x_j, r(x_j))$ implies that the projection $\pi(y)$ on $I\!R^{N-k}$ is in $Q^\gamma(x_j, r(x_j))$, we get

$$\sum_j \chi_{Q(x_j, r(x_j))}(y) \le \kappa \qquad \text{for every} \quad y \in I\!R^N$$

with the same κ as above, where χ_Q is the characteristic function on Q. We define

$$I = \{j \in I\!N \mid Q(x_j, r(x_j)) \cap D^{3\tilde{n}} \ne \emptyset\}.$$

From (7) it follows that $Q(x_j, r(x_j)) \subset D^{\tilde{n}}$ if $j \in I$ and we get

$$\Gamma_R^n \subset \bigcup_{j\in I} Q(x_j, r(x_j)) \subset D^{\tilde{n}} \qquad \text{for every} \quad n \geq 3\tilde{n}\,.$$

This shows that we can use the estimates (8)–(10) for $n \geq 3\tilde{n}$ and $j \in I$.

Now let K_1 denote the Sobolev constant for the trace operator in the unit cube $Q(0,1)$ centered at the origin, i. e.

$$\|u\|_{q,Q^\gamma(0,1)} \leq K_1 \|u\|_{1,p,Q(0,1)}\,.$$

Then by translation to x and dilation by r we obtain

$$\begin{aligned}
\left(\int\limits_{Q^\gamma(x,r)} |u(\eta)|^q \, d\eta\right)^{1/q} &\leq r^{\frac{N-k}{q}} K_1 \|u\|_{1,p,Q(0,1)} \\
&= K_1 r^{\frac{N-k}{q}} \left(r^{-N} \int\limits_{Q(x,r)} |u(y)|^p \, dy + r^{-N+p} \int\limits_{Q(x,r)} |\nabla u(y)|^p \, dy \right)^{1/p}
\end{aligned}$$

and from (8)–(10) we get for $j \in I$ (denoting $r_j = r(x_j)$, $Q_j = Q(x_j, r_j)$)

$$\begin{aligned}
\int_{Q_j^\gamma} |u(\eta)|^q w(\eta)\, d\eta &\leq \left(b_0^{1/q}(x_j) K_1 r_j^{\frac{N-k}{q} - \frac{N}{p} + 1}\right)^q \left(r_j^{-p} \int_{Q_j} |u(y)|^p \, dy + \int_{Q_j} |\nabla u(y)|^p \, dy \right)^{q/p} \\
&\leq K_1^q \left(\frac{b_0^{1/q}(x_j)}{b_1^{1/p}(x_j)} r_j^{\frac{N-k}{q} - \frac{N}{p} + 1} \right)^q \cdot \\
&\qquad \left(c_r^p \int_{Q_j} |u(y)|^p \frac{v_1(y)}{r^p(y)} \, dy + \int_{Q_j} |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p} \\
&\leq K_1^q \, \mathcal{B}_{n,k}^q \left(c_r^p K_0 \int_{Q_j} |u(y)|^p v_0(y)\, dy + \int_{Q_j} |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p} \\
&\leq K\, \mathcal{B}_{n,k}^q \left(\int_{Q_j} |u(y)|^p v_0(y) + |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p} ,
\end{aligned}$$

where $K = K_1^q \max\{c_r^q K_0^{q/p}; 1\}$. Taking the sum over $j \in I$ in the above estimate, we obtain for $n \geq 3\tilde{n}$ (since $q/p \geq 1$)

$$\|u\|^q_{q,\Gamma_R^n,w} \leq K \mathcal{B}_{n,k}^q \left(\sum_j \int_{Q_j} |u(y)|^p v_0(y) + |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p}$$

$$\leq \quad \kappa^{q/p} K \mathcal{B}_{n,k}^q \, \|u\|_{1,p,I\!R^N,v_0,v_1}^q \, .$$

Since the right hand side does not depend on R, we can take the limit $R \to \infty$ and get

$$\|u\|_{q,\Gamma^n,w}^q \leq C \, \mathcal{B}_{n,k}^q \, \|u\|_{1,p,I\!R^N,v_0,v_1}^q \, .$$

If $\mathcal{B}_{n,k} \to 0$ as $n \to \infty$, we see that condition (2) in Lemma 2 is fulfilled and the compactness of the trace operator follows. If $\lim_{n\to\infty} \mathcal{B}_{n,k} < \infty$, then (4) holds and we obtain continuity of the trace.
The general case $\Omega \subset I\!R^N$ now follows from Theorem 1, since

$$W^{1,p}(\Omega; v_0, v_1) \xrightarrow{\Lambda} W^{1,p}(I\!R^N; v_0, v_1) \overset{\gamma}{\hookrightarrow} L^q(\Gamma; w) \, .$$

is a composition of a continuous operator with γ compact, resp. continuous. □

If we take in Theorem 5 the intersection of Ω with $I\!R^N$, we obtain the following

COROLLARY 6. *Let* $1 \leq p \leq q < \infty$, $\dfrac{N}{q} - \dfrac{N}{p} + 1 \geq 0$ *and* r, v_0, v_1, w *satisfy (7)-(10).*
If $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w)$ *is compact for every* n *and* $\lim\limits_{n\to\infty} \mathcal{B}_{n,0} = 0$, *then* $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$ *is compact.*
If $W^{1,p}(\Omega_n; v_0, v_1) \hookrightarrow L^q(\Omega_n; w)$ *is continuous for every* n *and* $\lim\limits_{n\to\infty} \mathcal{B}_{n,0} < \infty$, *then* $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Omega; w)$ *is continuous.* □

Remark that we have obtained this result without further restrictions on the domain Ω. In particular for weight functions in A_p this corollary is a generalization of a result of Opic and Gurka [4].

4. PROOF OF THEOREMS 3 AND 4

Let $\Gamma = \partial\Omega$ be a C^1- smooth submanifold of dimension $(N-1)$ in $I\!R^N$. We assume that there exists a locally finite covering of Γ with open subsets $U_i \subset I\!R^N$ having the following properties:

U1 There is a global constant θ such that $\sum_i \chi_{U_i}(y) \leq \theta$ for every $y \in I\!R^N$.

U2 There exist C^1- diffeomorphisms $\varphi_i : U_i \to B_i \subset I\!R^N$ such that $0 \in B_i$ and $\varphi_i^{-1}(I\!R^{N-1} \times \{0\}) = U_i \cap \Gamma$.

U3 The partial derivatives of the coordinate functions φ_i and φ_i^{-1} are uniformly bounded by a constant K (not depending on i).

Remark. Conditions U1 and U3 are necessary to control the behaviour of Γ at infinity. It is easy to see that these conditions are satisfied for any compact C^1- submanifold of $I\!R^N$. To give an example of an unbounded domain, the boundary of which satisfies the above assumptions, consider an infinite cylinder $Q \times I\!R$, where $Q \subset I\!R^{N-1}$ is smooth and bounded. Then there exists a finite covering $\{U_i, \varphi_i\}_{i=1}^m$ of the compact manifold $\partial Q \times [0,1]$ with properties U1–U3. By translation of U_i and φ_i in the unbounded direction x_N, we obtain a covering of the whole boundary $\partial Q \times I\!R$, which obviously satisfies U1–U3.

Denote $B_i^\gamma = B_i \cap I\!R^{N-1} \times \{0\}$, and let $\rho = r \circ \varphi_i^{-1}$ be the weight on B_i induced by r. Then there exists a locally finite covering of B_i^γ with cubes in $I\!R^{N-1}$, $Q_j^\gamma = Q^\gamma(\xi_j, \rho(\xi_j))$, $\xi_j \in B_i^\gamma$. As in the last section we denote $Q_j = Q(\xi_j, \rho(\xi_j))$ the corresponding cube in $I\!R^N$ and we have

$$\sum_j \chi_{Q_j}(z) \le \kappa \qquad \text{for every} \quad z \in I\!R^N .$$

where κ only depends on N. Next we write $\tilde{v}_0$, $\tilde{v}_1$, $\tilde{w}$, $\tilde{b}_0$, $\tilde{b}_1$ for the transformed functions on B_i, i. e. $\tilde{v}_0 = v_0 \circ \varphi_i^{-1}$ and so on.
From assumption U3 we obtain the following mapping properties for φ_i.

<u>LEMMA 7.</u> *If U3 holds, then*
(i) The Jacobi determinants of φ_i *and* φ_i^{-1} *are uniformly bounded by* $K_2 = K^N N!$.
(ii) For every $x \in U_i \cap \Gamma^{\tilde{n}}$ *we have*

$$\varphi_i^{-1}[Q(\xi, \rho(\xi))] \subset Q(x, K_3 r(x)), \quad \textit{where} \quad \xi = \varphi_i(x), \quad K_3 = NK .$$

<u>Proof.</u> Since the first statement is obvious, we only prove (ii). Denote $f = \varphi_i^{-1}$ and let f_k, $k = 1, \ldots, N$ be the components of f. By translation we may assume that $\xi = 0$ and $x = f(0) = 0$. For $z \in Q(0, \rho(0))$ we obtain by the mean value theorem

$$f_k(z) = \frac{\partial f_k}{\partial \xi_1}(\zeta) z_1 + \ldots + \frac{\partial f_k}{\partial \xi_N}(\zeta) z_N ,$$

where ζ lies on the straight line between 0 and z. From U3 we get

$$|f_k(z)| \le K(|z_1| + \ldots + |z_N|) \le NK\,|z|\,,$$

where $|z|$ denotes the maximum norm of z. Consequently, $|f(z)| \le NK\,\rho(0) = NK\,r(0)$ for every $z \in Q(0,\rho(0))$. □

In the sequel we may assume without restriction that the constant K_3 in Lemma 7 is equal to 1. For if φ_i is a family of coordinate functions with properties U1–U3, then $\psi_i(x) = K_3\varphi_i(x)$ has the same properties (with different constants K and K_2) and $\psi_i^{-1}(z) = \varphi_i^{-1}(z/K_3)$ satisfies $|\psi_i^{-1}(z)| \le \rho(0) = r(0)$ for every $z \in Q(0,\rho(0))$.

Lemma 7 guarantees that the following estimates analogous to (8)–(10) are valid (with constants c_r and K_0 from (8) and (9)).

$$c_r^{-1} \le \rho(\xi)/\rho(z) \le c_r \quad \text{for every} \quad \xi = \varphi_i(x),\, x \in U_i \cap \Gamma^{\tilde n},\, z \in Q(\xi,\rho(\xi)) \tag{11}$$

$$\tilde v_1(z)\rho^{-p}(z) \le K_0\tilde v_0(z) \quad \text{for a. e.} \quad z = \varphi_i(x),\, x \in U_i \cap D^{\tilde n} \tag{12}$$

$$\tilde w(z) \le \tilde b_0(\xi)\,,\ \tilde b_1(\xi) \le \tilde v_1(z) \quad \text{for every} \quad \xi = \varphi_i(x),\, x \in U_i \cap \Gamma^{\tilde n},\, \text{a.e. } z \in Q(\xi,\rho(\xi)) \tag{13}$$

Remark that the cubes used in the proof below are always centered at points in $\varphi_i(U_i \cap \Gamma^{\tilde n})$, so that in our situation it is sufficient that (11) and (13) are satisfied for $x \in \Gamma^{\tilde n}$ (and not for all $x \in D^{\tilde n}$).

As first step we want prove the following estimate (for $n \ge 3\tilde n$):

$$\int_{U_i\cap\Gamma_R^n} |u(\eta)|^q w(\eta)\, d\eta \le C\,\mathcal{B}_{n,1}^q \left(\int_{U_i} |u(y)|^p v_0(y) + |\nabla u(y)|^p v_1(y)\, dy\right)^{q/p}. \tag{14}$$

Denoting $\rho_j = \rho(\xi_j)$, we obtain as in Section 3 for $n \ge 3\tilde n$

$$\begin{aligned}
\int\limits_{U_i\cap\Gamma_R^n} |u(\eta)|^q w(\eta)\, d\eta \;&\le\; K_2 \sum_j \int_{Q_j^\gamma} |u(\varphi_i^{-1}(\xi))|^q \tilde w(\xi))\, d\xi \\
&\le\; K_2 K_1^q \sum_j \left(\frac{\tilde b_0^{1/q}(\xi_j)}{\tilde b_1^{1/p}(\xi_j)} \rho_j^{\frac{N-1}{q}-\frac{N}{p}+1} \right)^q \cdot \\
&\qquad \left(c_r^p K_0 \int_{Q_j} |u(\varphi_i^{-1}(z))|^p \tilde v_0(z)\, dz + \int_{Q_j} |\nabla(u\circ\varphi_i^{-1})(z)|^p \tilde v_1(z)\, dz \right)^{q/p}
\end{aligned}$$

Since $\tilde b_0(z_j) = b_0(x_j)$, $\tilde b_1(z_j) = b_1(x_j)$, $\rho(z_j) = r(x_j)$ with $x_j \in U_i \cap \Gamma_R^n \subset D_R^n$, we finally obtain, setting $y = \varphi_i^{-1}(z)$,

$$\begin{aligned}
\int\limits_{U_i \cap \Gamma_R^n} |u(\eta)|^q w(\eta)\, d\eta &\le K_2 K_1^q \mathcal{B}_{n,1}^q \max\{K_0 c_r^p; 1\} \kappa^{q/p} \cdot \\
&\qquad \left(\int_{B_i} |u(\varphi_i^{-1}(z))|^p \tilde{v}_0(z) + |\nabla (u \circ \varphi_i^{-1})(z)|^p \tilde{v}_1(z)\, dz \right)^{q/p} \\
&\le C\, \mathcal{B}_{n,1}^q \left(\int_{U_i} |u(y)|^p v_0(y) + |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p},
\end{aligned}$$

where we have used U3 to estimate the derivatives of φ_i^{-1} and the constant C depends on K, K_0, K_1, K_2, c_r and κ. This proves (14).

Now let $I = \{i \in I\!N \mid U_i \cap \Gamma^{\tilde{n}} \neq \emptyset\}$; then from U1 and (14) we obtain

$$\begin{aligned}
\int_{\Gamma_R^n} |u(\eta)|^q w(\eta)\, d\eta &\le \sum_{i \in I} \int_{U_i \cap \Gamma_R^n} |u(\eta)|^q w(\eta)\, d\eta \\
&\le C\, \mathcal{B}_{n,1}^q \sum_{i \in I} \left(\int_{U_i} |u(y)|^p v_0(y) + |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p} \\
&\le C\, \mathcal{B}_{n,1}^q \left(\sum_{i \in I} \int_{U_i} |u(y)|^p v_0(y) + |\nabla u(y)|^p v_1(y)\, dy \right)^{q/p} \\
&\le \theta^{q/p} C\, \mathcal{B}_{n,1}^q \|u\|_{1,p,I\!R^N, v_0, v_1}^q .
\end{aligned}$$

In the limit $R \to \infty$ we get for $n \ge 3\tilde{n}$

$$\|u\|_{q,\Gamma^n,w}^q \le K\, \mathcal{B}_{n,1}^q \|u\|_{1,p,I\!R^N,v_0,v_1}^q$$

and we can again apply Lemma 2 to the trace operator

$$W^{1,p}(I\!R^N; v_0, v_1) \hookrightarrow L^q(\Gamma; w)\,.$$

The assertions of Theorems 3 and 4 for general domains $\Omega \subset I\!R^N$ then follow again from Theorem 1. $\square$

5. EXAMPLES

Let $B_1 = \{x \in I\!R^N \mid |x| < 1\}$ be the unit ball in $I\!R^N$. We define

$$v_0(x) = |x|^{\beta - p}, \quad v_1(x) = |x|^{\beta}, \quad w(x) = |x|^{\alpha} \qquad \text{for} \quad x \in I\!R^N \setminus B_1, \quad \alpha, \beta \in I\!R$$

and $v_0 = v_1 = w \equiv 1$ in B_1. Then v_0, v_1 and w belong to A_p $(1 < p < \infty)$ if α, β and $\beta - p$ belong to the interval $(-N, N(p-1))$; they are in A_1 if α, β and $\beta - p$ are in $(-N, 0]$ (cf. [10]).

We can now choose $r(x) = |x|/3$ and $b_0(x) = (4/3)^\alpha w(x)$, $b_1(x) = (2/3)^\beta v_1(x)$.
If

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N-1}{q} - \frac{N}{p} + 1 \le 0$$

then $\lim_{n\to\infty} \mathcal{B}_{n,1} < \infty$ and Theorem 3 shows that, if

$$\frac{N-1}{q} - \frac{N}{p} + 1 \ge 0\,,$$

then $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ is continuous.

If these inequalities are strict, i. e.

$$\frac{\alpha}{q} - \frac{\beta}{p} + \frac{N-1}{q} - \frac{N}{p} + 1 < 0 \qquad \text{and} \qquad \frac{N-1}{q} - \frac{N}{p} + 1 > 0\,,$$

then we conclude from Theorem 4 that $W^{1,p}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ is compact. Observe that the second inequality means that q is strictly smaller than the critical Sobolev exponent $p(N-1)/(N-p)$ for the usual trace operator in $W^{1,p}$.

6. AN APPLICATION TO NONLINEAR BOUNDARY VALUE PROBLEMS

We consider the following semilinear elliptic problem in an unbounded domain $\Omega \subset \mathbb{R}^N$ $(N \ge 3)$ with smooth boundary Γ

$$\begin{aligned} Au \equiv -\Delta u + b(x)u &= f(x,u) \qquad \text{in} \quad \Omega\,, & (15)\\ \partial_{\mathbf{n}} u &= \varphi(\xi, u) \qquad \text{on} \quad \Gamma\,, & (16) \end{aligned}$$

where b is a positive measurable function and $\partial_{\mathbf{n}}$ denotes the outer normal derivative on Γ. We define a bilinear form associated with A by

$$\langle u, v\rangle_A = \int_\Omega \nabla u \nabla v + b(x) uv\, dx\,.$$

A weak solution of (15)–(16) is then a critical point of the functional

$$J(u) = \frac{1}{2}\langle u, u\rangle_A - \int_\Omega F(x,u)\, dx - \int_\Gamma \Phi(\xi, u)\, d\xi\,,$$

where $F(x,u)=\int_0^u f(x,t)\,dt$ and $\Phi(\xi,u)=\int_0^u \varphi(\xi,t)\,dt$.

In this section we always assume that the weight functions v_0, v_1, w are choosen such that we can apply Theorem 4 and Corollary 6 to ensure that the embedding $W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w)$ and the trace $W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^q(\Gamma; w)$ are compact for $2 < p < 2N/(N-2)$, $2 < q < 2(N-1)/(N-2)$ and continuous for $2 \le p \le 2N/(N-2)$, $2 \le q \le 2(N-1)/(N-2)$, respectively. Then we have the estimates $\|u\|_{p,\Omega,w} \le C\|u\|_{v_0,v_1}$ and $\|u\|_{q,\Gamma,w} \le C\|u\|_{v_0,v_1}$, where we used the abbreviation $\|u\|_{v_0,v_1} = \|u\|_{1,2,\Omega,v_0,v_1}$. Furthermore, we make the following assumptions:

1^o We assume that A defines a continuous bilinear form $\langle\cdot,\cdot\rangle_A$ on $W^{1,2}(\Omega; v_0, v_1)$ and satisfies the ellipticity condition (with some constant $K > 0$)

$$\langle u,u\rangle_A \ge 2K\|u\|^2_{v_0,v_1} \qquad \text{for every} \quad u \in W^{1,2}(\Omega; v_0, v_1)\,.$$

2^o f and φ are Caratheodory functions on $\Omega \times I\!R$ and $\Gamma \times I\!R$, respectively, $f(x,0) = \varphi(\xi,0) = 0$, $\lim_{t\to 0} f(x,t)/t = \lim_{t\to 0}\varphi(\xi,t)/t = 0$ uniformly in x and ξ and there exist an open, nonempty subset $O \subset \Omega$ and a number $R > 0$ such that $F(x,u) > 0$ for every $u \ge R$, $x \in O$.

3^o $|f(x,t)| \le f_0(x) + f_1(x)|t|^{p-1}$ for $x \in \Omega$, $t \in I\!R$, where $2 < p < 2N/(N-2)$ and f_0, f_1 are positive functions satisfying

$$f_0 \in L^{p/(p-1)}(\Omega; w^{1/(1-p)})\,, \quad f_0 \le Cw\,, \quad f_1 \le Cw$$

and analogously $|\varphi(\xi,t)| \le \varphi_0(\xi) + \varphi_1(\xi)|t|^{q-1}$ for $\xi \in \Gamma$, $t \in I\!R$, $2 < q < 2(N-1)/(N-2)$, φ_0, φ_1 are positive functions satisfying

$$\varphi_0 \in L^{q/(q-1)}(\Gamma; w^{1/(1-q)})\,, \quad \varphi_0 \le Cw\,, \quad \varphi_1 \le Cw$$

with appropriate constant C.

4^o There exists a $\theta \in I\!R$, $0 \le \theta < 1/2$ such that $F(x,t) \le \theta f(x,t)t$, $\Phi(\xi,t) \le \theta\varphi(\xi,t)t$ for a. e. $x \in \Omega$, $\xi \in \Gamma$ and every $t \in I\!R$.

We denote by N_f, N_F, N_φ, N_Φ the corresponding Nemytskii operators. Under the assumptions above we have the following results.

LEMMA 8. *The operators* $N_f : L^p(\Omega; w) \to L^{p/(p-1)}(\Omega; w^{1/(1-p)})$, $N_F : L^p(\Omega; w) \to L^1(\Omega)$, $N_\varphi : L^q(\Gamma; w) \to L^{q/(q-1)}(\Gamma; w^{1/(1-q)})$ *and* $N_\Phi : L^q(\Gamma; w) \to L^1(\Gamma)$ *are bounded and continuous.*

Proof. We only prove the assertions for N_f and N_F, since the arguments for N_φ and N_Φ are identical. For $u \in L^p(\Omega; w)$ we get (setting $p' = p/(p-1)$)

$$\begin{aligned}
\int_\Omega |f(x,u)|^{p'} w^{1/(1-p)} dx &\le C \left(\int_\Omega f_0(x)^{p'} w^{1/(1-p)} dx + \int_\Omega f_1(x)^{p'} |u|^{p'(p-1)} w^{1/(1-p)} dx \right) \\
&\le C + C \int_\Omega |u|^p w^{p/(p-1)} w^{1/(1-p)} dx \\
&= C + C \|u\|^p_{p,\Omega,w} \,.
\end{aligned}$$

Therefore, N_f is bounded. The continuity now follows from standard properties of Nemytski operators (cf. [11]).

In the same way we obtain (using Hölder's inequality)

$$\begin{aligned}
\int_\Omega |F(x,u)| dx &\le \int_\Omega f_0(x)|u| dx + \int_\Omega f_1(x)|u|^p dx \\
&\le \left(\int_\Omega f_0(x)^{p'} w^{1/(1-p)} dx \right)^{1/p'} \left(\int_\Omega |u|^p w \, dx \right)^{1/p} + C \int_\Omega |u|^p w \, dx \\
&\le C\|u\|_{p,\Omega,w} + C\|u\|^p_{p,\Omega,w} \,.
\end{aligned}$$

This completes the proof. □

LEMMA 9. *The functional* J *is Fréchet-differentiable in* $W^{1,2}(\Omega; v_0, v_1)$, *and the derivative is given by*

$$\langle J'(u), v \rangle = \langle u, v \rangle_A - \int_\Omega f(x,u) v \, dx - \int_\Gamma \varphi(\xi, u) v \, d\xi \,. \tag{17}$$

Proof. Since (17) is the Gateaux derivative of J in u in direction v, we only have to show the continuity of J', i. e. the continuity of the mapings $u \mapsto \langle u, \cdot \rangle_A$, $u \mapsto \int_\Omega F(x,u) \cdot dx$ and $u \mapsto \int_\Gamma \varphi(\xi, u) \cdot dx$ in the dual norm. The first mapping is clearly continuous since we assumed that $\langle \cdot, \cdot \rangle_A$ is a continuous bilinear form. The second mapping is regarded as composition of the following mappings:

$$W^{1,2}(\Omega; v_0, v_1) \overset{i}{\hookrightarrow} L^p(\Omega; w) \xrightarrow{N_f} L^{p/(p-1)}(\Omega; w^{1/(1-p)}) \xrightarrow{j} \left(W^{1,2}(\Omega; v_0, v_1) \right)' ,$$

where $\langle j(g), v\rangle = \int_\Omega gv\,dx$. Since we know that i and N_f are continuous, it suffices to show that j is a bounded operator. By Hölder's inequality we get

$$\begin{aligned}\int_\Omega |gv|dx &\leq \left(\int_\Omega |g|^{p'} w^{1/(1-p)} dx\right)^{1/p'} \left(\int_\Omega |v|^p w\,dx\right)^{1/p} \\ &\leq C\|g\|_{p/(p-1),\Omega,w^{1/(1-p)}}\|v\|_{v_0,v_1}.\end{aligned}$$

Analogously we write the third mapping as

$$W^{1,2}(\Omega; v_0, v_1) \xhookrightarrow{\gamma} L^q(\Gamma; w) \xrightarrow{N_\varphi} L^{q/(q-1)}(\Gamma; w^{1/(1-q)}) \xrightarrow{\ell} \left(W^{1,2}(\Omega; v_0, v_1)\right)'.$$

Again γ and N_φ are continuous and $\langle \ell(h), v\rangle = \int_\Gamma h\,\gamma(v)\,d\xi$ satisfies

$$\begin{aligned}\int_\Gamma |h\,\gamma(v)|dx &\leq \left(\int_\Gamma |h|^{q'} w^{1/(1-q)} dx\right)^{1/q'} \left(\int_\Gamma |\gamma(v)|^q w\,dx\right)^{1/q} \\ &\leq C\|h\|_{q/(q-1),\Gamma,w^{1/(1-q)}}\|v\|_{v_0,v_1},\end{aligned}$$

which proves the lemma. □

LEMMA 10. *J satisfies the Palais-Smale condition on $W^{1,2}(\Omega; v_0, v_1)$.*

Proof. First we show that every Palais-Smale sequence is bounded. Let $u_k \in W^{1,2}(\Omega; v_0, v_1)$ be such a sequence, i. e. $J'(u_k) \to 0$, $|J(u_k)| \leq M$. From $|\langle J'(u_k), u_k/\|u_k\|_{v_0,v_1}\rangle| \leq \varepsilon$ for k large, we obtain

$$-\langle u_k, u_k\rangle_A + \int_\Omega f(x, u_k)u_k\,dx + \int_\Gamma \varphi(\xi, u_k)u_k\,d\xi \leq \varepsilon\|u_k\|_{v_0,v_1}.$$

Multiplying this equation with θ from 4^o and adding this to the inequality $\frac{1}{2}\langle u_k, u_k\rangle_A - \int_\Omega F(x, u_k)\,dx - \int_\Gamma \Phi(\xi, u_k)\,d\xi \leq M$, we obtain

$$\begin{aligned}(\frac{1}{2} - \theta)\langle u_k, u_k\rangle_A &+ \int_\Omega \theta f(x, u_k)u_k - F(x, u_k)\,dx \\ &+ \int_\Gamma \theta\varphi(\xi, u_k)u_k - \Phi(\xi, u_k)\,d\xi - \theta\varepsilon\|u_k\|_{v_0,v_1} \quad\leq\quad M. \end{aligned} \tag{18}$$

From 4^o we get

$$(\frac{1}{2} - \theta)\langle u_k, u_k\rangle_A - \theta\varepsilon\|u_k\|_{v_0,v_1} \leq M.$$

Now 1^o implies that $\langle u, u\rangle_A/\|u\|_{v_0,v_1} \to \infty$ as $\|u\|_{v_0,v_1} \to \infty$, which shows that $\|u_k\|_{v_0,v_1}$ must be bounded.

From Theorem 4 and Corollary 6 we get a subsequence (again denoted by u_k) which is convergent in $L^p(\Omega; w)$ with traces convergent in $L^q(\Gamma; w)$. For this subsequence we get

$$\begin{aligned}
\langle u_n - u_k, u_n - u_k\rangle_A &\leq |\langle u_n, u_n - u_k\rangle_A|\, |\langle u_k, u_n - u_k\rangle_A| \\
&\leq 2\varepsilon \|u_n - u_k\|_{v_0,v_1} + \\
&\quad \left|\int_\Omega f(x, u_n)(u_n - u_k)dx\right| + \left|\int_\Omega f(x, u_k)(u_n - u_k)dx\right| + \\
&\quad \left|\int_\Gamma \varphi(\xi, u_n)(u_n - u_k)d\xi\right| + \left|\int_\Gamma \varphi(\xi, u_k)(u_n - u_k)d\xi\right| .
\end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned}
\left|\int_\Omega f(x, u_n)(u_n - u_k)dx\right| &\leq \int_\Omega |f(x, u_n)w^{-1/p}|\, |(u_n - u_k)w^{1/p}|\, dx \\
&\leq \left(\int_\Omega |f(x, u_n)w^{-1/p}|^{p'} dx\right)^{1/p'} \left(\int_\Omega |u_n - u_k|^p w\, dx\right)^{1/p} .
\end{aligned}$$

and

$$\left|\int_\Gamma \varphi(\xi, u_n)(u_n - u_k)d\xi\right| \leq \left(\int_\Gamma |\varphi(\xi, u_n)w^{-1/q}|^{q'} d\xi\right)^{1/q'} \left(\int_\Gamma |u_n - u_k|^q w\, d\xi\right)^{1/q} .$$

Since $\|u_n - u_k\|_{p,\Omega,w}$ and $\|u_n - u_k\|_{q,\Gamma,w}$ tend to zero and $\int_\Omega |f(x, u_n)|^{p'} w^{1/(1-p)}dx$ and $\int_\Gamma |\varphi(\xi, u_n)|^{q'} w^{1/(1-q)}d\xi$ are bounded (by Lemma 8), it follows that $\|u_n - u_k\|_{v_0,v_1}$ tends to zero. □

By means of the Mountain Pass Lemma (cf. [9]) we can now prove

<u>THEOREM 11.</u> *If f and φ satisfy conditions 2^o–4^o, and A defines a continuous bilinear form on $W^{1,2}(\Omega; v_0, v_1)$, which satisfies the ellipticity condition 1^o, then there exists a nontrivial solution of (15), (16).*

<u>Proof.</u> From 2^o–3^o we obtain to every $\varepsilon > 0$ a constant $C_\varepsilon \geq 0$ such that $F(x,t) \leq \varepsilon f_0(x)t^2 + C_\varepsilon f_1(x)|t|^p$, $2 < p < 2N/(N-2)$ and $\Phi(\xi, t) \leq \varepsilon\varphi_0(\xi)t^2 + C_\varepsilon\varphi_1(\xi)|t|^q$, $2 < q < 2(N-1)/(N-2)$. Consequently

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\langle u, u\rangle_A - \int_\Omega \left(\varepsilon f_0(x)u^2 + C_\varepsilon f_1(x)|u|^p\right) dx - \int_\Gamma \left(\varepsilon\varphi_0(\xi)u^2 + C_\varepsilon\varphi_1(\xi)|u|^q\right) d\xi \\
&\geq K\|u\|^2_{v_0,v_1} - \varepsilon C\|u\|^2_{p,\Omega,w} - C_\varepsilon C\|u\|^p_{p,\Omega,w} - \varepsilon C\|u\|^2_{q,\Gamma,w} - C_\varepsilon C\|u\|^q_{q,\Gamma,w} \\
&\geq (K - \varepsilon\tilde{C})\, \|u\|^2_{v_0,v_1} - \tilde{C}_\varepsilon\|u\|^p_{v_0,v_1} - \tilde{C}_\varepsilon\|u\|^q_{v_0,v_1} .
\end{aligned}$$

For ε small we can now choose a $\rho > 0$ such that $J(u) \geq \delta > 0$ if $\|u\|_{v_0,v_1} = \rho$.

On the other hand, by 2^o and 4^o, we can choose a positive $g \in W^{1,2}(\Omega; v_0, v_1)$ with compact support satisfying $g \geq R > 0$ in $O \subset \Omega$, such that $|F(x, \lambda g)| \geq h(x)|\lambda g|^p$, $p > 2$ for $\lambda \geq 1$ with $h > 0$ in O and get

$$J(\lambda g) \leq \frac{1}{2}\lambda^2 \langle g, g\rangle_A - |\lambda|^p \int_O h(x)|g|^p \, dx + c\,.$$

For $\lambda \to \infty$ the right hand side of this inequality tends to $-\infty$.

Therefore, we have shown, that the functional J satisfies the Palais-Smale condition, $J(0) = 0$, $J(u) \geq \delta > 0$ if $\|u\|_{v_0,v_1} = \rho$ (small enough) and $J(g) \leq 0$ for some g, $\|g\|_{v_0,v_1} > \rho$. The Mountain Pass Lemma now implies the existence of a critical point of J. □

As an example we consider the weight functions from Section 5 with $\beta = 0$ in $W^{1,2}$:

$$v_0(x) = |x|^{-2}\,, \quad v_1(x) = 1\,, \quad w(x) = |x|^\alpha \qquad \text{for} \quad x \in I\!R^N \setminus B_1\,, \quad \alpha \in I\!R$$

and $v_0 = v_1 = w \equiv 1$ in B_1. They are in A_2 for $-N < \alpha < N$.

Then A defines a continuous bilinear form and satisfies 1^o if there are constants $c \leq C$ such that

$$c\, v_0(x) \leq b(x) \leq C\, v_0(x) \qquad \text{for a. e.} \quad x \in \Omega\,.$$

We assume that f and φ satisfy assumptions 2^o–4^o. To get compactness of the embedding $W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^p(\Omega; w)$ and of the trace operator $W^{1,2}(\Omega; v_0, 1) \hookrightarrow L^p(\Gamma; w)$, we have to choose α such that

$$\frac{\alpha}{p} + \frac{N}{p} - \frac{N}{2} + 1 < 0 \qquad \text{and} \qquad \frac{\alpha}{q} + \frac{N-1}{q} - \frac{N}{2} + 1 < 0$$

where $p \in (2, 2N/(N-2))$ and $q \in (2, 2(N-1)/(N-2))$ are defined by 3^0. This inequality is for example satisfied for $\alpha = -2$. Thus Theorem 11 shows that there exists a nontrivial solution of (15)–(16) in $W^{1,2}(\Omega; v_0, 1)$.

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