SERIE A — MATHEMATIK

Neighborhood conditions for balanced independent sets in bipartite graphs

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Abstract

Let G be a balanced bipartite graph of order $2n$ and minimum degree $\delta(G) \geq 3$. If for every balanced independent set S of four vertices in G the neighborhood union $|N(S)| > n$ then G is traceable, the circumference is at least $2n-2$ and G contains a 2-factor (with one exceptional graph for the latter statement). If $|N(S)| > n+2$ then G is hamiltonian.

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The investigation of neighborhood union conditions for certain extremal problems was initiated by Faudree, Gould, Jacobson and Schelp [3].

Theorem 1 (Faudree et. al. [3]) *Let G be a 2-connected graph of order $n \geq 3$. If for every pair of non-adjacent vertices u and v the neighborhood union*

(a) $|N(u) \cup N(v)| \geq (2n-1)/3$ *then G is hamiltonian.*

(b) $|N(u) \cup N(v)| \geq (n-1)/2$ *then G is traceable.*

(c) $|N(u) \cup N(v)| \geq (2n+1)/3$ *then G is hamilton-connected.*

Let $\mathcal{G}$ be the class of 2-connected graphs, which contain three vertices which do not lie on a common cycle (in particular, these graphs cannot be hamiltonian). This class was structurally characterized by Watkins and Mesner [5]. Answering a conjecture of B. Jackson [4] affirmatively, Broersma, van den Heuvel and Veldman strengthened Theorem 1(a).

Theorem 2 (Broersma, van den Heuvel and Veldman [2])
Let G be a 2-connected graph of order $n \geq 3$. If for every pair of non-adjacent vertices u and v

$$|N(u) \cup N(v)| \geq n/2,$$

then G is either hamiltonian, or the Petersen graph, or $G \in \mathcal{G}$.

The purpose of this paper is to investigate a certain neighborhood union condition for bipartite graphs. Let G be a balanced bipartite graphs, i.e. a graph with a bipartition into two independent vertex sets of the same cardinality (if the graph is connected, which is implied by most of our conditions, then the bipartition is unique). We investigate conditions on the neighborhood union $N(S)$ of any balanced independent set S of four vertices, i.e. an independent set containing two vertices from each side of the bipartition.

Theorem 3 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| > n + 2$ then G is hamiltonian.*

We conjecture, that the neighborhood union condition can be slightly relaxed. Let H_{14} be the graph depicted in Figure 1.

Conjecture 1 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| > n$ then G is hamiltonian or $G = H_{14}$.*

The following results for the same neighborhood union condition give some support to the conjecture, by showing that graphs satisfying the hypothesis of the conjecture are at least very close to being hamiltonian.

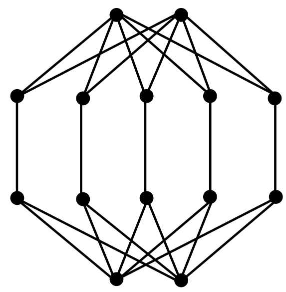

Figure 1: The graph H_{14}.

Theorem 4 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| > n$ then G is hamiltonian or G contains a cycle C of length $2n-2$ with an edge in $G - C$.*

As an immediate consequence we get:

Corollary 1 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| > n$ then G is traceable.*

Theorem 5 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 3$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| > n$ then G has a 2-factor, unless $G = H_{14}$.*

We denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a (simple, finite, undirected) graph G. Let H be a fixed subgraph of G. Then $G - H$ denotes the subgraph induced by $V(G) \setminus V(H)$. For a cycle C of G we always assume an orientation, which is arbitrary but fixed unless the orientation is explicitly specified. The successor and predecessor of a vertex v of C according to the orientation of C is v^+ and v^-, respectively. The cyclic distance of two vertices of C is their distance in C (and not in G). For undefined graph theoretical concepts the reader is referred to introductory literature, e.g. [1].

In the proofs we need that the graphs are connected and contain a perfect matching.

Lemma 1 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 2$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| > n$ then G is connected.*

Proof. Take a balanced bipartition $X \cup Y$ of $V(G)$ and suppose that there are distinct components G_1 and G_2 of G. As $\delta(G) \geq 2$ each component has at least two vertices in X and in Y. We may assume that $|V(G_1) \cap X| \geq |V(G_1) \cap Y|$. Then

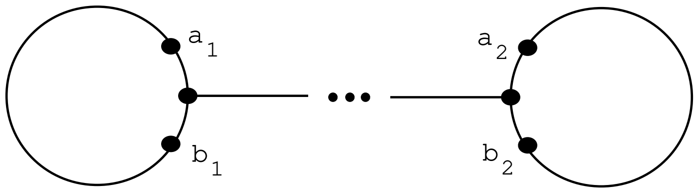

Figure 2: The dumb-bell.

every set S containing two vertices of $V(G_1) \cap X$ and two vertices of $V(G_2) \cap Y$ is a balanced independent set with $|N(S)| \leq n$, contradicting the hypothesis. $\square$

With a slightly more careful reasoning it can be shown that the graphs are actually 2-connected.

Lemma 2 *Let G be a balanced bipartite graph of order $2n$ with $\delta(G) \geq 1$. If for every balanced independent set S with $|S| = 4$ we have $|N(S)| \geq n$ then G has a perfect matching.*

Proof. Let $X \cup Y$ be a balanced bipartition of $V(G)$. By the König-Hall Theorem (see e.g. [1, p. 128]) it suffices to show, that for every subset $A \subseteq X$ we have $|N(A)| \geq |A|$. Indeed, if $|N(A)| < |A|$ then $2 \leq |A| \leq n-1$ as $\delta(G) \geq 1$. So every set S containing two vertices of A and two vertices of $Y - N(A)$ is a balanced independent set of four vertices with $|N(S)| < n$, a contradiction. $\square$

In the proofs the following subgraphs, called **dumb-bell**, consisting of two disjoint cycles Γ_1 and Γ_2 which are joined by a path P (cf. Figure 2) will play a significant role. The neighbors in Γ_i of the vertex shared by Γ_i and P are denoted by a_i and b_i, respectively. Moreover, we call a spanning dumb-bell of a graph **maximal**, if the path P is as long as possible among all spanning dumb-bells.

Lemma 3 *Let G be a balanced bipartite graph of order $2n$ and let D be a maximal spanning dumb-bell of G with $S = \{a_1, b_1, a_2, b_2\}$. If $|N(S)| > n$ and G is not hamiltonian, then G contains a cycle C of length $2n-2$ and an edge in $G - C$. If $|N(S)| > n+2$ then G is hamiltonian.*

Proof. Fix an orientation of Γ_i such that its intersection vertex with P is $a_i^- = b_i^+$. Fix a perfect matching M of D which contains the edges $a_i a_i^+$. If for any of the edges of M different from $b_i^- b_i^{--}$ both ends are adjacent to S then a hamiltonian cycle results. This can be easily seen for the edges in $P \cap M$ and for the edges containing a_i and b_i. For the remaining edges of $\Gamma_i \cap M$ this is due to the fact that, by the maximality, the vertices a_i and b_i have exactly two neighbors in Γ_i. $\square$

We will start with the proof of Theorem 5.

Proof of Theorem 5.

As G contains a perfect matching by Lemma 2, it certainly has a spanning subgraph consisting of vertex disjoint even length cycles and odd length paths. Let H be such a spanning subgraph, which has

(i) the largest number of vertices contained in cycle components,

(ii) subject to (i) the least number of components, and

(iii) subject to (i) and (ii) a path component of largest order.

Let M be a perfect matching of H and let H' be the subgraph of H which is spanned by the cycles.

It remains to show that $H' = H$. Assume, to the contrary, that $H' \neq H$. Let Q be a path component of largest order of $H - H'$, and let v_1 and v_2 be the endvertices of Q. By (i) and (iii) both v_1 and v_2 have exactly one neighbor in $G - H'$, so each of them has at least two neighbors in H' as $\delta(G) \geq 3$. If v_1 and v_2 have neighbors in different cycles of H' then we obtain a dumb-bell containing Q and two cycles C and C' of H'. Let D be a maximal dumb-bell with vertex set $V(Q) \cup V(C) \cup V(C')$ and $S = \{a_1, b_1, a_2, b_2\}$ of D. Observe that:

$$\text{one end of every edge } e \in M \cap E(G - D) \text{ has no neighbor in } S. \qquad (*)$$

Otherwise, if $e \in E(H - H')$ then the subgraph spanned by D and e is hamiltonian, and if $e \in E(H')$ then the subgraph spanned by D and the component of H' containing e is hamiltonian. In any case (i) is violated. Hence we conclude $|N(S) \cap V(D)| > |D|/2$, and by Lemma 3 applied to the subgraph induced by D this implies, that (ii) is violated.

So we may assume that v_1 and v_2 have all neighbors in one cycle, say C. Let w_1 and w_2 be neighbors of v_1 and v_2, resp., which have largest cyclic distance on C. By (i) the cyclic distance must be at least $|Q| + 1 \geq 3$. If the cyclic distance exceeds 3 then $S = \{w_1^+, w_1^-, w_2^+, w_2^-\}$ is a balanced independent set, as any edge between them causes one or two cycles spanning $V(Q) \cup V(C)$, which contradicts (i). Now fix a perfect matching M' of G which contains the edges v_1w_1 and v_2w_2 and takes the remaining edges from $E(H)$. It is easy to see that such a matching exists. On the other hand, if for any edge of M' both ends have a neighbor in S then we obtain a subgraph of independent cycles covering H' and Q, which again contradicts (i).

So, finally, assume that w_1 and w_2 have cyclic distance 3. It is easy to observe that in this case $|Q| = 2$, $|C| = 12$, $C = x_1, x_2, \ldots, x_{12}, x_1$, such that v_1 is adjacent to x_1 and x_7 and v_2 is adjacent to x_4 and x_{10}. Consider the edges $E' = \{v_1v_2, x_2x_3, x_5x_6, x_8x_9, x_{11}x_{12}\}$. Note that for each edge $e' \in E'$ there is a 12-cycle in $V(Q) \cup V(C)$ avoiding e'. Furthermore, any edge joining two edges of E' would cause a 2-factor of $V(Q) \cup V(C)$, contradicting (i). So, in particular, we may assume that $S = \{x_2, x_5, x_8, x_{11}\}$ is a balanced independent set and $|N(S) \cap (V(Q) \cup V(C))| \leq 8$.

If $n \geq 8$ then there must be an edge f joining S to another component of H. If f joins S to a path Q' in $H - H'$ then the graph induced by $V(Q) \cup V(Q') \cup V(C)$ has a spanning subgraph consisting of a 12-cycle and a length 3 path since $|Q'| \leq |Q| = 2$. This contradicts (ii). If f joins S to another cycle C' then we obtain a spanning dumb-bell of $V(Q) \cup V(C) \cup V(C')$. As above, for a maximal dumb-bell (*) holds, and we again conclude with Lemma 3 that (ii) is violated. So we may assume that $n = 7$, and since $\delta(G) \geq 3$, every vertex of $\{x_1, x_4, x_7, x_{10}\}$ has a neighbor in each edge of E. So the resulting graph is indeed H_{14}. $\square$

We prove the remaining statements simultaneously.

Proof of Theorems 3 and 4.
It remains to show, that if G is non-hamiltonian with $\delta(G) \geq 3$ and $|N(S)| > n$ for every balanced independent set S of four vertices, then G contains a spanning dumb-bell, as the results can be immediately derived from Lemma 3.

Indeed, if $G = H_{14}$ then G contains a spanning dumb-bell. Otherwise, by Theorem 5, G contains a spanning subgraph consisting of vertex disjoint cycles. Since G is non-hamiltonian and connected, it contains a spanning subgraph, consisting of a dumb-bell and vertex disjoint cycles. Let H be such a spanning subgraph, where the order of the dumb-bell D is as large as possible, and subject to this requirement, the path P of D is as long as possible. Set $S = \{a_1, b_1, a_2, b_2\}$. If $H - D$ contains a cycle then the vertices of S have all their neighbors in D, since the order of D is maximal. By Lemma 3 this implies that the subgraph induced by D is hamiltonian, which again contradicts the maximality of the order of D. So D must be a spanning dumb-bell, as required. $\square$

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