UNIFORM RECONSTRUCTION OF GAUSSIAN PROCESSES

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ABSTRACT. We consider a Gaussian process X with smoothness comparable to the Brownian motion. We analyze reconstructions of X which are based on observations at finitely many points. For each realization of X the error is defined in a weighted supremum norm; the overall error of a reconstruction is defined as the pth moment of this norm. We determine the rate of the minimal errors and provide different reconstruction methods which perform asymptotically optimal. In particular, we show that linear interpolation at the quantiles of a certain density is asymptotically optimal.

KEYWORDS: Brownian motion, Sacks-Ylvisaker conditions, asymptotically optimal designs, reproducing kernel Hilbert space, regular sequence, uniform norm.

1. INTRODUCTION

We study the problem of reconstructing the path of a stochastic process on the basis of observations at finitely many sampling points. This problem is of practical interest, for instance, in the fields of computer experiments, see Sacks, Welch, Mitchell and Wynn (1989), and earth sciences, see Christakos (1992).

In this paper we consider a process

$$X(t) = m(t) + Z(t), \qquad 0 \le t \le 1,$$

where m denotes the continuous mean response function of X and Z is a centered Gaussian process with continuous paths. A reconstruction of X which is based on observations at

$$T = \{t_1, \dots, t_n\} \subset [0,1]$$

is of the form

$$\widehat{X}_T = \phi(X(t_1), \dots, X(t_n))$$

where ϕ is a (measurable) mapping $\mathbb{R}^n \to C([0,1])$. The set T is called a sampling design.

It is natural to compare a realization of the process X and a corresponding reconstruction $\widehat{X}_T$ in some L_q-norm. In this paper we use the uniform norm and measure the quality of $\widehat{X}_T$ by

$$e_p(\widehat{X}_T) = \left(E \Big(\sup_{0 \le t \le 1} w(t) \, |X(t) - \widehat{X}_T(t)| \Big)^p \right)^{1/p}$$

where w is a positive and continuous weight function and $1 \le p < \infty$.

For a fixed design T the optimal reconstruction $\widehat{X}_T$ is given pointwise as the optimal estimator of $X(t)$ from observations $X(t_i)$. Basically it remains to study the design problem, i.e., the optimal choice of the sampling points t_i.

The Gaussian process X is completely specified by its mean m and its covariance kernel R, and the smoothness of m and R determine how well X may be reconstructed from n observations. Here we assume that R satisfies Sacks-Ylvisaker regularity conditions of order zero. The conditions imply

$$c_1 \delta \le \sup_{|t-s|<\delta} E(Z(s) - Z(t))^2 \le c_2 \delta$$

with positive constants c_i for the quadratic mean smoothness of the process Z.

For sequences of positive real numbers a_n and b_n we write

$$a_n \approx b_n \qquad \text{iff} \qquad \lim_{n \to \infty} a_n / b_n = 1.$$

We show that

$$\inf_{|T|=n} \inf_{\widehat{X}_T} e_p(\widehat{X}_T) \approx \left(\frac{1}{2} \int_0^1 w^2(t) \, dt \right)^{1/2} \left(\frac{\ln n}{n} \right)^{1/2}.$$

Here minimization is with respect to all reconstructions $\widehat{X}_T$ which are based on any n-point design T. Furthermore, we show that quantiles of the squared weight w^2 define an asymptotically optimal sequence of designs.

Clearly the optimal reconstruction $\widehat{X}_T$ depends on the particular mean m and covariance kernel R. We show, however, that linear interpolation based on the quantiles of w^2 is asymptotically optimal if m is Hölder continuous with exponent $1/2$. Under stronger assumptions on m optimal reconstructions for the centered case $m = 0$ and best linear unbiased predictors perform asymptotically optimal as well.

Similar results have been obtained for the zero mean Brownian motion, see Ritter (1990), and, more generally, for linear and product type kernels, see Müller-Gronbach (1994). Our method of proof reduces the general case to the case of a weighted Brownian motion by applying a comparison theorem for Gaussian processes and results from Müller-Gronbach (1995b) concerning the covariance structure of the error process.

Su and Cambanis (1993) and Müller-Gronbach (1995a) study reconstruction with respect to the L_2-norm for essentially the same class of processes as considered here. In this case, too, the above reconstruction methods perform asymptotically optimal, however sampling points have to be quantiles of the weight w. The order of the minimal errors is $1/\sqrt{n}$. See also Lee (1986) and Abt (1992) for the particular case of a zero mean Brownian motion or Brownian bridge and $w = 1$.

For reconstruction with respect to the maximum mean squared error similar results have been established by Müller-Gronbach (1995b). While the minimal errors are of order $1/\sqrt{n}$ as for L_2-reconstruction, the sampling points have to be chosen as for L_∞-reconstruction.

L_q-reconstruction with $1 \le q < \infty$ for classes of processes with higher regularity is studied in Speckman (1979). In this case regularity of order r yields minimal errors of order $1/n^{r+1/2}$. Furthermore, Wasilkowski (1992) has shown that $\sqrt{\ln n}/n^{r+1/2}$ is the order of the minimal errors for L_∞-reconstruction of the r-fold integrated Brownian motion.

Much less is known in the multivariate case, i.e., for random fields $X(t)$ where $t \in [0,1]^k$ and $k > 1$. Results on L_2-reconstruction are obtained in Micchelli and Wahba (1981), Woźniakowski (1992), Wasilkowski (1993), Ritter and Wasilkowski (1995), Ritter, Wasilkowski and Woźniakowski (1993, 1995), and Müller-Gronbach and Schwabe (1995).

In the following section we specify the assumptions on the covariance kernel R, introduce the above reconstruction methods and state the corresponding results on asymptotic optimality. The proofs are given in section 3.

Throughout the paper we will use certain standard facts about the reproducing kernel Hilbert space $(H(R), (\cdot,\cdot)_R)$ associated with the covariance kernel R. Recall that $H(R)$ is the unique Hilbert space of real-valued functions on $[0,1]$ such that

(i) $R(t,\cdot) \in H(R)$ for each $t \in [0,1]$,
(ii) $(f, R(t,\cdot))_R = f(t)$ for each $t \in [0,1]$ and $f \in H(R)$.

The mapping $R(\cdot,t) \mapsto Z(t)$ extends to an isomorphism between $H(R)$ and the Hilbert space $L_2(Z)$ which is spanned by the process Z. For further readings we refer to Aronszajn (1950), Parzen (1967) and Wahba (1990).

2. Results

Regularity in quadratic mean of the stochastic process Z is specified by the regularity of its covariance kernel at the diagonal in $[0,1]^2$. We denote one-sided limits at the diagonal in the following way. Let

$$\Omega_+ = \{(s,t) \in]0,1[^2 : s > t\}, \qquad \Omega_- = \{(s,t) \in]0,1[^2 : s < t\},$$

and let $\operatorname{cl} A$ denote the closure of a set A. Suppose that L is a continuous function on $\Omega_+ \cup \Omega_-$ such that $L|_{\Omega_j}$ is continuously extendable to $\operatorname{cl}\Omega_j$ for $j \in \{+,-\}$. By L_j we denote the extension of L to $[0,1]^2$ which is continuous on $\operatorname{cl}\Omega_j$ and on $[0,1]^2 \setminus \operatorname{cl}\Omega_j$.

The covariance kernel R satisfies the Sacks-Ylvisaker conditions of order zero if the following three conditions hold.

(A) $R \in C([0,1]^2)$, *the partial derivatives of* R *up to order two are continuous on* $\Omega_+ \cup \Omega_-$ *and continuously extendable to* $\operatorname{cl}\Omega_+$ *as well as to* $\operatorname{cl}\Omega_-$.

(B)
$$R_-^{(1,0)}(t,t) - R_+^{(1,0)}(t,t) = 1, \qquad 0 \le t \le 1.$$

(C) $R_+^{(2,0)}(t,\cdot) \in H(R)$ *for all* $0 \le t \le 1$ *and*

$$\sup_{0\le t\le 1} \|R_+^{(2,0)}(t,\cdot)\|_R < \infty.$$

Sacks and Ylvisaker (1966) have introduced these conditions to study estimation of weighted integrals of stochastic processes. A detailed discussion and various examples are given there. The conditions are satisfied in particular if $R(s,t)$ is given by $\min(s,t)$, $(1-|s-t|)/2$, or $\exp(-|s-t|)/2$. Similar conditions, also for kernels of higher regularity, are used by several authors, see, e.g., Sacks and Ylvisaker (1970), Su and Cambanis (1993), Müller-Gronbach (1995a, 1995b), Müller-Gronbach and Schwabe (1995), Ritter (1995), and Ritter *et al.* (1995).

Due to (A) the Gaussian process Z is continuous with probability one. If R fulfills (A) and corresponds to a stationary process then the difference in (B) is always a nonnegative constant. For arbitrary kernels which satisfy (A) and (C) this difference is constant, too. For convenience we assume that the constant equals one.

In the sequel we always assume that (A), (B), and (C) are satisfied.

Consider a fixed design $T = \{t_1, \ldots, t_n\}$ and let opr_T denote the orthogonal projector onto the subspace of $L_2(Z)$ which is spanned by $Z(t_1), \ldots, Z(t_n)$. The optimal reconstruction which uses the design T is given by

$$\widehat{X}_T^*(t) = m(t) + opr_T Z(t), \qquad 0 \le t \le 1, \tag{1}$$

i.e.,

$$e_p(\widehat{X}_T^*) = \inf_{\widehat{X}_T} e_p(\widehat{X}_T).$$

The optimality of $\widehat{X}_T^*$ holds for reconstruction with respect to the L_q-norm for any $1 \le q \le \infty$. This follows from the fact that the conditional distribution of Z given $Z(t_1), \ldots, Z(t_n)$ is symmetric with respect to the conditional mean $opr_T Z(\cdot)$. See, e.g., Traub, Wasilkowski, and Woźniakowski (1988).

Note that (1) can be written in terms of the observations $X(t_i)$. Assume that

$$R_T = (R(t_i, t_j))_{1\le i,j\le n}$$

is invertible, which involves no loss of generality. Then

$$\widehat{X}_T^*(t) = m(t) + a_T(t)'\, R_T^{-1}\, Z_T \tag{2}$$

where

$$a_T(t) = (R(t,t_1), \ldots, R(t,t_n))'$$

and

$$Z_T = (X(t_1) - m(t_1), \ldots, X(t_n) - m(t_n))'.$$

Let ψ be a positive and continuous function on $[0,1]$. Define quantiles $t_{i,n} \in [0,1]$ for $i = 1, \ldots, n$ and $n \ge 2$ by

$$\int_0^{t_{i,n}} \psi(t)\, dt = \frac{i-1}{n-1} \int_0^1 \psi(t)\, dt.$$

The corresponding sequence of designs

$$T_n = \{t_{1,n}, \ldots, t_{n,n}\}$$

is called the regular sequence generated by ψ and denoted by $(T_n)_n = \mathrm{RS}(\psi)$. Regular sequences are easy to describe and implement, and they are also well suited for an error analysis. It turns out that regular sequences already lead to asymptotically optimal designs.

Theorem 1. *The minimal errors satisfy*

$$\inf_{|T|=n} \inf_{\widehat{X}_T} e_p(\widehat{X}_T) \approx \left(\frac{1}{2}\int_0^1 w^2(t)\,dt\right)^{1/2} \left(\frac{\ln n}{n}\right)^{1/2}.$$

Furthermore, the regular sequence

$$(T_n)_n = \mathrm{RS}(w^2)$$

is asymptotically optimal, i.e.,

$$e_p(\widehat{X}^*_{T_n}) \approx \inf_{|T|=n} \inf_{\widehat{X}_T} e_p(\widehat{X}_T).$$

Remark 1. Regular sequences are known to be asymptotically optimal for several other problems. A partial list of papers includes Sacks and Ylvisaker (1966, 1968, 1970), Eubank, Smith, and Smith (1982) and Ritter (1995) for estimation of weighted integrals and Speckman (1979), Su and Cambanis (1993), and Müller-Gronbach (1994, 1995a, 1995b) for reconstruction.

Clearly, a reconstruction $\widehat{X}^*_T$ can be calculated on the basis of observations $X(t_i)$ only if the mean m and the covariance kernel R are known. Even in this case, the covariance matrix R_T has to be inverted, which may be a tedious task. Assume that

$$0 = t_1 < \ldots < t_n = 1.$$

A very simple reconstruction which overcomes the above difficulties is given by the linear interpolator

$$\widehat{X}^L_T(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i} X(t_i) + \frac{t-t_i}{t_{i+1}-t_i} X(t_{i+1}), \qquad t_i \le t \le t_{i+1}. \tag{3}$$

If $R^{(0,2)} = 0$ outside the diagonal and $m = 0$, then linear interpolation is optimal. In particular this holds for the covariance kernel of a Brownian motion and a Brownian bridge and for the triangular covariance kernel $R(s,t) = 1 - |t-s|$. However, $\widehat{X}^L_T = \widehat{X}^*_T$ does not hold in general; take $R(s,t) = \exp(-|s-t|)$ as a counterexample.

Theorem 2. *Assume that the mean m of X is Hölder continuous with exponent $1/2$. Then linear interpolation based on the regular sequence*

$$(T_n)_n = \mathrm{RS}(w^2)$$

is asymptotically optimal, i.e.,

$$e_p(\widehat{X}^L_{T_n}) \approx \inf_{|T|=n} \inf_{\widehat{X}_T} e_p(\widehat{X}_T).$$

If the covariance kernel R is known, there are at least two more methods of reconstruction which perform asymptotically optimal under appropriate conditions on the mean function m. Let

$$X_T = (X(t_1), \dots, X(t_n))'$$

and define

$$\widehat{X}^0_T(t) = a_T(t)'\, R_T^{-1}\, X_T, \qquad 0 \le t \le 1. \tag{4}$$

This method ignores the mean function m, see (2). Let P_T denote the orthogonal projector onto the subspace of $H(R)$ which is spanned by $R(\cdot, t_1), \dots, R(\cdot, t_n)$. If $m \in H(R)$ then (4) can be written as

$$\widehat{X}^0_T(t) = P_T m(t) + opr_T Z(t), \tag{5}$$

which shows the connection between $\widehat{X}^0_T$ and $\widehat{X}^*_T$ more clearly.

In general, neither $\widehat{X}^L_T$ nor $\widehat{X}^0_T$ are unbiased estimators for m. If the mean function m is linearly parameterized, i.e.,

$$m(t) = \sum_{j=1}^{\ell} \beta_j f_j(t) \tag{6}$$

with known and linearly independent functions $f_1, \dots, f_\ell$ on $[0,1]$, we therefore also consider the best linear unbiased predictors $\widehat{X}^B_T$ given by

$$\widehat{X}^B_T(t) = \widehat{\beta}'_T\, f(t) + a_T(t)'\, R_T^{-1}\, (X_T - F_T\, \widehat{\beta}_T), \qquad 0 \le t \le 1.$$

Here $\widehat{\beta}_T$ denotes the weighted least squares estimator for $\beta = (\beta_1, \dots, \beta_\ell)'$ on the basis of the observations X_T, and

$$f = (f_1, \dots, f_\ell)'$$

as well as

$$F_T = (f_j(t_i))_{1 \le i \le n,\, 1 \le j \le \ell}$$

Similar to $\widehat{X}^0_T$, we can write $\widehat{X}^B_T$ in terms of $H(R)$ if $f_1, \dots, f_\ell \in H(R)$. Let

$$A_T = ((P_T f_i, f_j)_R)_{1 \le i,j \le \ell}.$$

Observing that $A_T = F'_T\, R_T^{-1}\, F_T$ and $(m(t_1), \dots, m(t_n))' = F_T\, \beta$ it follows that

$$\widehat{\beta}_T = A_T^{-1}\, F'_T\, R_T^{-1}\, X_T = \beta + A_T^{-1}\, F'_T\, R_T^{-1}\, Z_T$$

and therefore

$$\widehat{X}^B_T(t) = \widehat{X}^*_T(t) + (f - P_T f)'(t)\, A_T^{-1}\, F'_T\, R_T^{-1}\, Z_T. \tag{7}$$

The matrix $A = ((f_i, f_j)_R)_{1\le i,j\le \ell}$ is positive definite since $f_1, \dots, f_\ell$ are linearly independent. Moreover, by continuity of R, the matrices A_{T_n} converge to A if the sequence of designs T_n gets dense in $[0,1]$. In particular, $\widehat{X}^B_{T_n}$ is well defined for large n if $(T_n)_n = RS(w^2)$.

Theorem 3. *Let* $(T_n)_n = RS(w^2)$.

(i) *If m is of the form*

$$m(t) = \int_0^1 R(s,t)\varphi(s)\, ds, \qquad 0 \le t \le 1,$$

with φ being continuous, then

$$e_p(\widehat{X}^0_{T_n}) \approx \inf_{|T|=n} \inf_{\widehat{X}_T} e_p(\widehat{X}_T).$$

(ii) *If m is of the form* (6) *and*

$$f_j(t) = \int_0^1 R(s,t)\varphi_j(s)\, ds, \qquad 0 \le t \le 1,$$

with $\varphi_1, \dots, \varphi_\ell$ being continuous, then

$$e_p(\widehat{X}^B_{T_n}) \approx \inf_{|T|=n} \inf_{\widehat{X}_T} e_p(\widehat{X}_T).$$

Remark 2. According to Theorems 2 and 3 an unknown mean m of a certain form has no effect asymptotically. For reconstruction with respect to the L_2-norm and the maximum mean squared error similar results are obtained in Su and Cambanis (1993) and Müller-Gronbach (1995b).

The functions m which are considered in (i) form a dense subset in the reproducing kernel Hilbert space $H(R)$. Furthermore, they are Lipschitz continuous while an arbitrary function from $H(R)$ is merely Hölder continuous with exponent 1/2. Therefore the assumptions from Theorem 3 are stronger than those from Theorem 2.

Remark 3. The foregoing theorems deal with the optimal regular sequence $(T_n)_n = \mathrm{RS}(w^2)$. Clearly equidistant designs $(T_n)_n = \mathrm{RS}(1)$ are asymptotically optimal for constant weight functions w, and they yield errors of order $\sqrt{\ln n/n}$ for any w. However, they do not yield optimal asymptotic constants in general.

So far the optimal regular sequence depends only on the weight function w and not on local properties of the process X. In a more general approach we consider processes

$$X(t) = m(t) + g(t)\, Z(f(t)) \qquad 0 \le t \le 1,$$

where Z is zero mean Gaussian with covariance kernel satisfying (A), (B), and (C), as previously. Extensions of the above results to this case are straightforward under suitable smoothness conditions on the additional weight function g and the parameter transformation f. For example, if f' and g are positive and Hölder continuous with exponent 1/2, then Theorem 2 holds with

$$(T_n)_n = \mathrm{RS}(w^2 \cdot g^2 f')$$

and

$$e_p(\widehat{X}^L_{T_n}) \approx \left(\frac{1}{2}\int_0^1 (w^2 \cdot g^2 f')(t)\,dt\right)^{1/2} \left(\frac{\ln n}{n}\right)^{1/2}.$$

The problem how to estimate $g^2 f'$ from data $X(t_i)$ in order to reconstruct X is addressed in a forthcoming paper.

Remark 4. In this paper we study reconstructions which are based on observations $X(t_i)$. More generally, we may allow observations $\lambda_i(X)$ where $\lambda_1, \dots, \lambda_n$ are bounded linear functionals on $C([0,1])$. For instance, this covers weighted integrals

$$\lambda_i(X) = \int_0^1 X(s)\,\xi_i(s)\,ds$$

and in particular Fourier or wavelet coefficients of X. Due to Maiorov and Wasilkowski (1995) one cannot improve the order of minimal errors in this way. The error of any reconstruction which uses n functionals λ_i is bounded from below by $c\sqrt{\ln n/n}$ under Sacks-Ylvisaker conditions of order zero. Here $c > 0$ depends only on the covariance kernel R of X. The corresponding asymptotic constant for minimal errors is unknown.

Furthermore, an adaptive (sequential) selection of the sampling points t_i or functionals λ_i does not lead to smaller errors. See Traub *et al.* (1988) and Novak (1995) for a comprehensive presentation of general results.

Note that the optimal reconstructions $\widehat{X}^*_T$ are affine operators with finite dimensional range in $C([0,1])$. More generally, we may allow any method which uses functions from a finite dimensional subspace $U \subset C([0,1])$ to reconstruct X. This covers in particular best uniform approximation from U. Due to Maiorov (1993) there exists a sequence U_n of subspaces of dimension n such that the error

$$E(\inf_{u \in U_n} \sup_{0 \le t \le 1} |X(t) - u(t)|)$$

is of order $1/\sqrt{n}$. The proof is nonconstructive and subspaces U_n which yield errors of order less than $\sqrt{\ln n/n}$ are not known explicitly.

3. Proofs

Let B denote a centered Brownian bridge on $[0,1]$. The basic idea which underlies our proofs is to relate the error $e_p(\widehat{X}^*_T)$ to $e_p(\widehat{B}^*_T)$ by comparing the covariance kernels of the corresponding error processes. Sharp estimates on $e_p(\widehat{B}^*_T)$ are available since $B - \widehat{B}^*_T$ consists of independent Brownian bridges, connected according to the design T.

For two functions K_1 and K_2 on $[0,1]^2$ we write $K_1 \ll K_2$ if $K_2 - K_1$ is positive semidefinite on $[0,1]^2$.

Lemma 1. *Let Y_1 and Y_2 be centered Gaussian processes with continuous sample paths and covariance kernels K_1 and K_2, respectively. Then $K_1 \ll K_2$ implies*

$$E\Big(\sup_{0 \le t \le 1} |Y_1(t)|^p\Big) \le E\Big(\sup_{0 \le t \le 1} |Y_2(t)|^p\Big).$$

Proof. Since $K_1 \ll K_2$ we may apply Anderson's inequality to obtain

$$P((Y_1(0), Y_1(1/n), \dots, Y_1(1)) \in A) \geq P((Y_2(0), Y_2(1/n), \dots, Y_2(1)) \in A)$$

for each convex and symmetric set $A \subset \mathbb{R}^{n+1}$ and $n \geq 1$. Because of

$$E\Big(\sup_{0 \leq t \leq 1} |Y_i(t)|^p\Big) = \int_0^\infty \Big(1 - \lim_{n \to \infty} P\Big(\bigcap_{k=0}^{n} \{|Y_i(k/n)| \leq x^{1/p}\}\Big)\Big)\, dx$$

the statement follows. $\square$

Without loss of generality we henceforth consider designs

$$T = \{t_1, \dots, t_n\}$$

such that

$$0 = t_1 < \dots < t_n = 1.$$

Put

$$\Delta_{\max}(T) = \max_{1 \leq i \leq n-1} (t_{i+1} - t_i).$$

Lemma 2. *Let R^T and L^T denote the covariance kernels of the error processes $X - \widehat{X}_T^*$ and $B - \widehat{B}_T^*$, respectively. Then*

$$(1 - c\,\Delta_{\max}(T)) \cdot L^T \ll R^T \ll (1 + c\,\Delta_{\max}(T)) \cdot L^T,$$

where the constant c is positive and independent of T and n.

Proof. Let $t_i \leq t \leq t_{i+1}$. From Müller-Gronbach (1995b, Lemma 3) we know that

$$K^T(s,t) = L^T(s,t) - \int_{t_i}^{t_{i+1}} (R_+^{(2,0)}(u,\cdot), R(\cdot,s) - P_T R(\cdot,s))_R\, L^T(u,t)\, du. \tag{8}$$

For $1 \leq i \leq n-1$ choose $a_{i1}, \dots, a_{im_i} \in \mathbb{R}$ and $s_{i1}, \dots, s_{im_i} \in [t_i, t_{i+1}]$. Let

$$X_i = \sum_{k=1}^{m_i} a_{ik}(X(s_{ik}) - \widehat{X}_T^*(s_{ik})), \qquad B_i = \sum_{k=1}^{m_i} a_{ik}(B(s_{ik}) - \widehat{B}_T^*(s_{ik}))$$

and

$$\overline{X} = \sum_{i=1}^{n-1} X_i, \qquad \overline{B} = \sum_{i=1}^{n-1} B_i.$$

In order to prove Lemma 2 we will show that

$$(1 - c\,\Delta_{\max}(T)) \cdot E(\overline{B}^2) \leq E(\overline{X}^2) \leq (1 + c\,\Delta_{\max}(T)) \cdot E(\overline{B}^2) \tag{9}$$

with $c > 0$ only depending on the kernel R.

Let $\mathfrak{I}$ denote the isomorphism from $L_2(Z)$ to $H(R)$. By (8) we have

$$\begin{aligned} \sum_{j=1}^{n-1} E(X_i X_j) &= E(B_i^2) - \sum_{j=1}^{n-1} \int_{t_i}^{t_{i+1}} (R_+^{(2,0)}(u,\cdot), \mathfrak{I}X_j)_R\, E((B(u) - \widehat{B}_T^*(u))B_i)\, du \\ &= E(B_i^2) - \int_{t_i}^{t_{i+1}} (R_+^{(2,0)}(u,\cdot), \mathfrak{I}\overline{X})_R\, E((B(u) - \widehat{B}_T^*(u))B_i)\, du \end{aligned}$$

for each $1 \le i \le n-1$. Observing that $E(\overline{B}^2) = \sum_{i=1}^{n-1} E(B_i^2)$, it follows that

$$E(\overline{X}^2) = E(\overline{B}^2) - \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} (R_+^{(2,0)}(u,\cdot), \Im \overline{X})_R \, E((B(u) - \widehat{B}_T^*(u))B_i) \, du. \tag{10}$$

Put $c_0 = \sup_{0 \le t \le 1} \|R_+^{(2,0)}(u,\cdot)\|_R$. Since

$$\begin{aligned}\sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} |E((B(u) - \widehat{B}_T^*(u))B_i)| \, du &\le \sum_{i=1}^{n-1} (E(B_i^2))^{1/2} \int_{t_i}^{t_{i+1}} (L^T(u,u))^{1/2} \, du \\ &\le \sum_{i=1}^{n-1} (t_{i+1} - t_i)^{3/2} \, (E(B_i^2))^{1/2} \\ &\le \Delta_{\max}(T) \, (E(\overline{B}^2))^{1/2}\end{aligned}$$

and

$$\sup_{0 \le u \le 1} |(R_+^{(2,0)}(u,\cdot), \Im \overline{X})_R| \le c_0 \, (E(\overline{X}^2))^{1/2}$$

we conclude from (10) that

$$|E(\overline{X}^2) - E(\overline{B}^2)| \le c_0 \, \Delta_{\max}(T) \, (E(\overline{B}^2) \, E(\overline{X}^2))^{1/2}.$$

This implies

$$(1 + c_0 \, \Delta_{\max}(T))^{-2} \le \frac{E(\overline{X}^2)}{E(\overline{B}^2)} \le (1 + c_0 \, \Delta_{\max}(T))^2$$

and (9) follows with a suitable choice of c. $\square$

Proof of Theorem 1. Consider the particular case of a centered Brownian bridge. For a fixed design T let $Y = B - \widehat{B}_T^*$ denote the corresponding error process. We have

$$\begin{aligned}e_p(\widehat{B}_T^*)^p &= \int_0^\infty P\Big(\sup_{0 \le t \le 1} w(t) \, |Y(t)| > u^{1/p}\Big) \, du \\ &= \int_0^\infty \Big(1 - \prod_{i=1}^{n-1} P\Big(\sup_{t_i \le t \le t_{i+1}} w(t) \, |Y(t)| \le u^{1/p}\Big)\Big) \, du.\end{aligned}$$

Furthermore,

$$\begin{aligned}P\Big(\sup_{t_i \le t \le t_{i+1}} w(t) \, |Y(t)| \le u^{1/p}\Big) &\le P\Big(a_i \sup_{t_i \le t \le t_{i+1}} |Y(t)| \le u^{1/p}\Big) \\ &= F\left(\frac{u^{1/p}}{a_i \, (t_{i+1} - t_i)^{1/2}}\right)\end{aligned} \tag{11}$$

where F denotes the distribution function of $\sup_{0 \le t \le 1} |B(t)|$ and

$$a_i = \min_{t_i \le t \le t_{i+1}} w(t) > 0.$$

From Ritter (1990, Lemma 1) we know that

$$\prod_{i=1}^{n-1} F\left(\frac{u^{1/p}}{a_i \, (t_{i+1} - t_i)^{1/2}}\right) \le F^n\left(\frac{u^{1/p}}{(r/n)^{1/2}}\right)$$

where

$$r = r(T) = \sum_{i=1}^{n-1} a_i^2 \, (t_{i+1} - t_i).$$

Therefore

$$e_p(\widehat{B}_T^*)^p \geq \int_0^\infty \Big(1 - F^n\Big(u^{1/p} \, (n/r)^{1/2}\Big)\Big) \, du = (r/n)^{p/2} \int_0^\infty \Big(1 - F^n(u^{1/p})\Big) \, du.$$

Consider a sequence of designs T_n such that $e_p(\widehat{B}_{T_n}^*)$ tends to zero. Then $\Delta_{\max}(T_n)$ tends to zero, too, and

$$\lim_{n\to\infty} r(T_n) = \int_0^1 w^2(t) \, dt = c_1.$$

Since

$$\left(\int_0^\infty \Big(1 - F^n(u^{1/p})\Big) \, du\right)^{1/p} \approx (\ln n/2)^{1/2}, \tag{12}$$

see Ritter (1990, p. 351), we get the lower bound

$$\liminf_{n\to\infty} \inf_{|T|=n} e_p(\widehat{B}_T^*) \left(\frac{n}{\ln n}\right)^{1/2} \geq (c_1/2)^{1/2}.$$

The matching upper bound is obtained by the regular sequence $(T_n)_n = \mathrm{RS}(w^2)$. We have

$$t_{i+1,n} - t_{i,n} \leq \frac{c_1}{(n-1) \min\limits_{t_{i,n} \leq t \leq t_{i+1,n}} w^2(t)}$$

for the corresponding sampling points. Analogously to (11) we conclude that

$$P\Big(\sup_{t_{i,n} \leq t \leq t_{i+1,n}} w(t) \, |Y(t)| \leq u^{1/p}\Big) \geq F(u^{1/p} \, ((n-1)/c_1)^{1/2} \, q_n)$$

with

$$q_n = \inf_{1 \leq i \leq n-1} \frac{\min\limits_{t_{i,n} \leq t \leq t_{i+1,n}} w(t)}{\max\limits_{t_{i,n} \leq t \leq t_{i+1,n}} w(t)}.$$

Thus

$$\begin{aligned} e_p(\widehat{B}_{T_n}^*)^p &\leq \int_0^\infty \Big(1 - F^n\Big(u^{1/p} \, ((n-1)/c_1)^{1/2} \, q_n\Big)\Big) \, du \\ &= \Big(((n-1)/c_1)^{1/2} \, q_n\Big)^{-p} \int_0^\infty \Big(1 - F^n(u^{1/p})\Big) \, du. \end{aligned}$$

Clearly q_n tends to one, and due to (12) we have

$$\limsup_{n\to\infty} e_p(\widehat{B}_{T_n}^*) \left(\frac{n}{\ln n}\right)^{1/2} \leq (c_1/2)^{1/2},$$

which completes the proof in the case of a Brownian bridge.

Now we turn to an arbitrary process X. Due to Lemmata 1 and 2,

$$e_p(\widehat{X}_{T_n}^*) \approx e_p(\widehat{B}_{T_n}^*)$$

for any sequence of designs T_n such that $\Delta_{\max}(T_n)$ tends to zero. Thus the result for the Brownian bridge apply. $\square$

Proof of Theorem 2. Define a stochastic processes Z_T^L by

$$Z_T^L(t) = \frac{t_{i+1}-t}{t_{i+1}-t_i} Z(t_i) + \frac{t-t_i}{t_{i+1}-t_i} Z(t_{i+1})$$

for $t_i \le t \le t_{i+1}$ and $1 \le i \le n-1$. Since m is Hölder continuous with exponent $1/2$ we have

$$\sup_{t_i \le t \le t_{i+1}} \left| m(t) - \frac{t_{i+1}-t}{t_{i+1}-t_i} m(t_i) + \frac{t-t_i}{t_{i+1}-t_i} m(t_{i+1}) \right| \le c_2 (t_{i+1}-t_i)^{1/2}$$

with a positive constant c_2. Hence

$$\begin{aligned} e_p(\widehat{X}_T^L) \le e_p(\widehat{X}_T^*) + &\left(E\Big(\sup_{0 \le t \le 1} w(t)\, |opr_T Z(t) - Z_T^L(t)| \Big)^p \right)^{1/p} \\ &+ c_2\, (\Delta_{\max}(T))^{1/2} \sup_{0 \le t \le 1} w(t). \end{aligned}$$

Clearly $\Delta_{\max}(T_n)$ is of order $1/n$ for any regular sequence $(T_n)_n$. Let S^T and L^T denote the covariance kernels of $opr_T Z - Z_T^L$ and $B - \widehat{B}_T^*$, respectively. Observing Lemma 1 and Theorem 1, it is sufficient to show that

$$S^T \ll c_3\, \Delta_{\max}(T)\, L^T, \tag{13}$$

where the constant c_3 is positive and independent of T.

From Müller-Gronbach (1995b, p. 18) we know that

$$S^T(s,t) = (h_s^T, h_t^T)_R,$$

where

$$h_t^T(u) = \int_0^1 P_T R_+^{(2,0)}(v,\cdot)(u)\, L^T(v,t)\, dv.$$

Hence

$$S^T(s,t) = \int_0^1 \int_0^1 \Big(P_T R_+^{(2,0)}(u,\cdot), P_T R_+^{(2,0)}(v,\cdot) \Big)_R L^T(u,s)\, L^T(v,t)\, dv\, du.$$

For $1 \le i \le n-1$ let

$$V_i = \sum_{k=1}^{m_i} a_{ik} (opr_T Z(s_{ik}) - Z_T^L(s_{ik}))$$

and put

$$\overline{V} = \sum_{i=1}^{n-1} V_i.$$

By the same arguments used in the proof of Lemma 2 it follows that

$$\begin{aligned}
E(\overline{V}^2) &\leq c_0^2 \sum_{i,j=1}^{n-1} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} |E((B(v) - \widehat{B}_T^*(v))B_j)\, E((B(u) - \widehat{B}_T^*(u))B_i)|\, dv\, du \\
&\leq c_0^2 \Big(\sum_{i=1}^{n-1} (t_{i+1} - t_i)^{3/2} \,(E(B_i^2))^{1/2} \Big)^2 \\
&\leq c_0^2 \cdot \Delta_{\max}^2(T)\, E(\overline{B}^2),
\end{aligned}$$

which proves (13). $\square$

Proof of Theorem 3. Let $w_{\max} = \sup_{0\leq t\leq 1} w(t)$ and $R_{\max} = \sup_{0\leq t\leq 1} R(t,t)$. By (5),

$$\begin{aligned}
\sup_{0\leq t\leq 1} w(t)\, |X(t) - \widehat{X}_T^0(t)| &\leq \sup_{0\leq t\leq 1} w(t)\, |Z(t) - opr_T\, Z(t)| + \sup_{0\leq t\leq 1} w(t)\, |m(t) - P_T m(t)| \\
&\leq \sup_{0\leq t\leq 1} w(t)\, |X(t) - \widehat{X}_T^*(t)| + w_{\max} R_{\max}^{1/2}\, \|m - P_T m\|_R.
\end{aligned}$$

Hence

$$e_p(\widehat{X}_T^0) \leq e_p(\widehat{X}_T^*) + w_{\max} R_{\max}^{1/2}\, \|m - P_T m\|_R.$$

From Sacks and Ylvisaker (1968, Lemma 3.1) it is easy to see that

$$\|m - P_T m\|_R \leq c_4\, \Delta_{\max}(T), \tag{14}$$

where the constant c_4 is independent of T and n. Hence $\|m - P_{T_n} m\|_R$ is of order $1/n$ for any regular sequence $(T_n)_n$. Observing Theorem 1, part (i) follows.

Now let m satisfy the assumptions of part (ii). Assume without loss of generality that R_T and A_T are invertible. From (7) we get

$$e_p(\widehat{X}_T^B) \leq e_p(\widehat{X}_T^*) + w_{\max} \Big(E\Big(\sup_{0\leq t\leq 1} |(f - P_T f)'(t)\, A_T^{-1/2}\, U_T|^p \Big)\Big)^{1/p} \tag{15}$$

where

$$U_T = A_T^{-1/2}\, F_T'\, R_T^{-1}\, Z_T.$$

Let $\lceil p \rceil = \min\{k \in \mathbb{N} : k \geq p\}$. Since U_T has an n-dimensional standard normal distribution we have $U_T'\, U_T \sim \chi^2(n)$ and therefore

$$\begin{aligned}
(E(U_T'\, U_T)^p)^{1/p} &\leq \Big(E(U_T'\, U_T)^{\lceil p \rceil} \Big)^{1/\lceil p \rceil} \\
&= (n(n+2) \cdots (n + 2(\lceil p \rceil - 1)))^{1/\lceil p \rceil} \\
&\leq 3n
\end{aligned}$$

for each $n \geq \lceil p \rceil$. Furthermore, by (14),

$$\begin{aligned}
(f - P_T f)'(t)\, A_T^{-1}\, (f - P_T f)(t) &\leq \lambda_{\max}(A_T^{-1}) \cdot (f - P_T f)'(t)\, (f - P_T f)(t) \\
&\leq \lambda_{\max}(A_T^{-1}) \cdot R(t,t) \sum_{j=1}^{\ell} \|f_j - P_T f_j\|_R^2 \\
&\leq c_5 \cdot \lambda_{\max}(A_T^{-1})\, \Delta_{\max}^2(T),
\end{aligned}$$

where the constant c_5 is independent of T and n.

Now let $(T_n)_n = \mathrm{RS}(w^2)$ and assume $n \geq \lceil p \rceil$. Then

$$\begin{aligned}\Big(E\Big(\sup_{0\leq t\leq 1} |(f-P_{T_n}f)'(t)\, A_{T_n}^{-1/2}\, U_{T_n}|^p\Big)\Big)^{1/p} \\ &\leq \sup_{0\leq t\leq 1} \Big((f-P_{T_n}f)'(t)\, A_{T_n}^{-1}\, (f-P_{T_n}f)(t)\Big)^{1/2} \cdot \Big(E(U_{T_n}'\, U_{T_n})^p\Big)^{1/2p} \\ &\leq c_6 \cdot n^{-1/2}\, \lambda_{\max}(A_{T_n}^{-1})^{1/2},\end{aligned}$$

where the constant c_6 is independent of n. Since $\lambda_{\max}(A_{T_n}^{-1})$ tends to $\lambda_{\max}(A^{-1})$ part (ii) now follows from (15) and Theorem 1. $\square$

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