The Daugavet equation for operators on function spaces

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ABSTRACT. We prove the norm identity $\|Id + T\| = 1 + \|T\|$, which is known as the Daugavet equation, for weakly compact operators T on natural function spaces such as function algebras and L^1-predual spaces, provided a non-discreteness assumption is met. We also consider c_0-factorable operators and operators on C_Λ-spaces.

1. INTRODUCTION

In his 1963 paper [5] Daugavet proved the remarkable norm identity

$$\|Id + T\| = 1 + \|T\| \tag{1.1}$$

for a compact operator on $C[0,1]$; in the sequel (1.1) has become known as the *Daugavet equation*. (1.1) has proved useful in approximation theory; Stečkin used it in order to provide precise lower bounds for trigonometric approximations [14]. Daugavet's result was extended to other classes of operators on spaces of continuous functions $C(S)$, notably to weakly compact operators [8] and also to operators on L^1-spaces; we refer to [1] and [2] for a more detailed account of the history of the subject.

In this paper we suggest a systematic, yet simple approach to studying the Daugavet equation for operators on subspaces of $C(S)$-spaces. Since every Banach space embeds isometrically into a $C(S)$-space, some restriction on the embedding has to be imposed. Also, a natural obstruction for the Daugavet equation to hold for – say – compact operators on $C(S)$ is the presence of isolated points in S, because an isolated point of S immediately gives rise to a one-dimensional operator (i.e., an operator with one-dimensional range) T on $C(S)$ with $\|Id - T\| = 1$. In section 2 we formulate conditions on an isometric embedding J of a Banach space X into $C(S)$, labelled (N1) and (N2), which we will assume in order that X be "nicely embedded;" in addition we will require a non-discreteness condition (N3).

In the next section we will present a necessary and sufficient condition for the Daugavet equation on a nicely embedded space, and we check it for weakly compact operators and for operators factoring through a subspace of c_0; note that both these classes encompass the compact operators. Whereas section 2 has a deliberately technical flavour, we give applications to concrete spaces in section 3. In particular, we deal with function algebras, L^1-predual spaces and translation invariant spaces, and we establish the Daugavet equation for various classes of operators.

Condition (N2) will be given in terms of the L-structure of X^*. We recall the relevant definitions. A closed subspace F of a Banach space E is an *L-summand* if there is a projection Π from E onto F such that

$$\|\xi\| = \|\Pi(\xi)\| + \|\xi - \Pi(\xi)\| \qquad \forall \xi \in E.$$

Dual to this notion is the definition of an M-ideal: $F \subset E$ is an *M-ideal* if its annihilator $F^\perp \subset E^*$ is an L-summand. These concepts are studied in detail in [10]. Roughly speaking, our conditions (N1)–(N3) mean that the function space X has a rich and nondiscrete M-ideal structure.

We use standard notation such as $L(X)$ for the space of all bounded linear operators on a Banach space X, B_X for the closed unit ball of X and $\operatorname{ex} C$ for the set of extreme points of a convex set C.

2. Nicely embedded Banach spaces

In order to formulate the lemmas of this section succinctly, we need to introduce some vocabulary. Let S be a Hausdorff topological space, and let $C^b(S)$ be the sup-normed Banach space of all bounded continuous scalar-valued functions. The functional $f \mapsto f(s)$ on $C^b(S)$ is denoted by δ_s. We say that a linear map $J\colon X \to C^b(S)$ on a Banach space X is a *nice embedding* and that X is *nicely embedded* into $C^b(S)$ if J is an isometry such that for all $s \in S$ the following properties are satisfied:

(N1) *For $p_s := J^*(\delta_s) \in X^*$ we have $\|p_s\| = 1$.*
(N2) $\operatorname{lin}\{p_s\}$ *is an L-summand in X^*.*

The latter condition can also be rephrased by saying that the kernel of p_s is an M-ideal. We will discuss examples of nicely embedded Banach spaces in section 3.

Throughout this section we will stick to the following notation. Let $J\colon X \to C^b(S)$ be a nice embedding, $p_s = J^*(\delta_s)$, and let $T \in L(X)$. We put

$$q_s := (JT)^*(\delta_s) = T^*(p_s) \in X^*$$

and note that $s \mapsto q_s$ is weak* continuous and $\|T\| = \sup_s \|q_s\|$. Likewise, $s \mapsto p_s$ is weak* continuous. By (N2) there is a family of projections Π_s, $s \in S$, with $\operatorname{ran} \Pi_s = \operatorname{lin}\{p_s\}$ such that

$$\|x^*\| = \|\Pi_s(x^*)\| + \|x^* - \Pi_s(x^*)\| \qquad \forall x^* \in X^*$$

and a family of functionals $\pi_s \in X^{**}$, $s \in S$, such that

$$\Pi_s(x^*) = \pi_s(x^*)p_s \qquad \forall x^* \in X^*.$$

In particular, we have $\pi_s(p_s) = 1$.

We will also need the equivalence relation

$$s \sim t \text{ if and only if } \Pi_s = \Pi_t \tag{2.1}$$

on S. Then s and t are equivalent if and only if p_s and p_t are linearly dependent, which implies by (N1) that $p_t = \lambda p_s$ for some scalar of modulus 1. The equivalence classes of this relation are obviously closed.

In some of the lemmas to follow we will additionally have to assume

(N3) *None of the equivalence classes $Q_s = \{t \in S\colon s \sim t\}$ contains an interior point.*

If the set $\{p_s\colon s \in S\}$ is linearly independent, this simply means that

(N3$'$) *S does not contain an isolated point.*

By (N2), the p_s are linearly independent as soon as they are pairwise linearly independent.

We now give the basic criterion for an operator on a nicely embedded Banach space to satisfy the Daugavet equation.

Lemma 2.1 *Let X be nicely embedded into $C^b(S)$ so that* (N1) *and* (N2) *are valid, and let $T \in L(X)$. For $\varepsilon > 0$ put $U_\varepsilon = \{s \in S\colon \|q_s\| > \|T\| - \varepsilon\}$, which is an open subset of S. Then T satisfies the Daugavet equation* (1.1) *if and only if*

$$\sup_{s \in U_\varepsilon} \left(|1 + \pi_s(q_s)| - (1 + |\pi_s(q_s)|)\right) \geq 0 \qquad \forall \varepsilon > 0. \tag{2.2}$$

Proof. First assume (2.2). We observe

$$\begin{aligned} \|p_s + q_s\| &= \|\Pi_s(p_s + q_s)\| + \|(p_s + q_s) - \Pi_s(p_s + q_s)\| \\ &= |1 + \pi_s(q_s)| + \|(Id - \Pi_s)(q_s)\| \end{aligned}$$

and
$$1 + \|q_s\| = 1 + |\pi_s(q_s)| + \|(Id - \Pi_s)(q_s)\|$$
for all $s \in S$. Applying the assumption, for some $\varepsilon > 0$, we obtain from this
$$\begin{aligned}
\|Id + T\| &= \sup_{s\in S} \|p_s + q_s\| \ \geq \ \sup_{s\in U_\varepsilon} \|p_s + q_s\| \\
&= \sup_{s\in U_\varepsilon} (|1 + \pi_s(q_s)| + \|(Id - \Pi_s)(q_s)\|) \\
&\geq 1 + |\pi_s(q_s)| - \varepsilon + \|(Id - \Pi_s)(q_s)\| \\
&\qquad \text{for some } s \in U_\varepsilon \\
&= 1 + \|q_s\| - \varepsilon \\
&> 1 + \|T\| - 2\varepsilon \qquad (\text{since } s \in U_\varepsilon).
\end{aligned}$$
This proves (1.1), since $\varepsilon > 0$ was arbitrary.

The proof of the converse implication follows the same lines. □

Corollary 2.2 *If X is nicely embedded into $C^b(S)$ so that* (N1) *and* (N2) *are valid and if $T \in L(X)$, then there is a scalar λ, $|\lambda| = 1$, such that λT satisfies the Daugavet equation* (1.1).

In the case of real Banach spaces (2.2) is equivalent to
$$\sup_{s\in U_\varepsilon} \pi_s(q_s) \geq 0 \qquad \forall \varepsilon > 0,$$
and Corollary 2.2 simply says that T or $-T$ satisfies the Daugavet equation.

It remains to give examples of classes of operators for which (2.2) is valid. This will be done in Lemmas 2.4 and 2.6. But first we single out a simple estimate that will be used in the proof of those lemmas.

Lemma 2.3 *Suppose X is nicely embedded into $C^b(S)$ such that* (N1) *and* (N2) *hold. If $t_1, \dots, t_k$ are pairwise nonequivalent points* (*for the equivalence relation $\sim$ of* (2.1)), *then*
$$\|x^*\| \geq \sum_{j=1}^{k} \|\Pi_{t_j}(x^*)\| \qquad \forall x^* \in X^*.$$

Proof. Let $\Pi = \sum_{j=1}^{k} \Pi_{t_j}$ be the L-projection with range $\operatorname{lin}\{p_{t_1}, \dots, p_{t_k}\}$. Then
$$\|x^*\| \geq \|\Pi(x^*)\| = \sum_{j=1}^{k} \|\Pi_{t_j}(x^*)\|,$$
which implies our claim. □

Lemma 2.4 *Suppose X is nicely embedded into $C^b(S)$ such that* (N1), (N2) *and* (N3) *hold, and let $T \in L(X)$ be an operator. If*

$$s \mapsto \pi_t(q_s) \text{ is continuous for all } t \in S, \tag{2.3}$$

then T satisfies (2.2) *and consequently the Daugavet equation* (1.1).

Proof. We consider the function f on $S \times S$ defined by $f(s,t) = \pi_t(q_s)$. Then our assumption on T means that $s \mapsto f(s,t)$ is continuous for all $t \in S$. Now we check condition (2.2) and argue by contradiction. If (2.2) were false, we would find an open set $U \neq \emptyset$ and some $\beta > 0$ such that

$$|1 + f(s,s)| - (1 + |f(s,s)|) < -2\beta \qquad \forall s \in U.$$

In particular,

$$|f(s,s)| > \beta \qquad \forall s \in U. \tag{2.4}$$

Let $s_1 \in U$ be arbitrary. By continuity of f in the first variable, there is an open neighbourhood $U_1 \subset U$ of s_1 such that

$$|f(u,s_1)| > \beta \qquad \forall u \in U_1. \tag{2.5}$$

Since the equivalence class Q_{s_1} does not contain interior points, we may pick some $s_2 \in U_1$, $s_2 \notin Q_{s_1}$. Consequently, we have by (2.4) and (2.5)

$$|f(s_2,s_2)| > \beta, \quad |f(s_2,s_1)| > \beta.$$

We proceed to find an open neighbourhood $U_2 \subset U_1$ of s_2 such that

$$|f(u,s_2)| > \beta \qquad \forall u \in U_2 \tag{2.6}$$

and some $s_3 \in U_2$, $s_3 \notin Q_{s_1} \cup Q_{s_2}$, with

$$|f(s_3,s_3)| > \beta, \quad |f(s_3,s_2)| > \beta, \quad |f(s_3,s_1)| > \beta$$

by (2.4), (2.6) and (2.5).

Continuing in the obvious manner, we arrive at a sequence $s_1, s_2, \ldots$ in S such that for each $k \in \mathbf{N}$

$$|f(s_k,s_j)| > \beta \qquad \forall j = 1, \ldots, k.$$

But by Lemma 2.3, since the $\Pi_{s_1}, \Pi_{s_2}, \ldots$ are different L-projections, we have for each $k \in \mathbf{N}$

$$\|T\| \geq \|q_{s_k}\| \geq \sum_{j=1}^{k} \|\Pi_{s_j}(q_{s_k})\| = \sum_{j=1}^{k} |f(s_k,s_j)| > k\beta,$$

which clearly contradicts the continuity of T.

Thus, the lemma is proved. $\square$

Remark 2.5 Weakly compact operators fulfill the continuity assumption (2.3) in Lemma 2.4. In fact, if $T \in L(X)$ is weakly compact, then T^* is weak*-weakly-continuous, and $s \mapsto q_s = T^* p_s$ is weakly continuous since $s \mapsto p_s$ is weak* continuous. We will meet non-weakly-compact operators that fulfill (2.3) in Example 3.2.

We will now discuss a different class of operators satisfying (2.2). We say that $T \in L(X)$ factors through a subspace of c_0 if there is a closed subspace $E \subset c_0$ together with continuous operators $T_1\colon X \to E$ and $T_2\colon E \to X$ such that $T = T_2 T_1$. As already noted in the introduction, every compact operator has this property.

Lemma 2.6 *Suppose X is nicely embedded into $C^b(S)$ such that* (N1), (N2) *and* (N3) *hold, and let $T \in L(X)$ be an operator factoring through a subspace of c_0. Assume further that S is a Baire space. Then T satisfies* (2.2) *and consequently the Daugavet equation* (1.1).

Proof. As indicated above, let us write

$$JT\colon X \xrightarrow{T_1} E \xrightarrow{T_2} X \xrightarrow{J} C^b(S).$$

We define functionals $x_n^* \in X^*$ $(n \in \mathbf{N})$ and $a_s^* \in E^*$ $(s \in S)$ by

$$\begin{aligned} x_n^*(x) &= (T_1 x)(n) \qquad (x \in X), \\ a_s^*(a) &= (JT_2 a)(s) \quad (a \in E). \end{aligned}$$

Observe that $\sup_n \|x_n^*\| = \|T_1\|$ and $\sup_s \|a_s^*\| = \|T_2\|$. Let $\nu_s \in \ell^1$ be a Hahn-Banach extension of $a_s^* \in E^*$. Then we have

$$(JTx)(s) = a_s^*(T_1 x) = \sum_{n=1}^{\infty} \nu_s(n) x_n^*(x) = \Bigl(\sum_{n=1}^{\infty} \nu_s(n) x_n^*\Bigr)(x)$$

and consequently

$$q_s = \sum_{n=1}^{\infty} \nu_s(n) x_n^*;$$

note that this series is absolutely norm-convergent.

In order to achieve the proof proper of Lemma 2.6, we define

$$S' = \{t \in S \colon \pi_t(q_s) = 0 \ \forall s \in S\},$$

and we claim that S' is dense in S, which clearly implies (2.2). In fact, from

$$\Pi_t(q_s) = \sum_{n=1}^{\infty} \nu_s(n) \Pi_t(x_n^*)$$

we infer that

$$S'' := \{t \in S \colon \Pi_t(x_n^*) = 0 \ \forall n \in \mathbf{N}\} \subset S',$$

and

$$S \backslash S'' = \bigcup_{n \in \mathbf{N}} \{t \in S \colon \Pi_t(x_n^*) \neq 0\}.$$

But by Lemma 2.3, each set $\{t \colon \Pi_t(x_n^*) \neq 0\}$ consists of at most countably many equivalence classes for $\sim$, for there are at most $\|T_1\|/\delta$ many nonequivalent t with $\|\Pi_t(x_n^*)\| \geq \delta$. Therefore $S \backslash S''$ is a countable union of (by assumption (N3)) nowhere dense sets. The Baire property yields that S'' and hence S' is dense. $\square$

3. APPLICATIONS

Now we will apply the results of the previous section to some natural classes of function spaces. The most obvious example of a nicely embedded space is of course $C_0(S)$, S a locally compact Hausdorff space, with $J =$ the natural inclusion into $C^b(S)$.

Proposition 3.1 *Let S be a locally compact Hausdorff space without isolated points. If $T \in L(C_0(S))$ is weakly compact or factors through a subspace of c_0, then T satisfies the Daugavet equation* (1.1). *More generally, it is enough that the function $s \mapsto (T^*\delta_s)(\{t\})$ on S is continuous for all $t \in S$ in order that T satisfies the Daugavet equation.*

Proof. Since here $s \sim t$ iff $s = t$, we see that (N3$'$) and hence (N3) are fulfilled so that the assertion follows from Lemma 2.4, Remark 2.5 and Lemma 2.6. (Observe that Π_t is the L-projection $\mu \mapsto \mu(\{t\})\delta_t$ on $M(S) \cong (C_0(S))^*$ in the present context so that $\pi_t(q_s) = (T^*\delta_s)(\{t\})$.) $\square$

Most of Proposition 3.1 is already known; we refer to [15] for a straightforward proof and to [2], [3], [8] and [12] for different other approaches.

Example 3.2 There are non-weakly-compact operators on $C(\mathbf{T})$ such that $s \mapsto (T^*\delta_s)(\{t\}) = 0$ for all $t \in \mathbf{T}$; a fortiori these are continuous functions. In fact, every convolution operator $T\colon f \mapsto f * \mu$ for a continuous (= diffuse) singular measure has this property: In this case $(T^*\delta_s)(\{t\}) = \mu(\{st^{-1}\}) = 0$, since μ is continuous; and a result due to Costé [6, p. 90] implies that μ would be absolutely continuous if T were weakly compact. I am grateful to W. Hensgen for pointing out this example to me.

Therefore we see that such convolution operators satisfy the Daugavet equation, a fact that can also be checked directly.

We now turn to algebras of functions. A *function algebra* A on a compact Hausdorff space K is a closed subalgebra of the space of complex-valued functions $C(K)$ separating the points of K and containing the constant functions. To each function algebra A on K one can associate a distinguished subset $\partial A \subset K$, called the *Choquet boundary*, defined by

$$\partial A = \{k \in K\colon \delta_k|_A \text{ is an extreme point of } B_{A^*}\}.$$

This notion will be instrumental in the proof of the following result.

Theorem 3.3 *Let A be a function algebra such that its Choquet boundary ∂A does not contain an isolated point. Then every operator $T \in L(A)$ which is weakly compact or factors through a subspace of c_0 satisfies the Daugavet equation* (1.1).

Proof. We will verify that A is nicely embedded into $C^b(\partial A)$ such that (N1), (N2) and (N3) are valid. We consider the mapping $J\colon A \to C^b(\partial A)$, $Jf = f|_{\partial A}$. It is a well-known consequence of the Krein-Milman theorem that J is an isometry, cf. [4, p. 180f.] for details. Since $J^*(\delta_k) = \delta_k|_A$, condition (N1) is fulfilled by construction, and (N2) is a result due to Hirsberg [11], see also [10, p. 15 and Th. V.4.2]. As in Proposition 3.1, (N3) reduces to (N3′) which is part of the assumption of Theorem 3.3. We finally observe that ∂A is homeomorphic to the extreme boundary of $\{\ell \in A^*\colon \|\ell\| = |\ell(\mathbf{1})| = 1\}$ (the state space of A). By a theorem of Choquet's [4, p. 146] ∂A is a Baire space.

Hence we can apply Lemma 2.4, Remark 2.5 and Lemma 2.6 to finish the proof of Theorem 3.3. $\square$

The Daugavet equation for weakly compact operators on function algebras was first established by Wojtaszczyk [16] using a different argument.

He also remarks that ∂A fails to contain an isolated point if A fails to contain nontrivial idempotents – a property shared by the disk algebra and the algebra of bounded analytic functions on the unit disk.

Again, it would have been enough to require (2.3) instead of the weak compactness of T.

The next class of Banach spaces we wish to investigate are the *L^1-predual spaces* X defined by the requirement that X^* is isometric to a space of integrable functions $L^1(\mu)$. We consider the equivalence relation $p \sim q$ iff p and q are linearly dependent on $\operatorname{ex} B_{X^*}$, and we equip the quotient space $\operatorname{ex} B_{X^*}/\!\sim$ with the quotient topology of the weak* topology. Of course, the following result contains Proposition 3.1 as a special case.

Theorem 3.4 *Let X be an L^1-predual space such that $\operatorname{ex} B_{X^*}/\!\sim$ does not contain an isolated point. Then every operator $T \in L(X)$ which is weakly compact or factors through a subspace of c_0 satisfies the Daugavet equation* (1.1).

Proof. Let $J\colon X \to C^b(\operatorname{ex} B_{X^*})$ be the canonical isometry. For $s \in \operatorname{ex} B_{X^*}$ we have $J^*(\delta_s) = s$; hence (N1) and (N2) are satisfied. (Observe that the linear span of an extreme point in an L^1-space is an L-summand.) Also, (N3) holds by assumption on $\operatorname{ex} B_{X^*}$. Finally, we again invoke Choquet's theorem [4, p. 146] to ensure that $\operatorname{ex} B_{X^*}$ is a Baire space. Thus, Lemma 2.4, Remark 2.5 and Lemma 2.6 yield Theorem 3.4. □

Example 3.5 Let $\Omega \subset \mathbf{R}^d$ be open and bounded and consider the sup-normed space

$$H(\Omega) = \{f \in C(\overline{\Omega})\colon f \text{ is harmonic on } \Omega\}.$$

This is an order unit space whose state space K is a Choquet simplex so that $H(\Omega)$ is an L^1-predual space (cf. [7]). The extreme points of K, that is up to scalar multiples the extreme points of $B_{H(\Omega)^*}$, can be identified with the set $\partial_r\Omega$ of regular boundary points (in the sense of potential theory) of Ω. Thus we conclude that every operator on $H(\Omega)$ which is weakly compact or factors through a subspace of c_0 satisfies the Daugavet equation, provided $\partial_r\Omega$ does not contain isolated points. It is classical that $\partial_r\Omega = \partial\Omega$ if the boundary of Ω is sufficiently smooth or if Ω is simply connected and $d = 2$.

Our final application deals with translation invariant spaces. Let G be an infinite compact abelian group with dual group Γ and Haar measure m;

we denote the group operation in Γ by $+$. For $\Lambda \subset \Gamma$ the space of Λ-spectral continuous functions is defined by

$$C_\Lambda = \{f \in C(G) \colon \widehat{f}(\gamma) = 0 \ \forall \gamma \notin \Lambda\},$$

where $\widehat{\gamma}$ denotes the Fourier transform of f. These spaces are known to be precisely the closed translation invariant subspaces of $C(G)$. Likewise, one defines spaces of Λ-spectral measures M_Λ and Λ-spectral integrable functions L^1_Λ.

A subset $\Lambda \subset \Gamma$ is called a *Riesz set* if $M_\Lambda \subset L^1(m)$; the chief example of a Riesz subset of $\widehat{\mathbf{T}} = \mathbf{Z}$ is $\mathbf{N}$. For an in-depth analysis of this class of sets and its relation to Banach space geometry we refer to [9], see also Chapter IV.4 in [10]. Here we consider a broader class of sets which we propose to call semi-Riesz sets: If M_{diff} denotes the space of diffuse measures on G, i.e., those which map singletons to 0, then $\Lambda \subset \Gamma$ is a *semi-Riesz set* if $M_\Lambda \subset M_{\mathrm{diff}}$. Obviously, Riesz sets are semi-Riesz.

Theorem 3.6 *Let G be a compact abelian group and suppose Λ is a subset of the dual group Γ such set $\Gamma \backslash (-\Lambda)$ is a semi-Riesz set. Then every operator $T \in L(C_\Lambda)$ which is weakly compact (or merely satisfies (2.3)) or factors through a subspace of c_0 satisfies the Daugavet equation (1.1).*

Proof. Let $J \colon C_\Lambda \to C(G)$ be the identical mapping. Then J^* is the quotient map onto

$$\begin{aligned} C_\Lambda^* \ \cong \ M(G)/(C_\Lambda)^\perp \ &\cong \ M(G)/M_{\Gamma\backslash(-\Lambda)} \\ &\cong \ (\ell^1(G) \oplus_1 M_{\mathrm{diff}})/M_{\Gamma\backslash(-\Lambda)} \\ &\cong \ \ell^1(G) \oplus_1 M_{\mathrm{diff}}/M_{\Gamma\backslash(-\Lambda)} \end{aligned}$$

with $\oplus_1$ denoting ℓ^1-direct sums, by assumption on $\Gamma\backslash(-\Lambda)$. Hence $J^*(\delta_g)$ can be identified with $e_g \in \ell^1(G)$, and we conclude that (N1) and (N2) are satisfied. Moreover, since the e_g are linearly independent and G, being a compact infinite group, does not contain any isolated point, (N3′) is fulfilled as well. It is left to apply Lemma 2.4, Remark 2.5 and Lemma 2.6. $\square$

We remark that some restriction on Λ is necessary in order to ensure the Daugavet equation for – say – compact operators on C_Λ, because for the class of Sidon sets Λ the spaces C_Λ are isomorphic to $\ell^1(\Lambda)$, and thus there are one-dimensional operators on C_Λ which fail the Daugavet equation [16, Cor. 1].

We finish this section with another negative result which is a counterpart of the one just quoted.

Proposition 3.7 *If X^* has the Radon-Nikodým property (in particular, if X^* is separable), then there is a one-dimensional operator on X failing the Daugavet equation.*

Proof. Since X^* has the Radon-Nikodým property, B_{X^*} contains a weak* strongly exposed point x_0^* [13, Th. 5.12], that is, there is $x_0 \in X$ such that $\operatorname{Re} x_0^*(x_0) = \|x_0^*\| = \|x_0\| = 1$ and

$$\|x_n^*\| \leq 1, \ \operatorname{Re} x_n^*(x_0) \to 1 \quad \Rightarrow \quad \|x_n^* - x_0^*\| \to 0. \tag{3.1}$$

Define $T \in L(X)$ by $T(x) = x_0^*(x)x_0$ and assume that $\|Id - T\| = 1 + \|T\| = 2$. Then $\|x_n^* - T^*x_n^*\| \to 2$ for some sequence $(x_n^*) \subset B_{X^*}$ and thus $\|T^*x_n^*\| \to 1$. Hence $|x_n^*(x_0)| \to 1$ and with no loss in generality $x_n^*(x_0) \to \alpha$ for some $|\alpha| = 1$. This implies $(\alpha^{-1}x_n^*)(x_0) \to 1$ and by (3.1) $\|\alpha^{-1}x_n^* - x_0^*\| \to 0$. Finally we obtain the contradiction

$$2 = \lim_{n\to\infty} \|x_n^* - T^*x_n^*\| = \|\alpha x_0^* - T^*(\alpha x_0^*)\| = 0. \qquad \square$$

In the setting of harmonic analysis this result tells us that for many sets $\Lambda \subset \Gamma$, in particular for Shapiro sets [9], there are one-dimensional operator on $C(G)/C_\Lambda$ failing the Daugavet equation, since for those sets $C(G)/C_\Lambda$ has a separable dual.

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