ON SOME CATEGORIES OF SEQUENTIAL SPACES

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Abstract. The sequential spaces dealt with in the paper have arbitrary ordinals as domains of sequences and are subjected, besides the axiom of convergence of constant sequences, to only one other, quite special axiom. Categories of these sequential spaces are studied and the results are then applied to categories of associated closure spaces.

It is a usual way to define a topological structure on a set X by introducing a convergence on X, i.e. a map which assigns to any point $x \in X$ a set of sequences (of points of X) converging to x and which is subjected to certain axioms. The topological structure on X is then given by a closure operation where the closure of a subset $A \subseteq X$ is the set of all limits of sequences contained in A. The most known topological structures obtained by this way are the Fréchet $\mathcal{S}^*$-spaces - see e.g. [4]. Another kind of such topological structures is studied in this paper. But here, in contrast to the classical access, domains of sequences are considered to be arbitrary ordinals (and not only the least infinite ordinal). Also the axiom prescribed besides the usual axiom of convergence of constant sequences is quite special. We show that closure spaces induced by these sequential spaces have a useful categorical property: they form an exponential supercategory of the category of finitely generated topological spaces.

Let X be a set, $\alpha > 0$ an ordinal, and $\pi : X \to \exp(X^\alpha)$ a map such that

(i) $(x_i | i < \alpha) \in \pi(x)$ whenever $x_i = x$ for all ordinals $i < \alpha$,

(ii) $(x_i | i < \alpha) \in \pi(x) \Leftrightarrow$ for each ordinal $j, 0 < j < \alpha$, there is a sequence $(y_i^j | i < \alpha) \in \pi(x_j)$ with $\{y_i^j; i < \alpha\} \subseteq \{x_i; i < j\}$, and there is a sequence $(y_i | i < \alpha) \in \pi(x)$ with $\{y_i; i < \alpha\} \subseteq \{x_i; i < \alpha\}$.

The pair (X, π) is called a *sequential space of type* α. Given a pair of sequential spaces of type α, (X, π) and (Y, ρ), by a *homomorphism* of (X, π) into (Y, ρ) we understand a map $f : X \to Y$ with $(x_i | i < \alpha) \in \pi(x) \Rightarrow (f(x_i) | i < \alpha) \in \rho(f(x))$. We denote by Seq_α the (finitely productive) category of sequential spaces of type α with homomorphisms as morphisms. (All categories are considered to be concrete categories of structured sets and structure-compatible maps.)

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By a *closure space* we mean a pair (X, u) where X is a set and $u : \exp X \to \exp X$ is a map fulfilling $u\emptyset = \emptyset$, $A \subseteq X \Rightarrow A \subseteq uA$, and $A \subseteq B \subseteq X \Rightarrow uA \subseteq uB$ (such spaces are studied e.g. in [2]). The symbol *Clo* will denote the category of closure spaces and continuous maps (i.e. maps $f : (X, u) \to (Y, v)$ with $f(uA) \subseteq vf(A)$ whenever $A \subseteq X$).

Let us assign to any sequential space $(X, \pi) \in Seq_\alpha$ the pair (X, u_π) where $u_\pi : \exp X \to \exp X$ is given as follows:
$A \subseteq X \Rightarrow u_\pi A = \{x \in X;$ there is a sequence $(x_i | i < \alpha) \in \pi(x)$ such that $x_i \in A$ for all $i < \alpha\}$.

Proposition 1. *The assignation $(X, \pi) \mapsto (X, u_\pi)$ determines a full concrete embedding of Seq_α into Clo for any ordinal $\alpha > 0$.*

Proof. The proof is trivial when using the following, quite obvious lemma:

Lemma 1. *Let $(X, \pi) \in Seq_\alpha$ ($\alpha > 0$ an ordinal), $x \in X$ and $(x_i | i < \alpha) \in X^\alpha$. Then $(x_i | i < \alpha) \in \pi(x)$ if and only if $x \in u_\pi\{x_i; i < \alpha\}$ and $x_j \in u_\pi\{x_i; i < j\}$ for each ordinal $j, 0 < j < \alpha$.*

The full concrete embedding mentioned will be denoted by F_α. The objects of $F_\alpha(Seq_\alpha)$ will be called the *sequential closure spaces of type* α. Clearly, objects of $F_1(Seq_1)$ are precisely the quasi-discrete closure spaces [3], i.e. the closure spaces (X, u) fulfilling $uA = \bigcup_{x \in A} u\{x\}$ for each subset $A \subseteq X$.

Proposition 2. *There exists a full concrete embedding of Seq_α into Seq_β whenever $0 < \alpha < \beta$.*

Proof. Let $(X, \pi) \in Seq_\alpha$, and let $\rho : X \to \exp(X^\beta)$ be given by $(x_i | i < \beta) \in \rho(x) \Leftrightarrow$ for each ordinal $j, 0 < j < \beta$, there is a sequence $(y_i^j | j < \alpha) \in \pi(x_j)$ with $\{y_i^j; i < \alpha\} \subseteq \{x_i; i < j\}$, and there is a sequence $(y_i | i < \alpha) \in \pi(x)$ with $\{y_i; i < \alpha\} \subseteq \{x_i; i < \beta\}$. For each $j, 0 < j < \beta$, let $(z_i^j | i < \beta) \in X^\beta$ be the sequence defined by $z_i^j = y_i^j$ for all $i < \alpha$ and $z_i^j = y_0^j$ for all i with $\alpha \leq i < \beta$. Similarly, let $(z_i | i < \beta) \in X^\beta$ be the sequence defined by $z_i = y_i$ for all $i < \alpha$ and $z_i = y_0$ for all i with $\alpha \leq i < \beta$. Then we have $(z_i^j | i < \beta) \in \rho(x_j)$, $\{z_i^j; i < \beta\} \subseteq \{x_i; i < j\}$ for each $j, 0 < j < \beta$, $(z_i | i < \beta) \in \rho(x)$ and $\{z_i; i < \beta\} \subseteq \{x_i; i < \beta\}$. Conversely, let $x \in X$ and $(x_i | i < \beta) \in X^\beta$ be such that for each $j, 0 < j < \beta$, there is a sequence $(z_i^j | i < \beta) \in \rho(x_j)$ with $\{z_i^j; i < \beta\} \subseteq \{x_i; i < j\}$, and there is a sequence $(z_i | i < \beta) \in \rho(x)$ with $\{z_i; i < \beta\} \subseteq \{x_i; i < \beta\}$. Then for each $j, 0 < j < \beta$, there is a sequence $(t_i^j | i < \alpha) \in \pi(x_j)$ such that $\{t_i^j; i < \alpha\} \subseteq \{z_i^j; i < \beta\}$ and there is a sequence $(t_i | i < \alpha) \in \pi(x)$ such that $\{t_i; i < \alpha\} \subseteq \{z_i; i < \beta\}$. Hence $\{t_i^j; i < \alpha\} \subseteq \{x_i; i < j\}$ for each $j, 0 < j < \beta$, and $\{t_i; i < \alpha\} \subseteq \{x_i; i < \beta\}$. This results in $(x_i | i < \beta) \in \rho(x)$. We have shown that ρ fulfils the axiom (ii). As the axiom (i) is fulfilled trivially, $(X, \rho) \in Seq_\beta$. The proof is complete.

Obviously, in the proof of Proposition 2 there holds the equality $F_\alpha(X, \pi) = F_\beta(X, \rho)$, which results in the following

Corollary 1. *$F_\alpha(Seq_\alpha)$ is a full subcategory of $F_\beta(Seq_\beta)$ whenever $0 < \alpha < \beta$.*

A sequential space (X, π) of type α will be called *diagonal* provided that the following condition is satisfied (cf. [8]):

if $((x_{ij}|j<\alpha)|i<\alpha)\in (X^\alpha)^\alpha$ is a sequence of sequences with $(x_{ij}|j<\alpha)\in\pi(y_i)$ for all $i<\alpha$, $(x_{ij}|i<\alpha)\in\pi(z_j)$ for all $j<\alpha$, $(y_i|i<\alpha)\in\pi(x)$ and $(z_j|j<\alpha)\in\pi(x)$, then $(x_{ii}|i<\alpha)\in\pi(x)$.
The full subcategory of Seq_α given by the diagonal sequential spaces (of type α) will be denoted by $\widetilde{Seq}_\alpha$.

We denote by *Fin* the full subcategory of *Clo* given by the finitely generated topological spaces [5], i.e. closure spaces (X,u) satisfying $uuA = uA = \bigcup\limits_{x\in A} u\{x\}$ whenever $A\subseteq X$.

Definition (cf. [9]). If $\mathcal{K}$ is a category with finite products and $\mathcal{L}$ its full isomorphism closed subcategory, then $\mathcal{K}$ is said to be an *exponential supercategory* of $\mathcal{L}$ provided that for any pair of objects $G\in\mathcal{L}$ and $H\in\mathcal{K}$ there exists an object $G^H\in\mathcal{L}$ with $|G^H| = Mor_{\mathcal{K}}(H,G)$ such that the pair (G^H,e), where $e: H\times G^H\to G$ is the evaluation map, is a couniversal map for G with respect to the functor $H\times - :\mathcal{K}\to\mathcal{K}$.

The object G^H from the definition is called a *power* of G and H and it is uniquely determined by G and H whenever $\mathcal{K}$ is transportable. Obviously, if $\mathcal{K}$ is an exponential supercategory of $\mathcal{L}$, then the powers fulfil the first exponential law, i.e. $(G^H)^M\simeq G^{H\times M}$ whenever $G\in\mathcal{L}$ and $H,M\in\mathcal{K}$. If $\mathcal{K}$ is an exponential supercategory of itself, then $\mathcal{K}$ is cartesian closed (see [6]).

Lemma 2. *Let $\alpha>1$ be an ordinal and $(X,\pi)\in\widetilde{Seq}_\alpha$ an object. Let $x\in X$, $(x_i|i<\alpha)\in\pi(x)$, and let $(y_j|j<\alpha)\in X^\alpha$ be the constant sequence with $y_j=x_0$ for each $j<\alpha$. Then $(y_j|j<\alpha)\in\pi(x)$ and $(y_j|j<\alpha)\in\pi(x_i)$ for each $i<\alpha$.*

Proof. Obviously, $(y_j|j<\alpha)\in\pi(x_0)$. Let i_0 be an ordinal, $0<i_0<\alpha$, and suppose that $(y_j|j<\alpha)\in\pi(x_i)$ for each $j<i_0$. Put $x_{ij}=x_0$ for each $i,j<\alpha$ and let $(z_i|i<\alpha)$ be the sequence given by $z_i=x_i$ for all $i<i_0$ and $z_i=x_0$ for all i with $i_0\leqslant i<\alpha$. Then clearly $(x_{ij}|j<\alpha)\in\pi(z_i)$ for all $i<\alpha$ and $(x_{ij}|i<\alpha)\in\pi(z_j)$ for all $j<\alpha$. But from $(x_i|i<\alpha)\in\pi(x)$ it also follows that $(z_i|i<\alpha)\in\pi(x_{i_0})$. Hence $(x_{ii}|i<\alpha)\in\pi(x_{i_0})$. Because $(x_{ii}|i<\alpha)=(y_j|j<\alpha)$, according to the principle of transfinite induction we have $(y_j|j<\alpha)\in\pi(x_i)$ for all $i<\alpha$. Consequently, $(x_{ij}|j<\alpha)\in\pi(x_i)$ for each $i<\alpha$ and $(x_{ij}|i<\alpha)\in\pi(x_j)$ for each $j<\alpha$. As $(x_i|i<\alpha)\in\pi(x)$, we have $(x_{ii}|i<\alpha)=(y_j|j<\alpha)\in\pi(x)$. The lemma is proved.

Theorem 1. *Seq_α is an exponential supercategory of $\widetilde{Seq}_\alpha$ for each ordinal $\alpha>0$.*

Proof. Let $G=(X,\pi)\in\widetilde{Seq}_\alpha$ and $H=(Y,\rho)\in Seq_\alpha$ be objects and put $G^H=(Z,\sigma)$ where $Z=\mathrm{Mor}(H,G)$ and $\sigma: Z\to\exp(Z^\alpha)$ is given by $(z_i|i<\alpha)\in\sigma(z)$ iff $(z_i(y)|i<\alpha)\in\pi(z(y))$ for each $y\in Y$. In virtue of the Lemma 2 it can easily be seen that $G^H\in Seq_\alpha$. But G^H is evidently diagonal, so that $G^H\in\widetilde{Seq}_\alpha$. Let $e: H\times G^H\to G$ be the evaluation map. Denote $(W,\tau)=H\times G^H$ and let $(w_i|i<\alpha)\in\tau(w)$. Then there are $(y_i|i<\alpha)\in\rho(y)$ and $(z_i|i<\alpha)\in\sigma(z)$ such that $(y_i,z_i)=w_i$ for each $i<\alpha$ and $(y,z)=w$. Since $z_i\in\mathrm{Mor}(H,G)$ for each $i<\alpha$, we have $(z_i(y_j)|j<\alpha)\in\pi(z_i(y))$ for each $i<\alpha$. But we have also $(z_i(y_j)|i<\alpha)\in\pi(z(y_j))$ for each $j<\alpha$. Further, $(z_i(y)|i<\alpha)\in\pi(z(y))$

and $(z(y_j)|j < \alpha) \in \pi(z(y))$ is evidently true. Thus, because of the diagonality of G, $(z_i(y_i)|i < \alpha) \in \pi(z(y))$. But $z_i(y_i) = e(y_i, z_i) = e(w_i)$ for each $i < \alpha$ and $z(y) = e(w)$. Therefore $e : H \times G^H \to G$ is a homomorphism. Now, if $K = (V, \nu) \in Seq_\alpha$ and $h : H \times K \to G$ is a homomorphism, then the map $\hat{h} : K \to Z$ given by $\hat{h}(v)(y) = h(y, v)$ is clearly a unique homomorphism of K into G^H with the property $h = e \circ (id_Y \times \hat{h})$. Hence the pair (G^H, e) is a co-universal map for G with regard to the functor $H \times - : Seq_\alpha \to Seq_\alpha$. This completes the proof.

Lemma 3. *Let $\alpha > 0$ be an ordinal, $(X, \pi) \in Seq_\alpha$ an object, and let $u_\pi u_\pi A = u_\pi A$ whenever $A \subseteq X$. Then $(x_i|i < \alpha) \in \pi(x)$ implies $x \in u_\pi\{x_0\}$ and $x_i \in u_\pi\{x_0\}$ for each ordinal $i < \alpha$.*

Proof. Clearly, $x_0 \in u_\pi\{x_0\}$ is valid. Let $i_0 < \alpha$ be an ordinal and suppose that $x_i \in u_\pi\{x_0\}$ is valid for each ordinal $i < i_0$. Then, by Lemma 1, $x_{i_0} \in u_\pi\{x_i; i < i_0\} \subseteq u_\pi u_\pi\{x_0\} = u_\pi\{x_0\}$. Thus, according to the principle of transfinite induction, $x_i \in u_\pi\{x_0\}$ for each $i < \alpha$. But, as $x \in u_\pi\{x_i : i < \alpha\}$, we have also $x \in u_\pi u_\pi\{x_0\} = u_\pi\{x_0\}$.

Remark 1. From Lemma 3 it immediately follows that for any object $(X, \pi) \in Seq_\alpha$ the condition $(X, u_\pi) \in Fin$ is equivalent to $u_\pi u_\pi A = u_\pi A$ for each $A \subseteq X$.

Theorem 2. *$F_\alpha(\widetilde{Seq_\alpha}) = Fin$ for each ordinal $\alpha > 0$.*

Proof. Let $(X, u_\pi) \in Fin$. Let $((x_{ij}|j < \alpha)|i < \alpha) \in (X^\alpha)^\alpha$ be a sequence of sequences such that $(x_{ij}|j < \alpha) \in \pi(y_i)$ for all $i < \alpha$, $(x_{ij}|i < \alpha) \in \pi(z_j)$ for all $j < \alpha$, $(y_i|i < \alpha) \in \pi(x)$ and $(z_j|j < \alpha) \in \pi(x)$. Then, by Lemma 3, for each $i < \alpha$ we have $y_i \in u_\pi\{x_{i0}\}$ and $x_{ij} \in u_\pi\{x_{i0}\}$ whenever $j < \alpha$, and for each $j < \alpha$ we have $x_{ij} \in u_\pi\{x_{0j}\}$ whenever $i < \alpha$. We also have $x \in u_\pi\{y_0\}$. Thus, $x \in u_\pi\{y_0\} \subseteq u_\pi u_\pi\{x_{00}\}$ and $x_{i_0 i_0} \in u_\pi\{x_{i_0 0}\} \subseteq u_\pi u_\pi\{x_{00}\} = u_\pi\{x_{00}\}$. Consequently, $x \in u_\pi\{x_{ii}; i < \alpha\}$ and $x_{i_0 i_0} \in u_\pi\{x_{ii}; i < i_0\}$ for each i_0, $0 < i_0 < \alpha$. Therefore, according to Lemma 1, $(x_{ii}|i < \alpha) \in \pi(x)$. Hence $(X, \pi) \in \widetilde{Seq_\alpha}$.
Conversely, let $(X, \pi) \in \widetilde{Seq_\alpha}$. Let $A \subseteq X$ and $x \in u_\pi u_\pi A$. Then there is a sequence $(x_i|i < \alpha) \in (u_\pi A)^\alpha$ such that $(x_i|i < \alpha) \in \pi(x)$. By Lemma 2, $x \in u_\pi\{x_0\}$. As $x_0 \in u_\pi A$, there exists a sequence $(y_i|i < \alpha) \in A^\alpha$ such that $(y_i|i < \alpha) \in \pi(x_0)$. Therefore, applying again Lemma 2, ve have $x_0 \in u_\pi\{y_0\}$. Consequently, $x \in u_\pi u_\pi\{y_0\} = u_\pi\{y_0\} \subseteq u_\pi A$. We have proved the inclusion $u_\pi u_\pi A \subseteq u_\pi A$. As the converse inclusion is trivially fulfilled, the equality $u_\pi u_\pi A = u_\pi A$ is valid. Thus, by Remark 1, $(X, u_\pi) \in Fin$ and the statement is proved.

Corollary 2. *$F_\alpha(Seq_\alpha)$ is an exponential supercategory of Fin for each ordinal $\alpha > 0$.*

Denote by *Clo** the full subcategory of *Clo* whose objects are precisely the closure spaces $(X, u) \in Clo$ with the property that for each $A \subseteq X$ and each $x \in uA$ there exists an ordinal $\alpha > 0$ and a sequence $(x_i|i < \alpha) \in A^\alpha$ such that $x_j \in u\{x_i; i < j\}$ for each $j, 0 < j < \alpha$, and $x \in u\{x_i; i < \alpha\}$.

Corollary 3 (cf. [10]). *Clo** *is an exponential supercategory of Fin.*

Proof. In the quasicategory of all categories and functors let us consider the diagram D whose objects are the categories $Seq_\alpha, \alpha > 0$, and whose arrows are the full concrete embeddings $F_\beta^{-1} \circ F_\alpha : Seq_\alpha \to Seq_\beta, 1 < \alpha \leqslant \beta$. Then D is a directed diagram and Clo^* is a direct limit of D. The corresponding natural sink is formed by the embeddings $F_\alpha, \alpha > 0$. Now Corollary 3 follows from Corollary 2.

Remark 2. a) The product of a pair of objects (X, u), (Y, v) in Clo^* is the space $(X \times Y, w)$, where $D \subseteq X \times Y \Rightarrow wD = \bigcup\{(x, y) \in X \times Y;$ there is an ordinal $\alpha > 0$ and a sequence $((x_i, y_i)|i < \alpha) \in D^\alpha$ such that $x_j \in u\{x_i; i < j\}$ and $y_j \in v\{y_i; i < j\}$ whenever $0 < j < \alpha$, $x \in u\{x_i; i < \alpha\}$ and $y \in v\{y_i; i < \alpha\}$. The power G^H of objects $G = (X, u) \in Fin$ and $H = (Y, v) \in Clo^*$ is given pointwise, i.e. by $G^H = (\mathrm{Mor}(H, G), p)$ where $p(C) = \{g \in \mathrm{Mor}(H, G);$ there exists $f \in C$ such that $g(y) \in u\{f(y)\}$ for each $y \in Y\}$ whenever $C \subseteq \mathrm{Mor}(H, G)$.

b) It is well known that *Fin* is an exponential supercategory of itself, i.e. a cartesian closed category. The Corollary can be considered as a generalization of this fact.

c) If a space $(X, u) \in Clo^*$ is idempotent, i.e. fulfils $A \subseteq X \Rightarrow uuA = uA$, then $(X, u) \in Fin$ (see Remark 1). On the other hand, the full subcategory of Clo^* given by the additive closure spaces, i.e. closure spaces $(X, u) \in Clo^*$ with $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$, is not finitely productive in Clo^*. Each of these two facts means no new results could be obtained by restricting our considerations to topological spaces.

d) It is obvious that Clo^* is a bicoreflective subcategory of Clo. For any object $(X, u) \in Clo$ its bicoreflection in Clo^* is the identity map $id_X : (X, u^*) \to (X, u)$ where u^* is given by $A \subseteq X \Rightarrow u^*A = \bigcup\{x \in X;$ there is an ordinal $\alpha > 0$ and a sequence $(x_i|i < \alpha) \in A^\alpha$ such that $x_j \in u\{x_i; i < j\}$ for each $j, 0 < j < \alpha$, and $x \in u\{x_i; i < \alpha\}\}$.

Examples. 1. Let $(X, \pi) \in Seq_2$ be the sequential space with $X = \{a, b, c\}$ and $\pi(a) = \{(a, a), (a, b)\}$, $\pi(b) = \{(b, b), (a, a), (a, b)\}$, $\pi(c) = \{(c, c), (a, b)\}$. Clearly, the closure space (X, u) with $u = u_\pi$, i.e. with $(X, u) = F_2(X, \pi)$, is given by $u\emptyset = \emptyset$, $u\{a\} = \{a, b\}$, $u\{b\} = \{b\}$, $u\{c\} = \{c\}$, $u\{a, b\} = u\{a, c\} = uX = X$, $u\{b, c\} = \{b, c\}$. Obviously, $(X, u) \notin F_1(Seq_1)$.

2. Let $(\omega + 1, u)$ be the closure space defined by $u\emptyset = \emptyset$, $uA = \omega$ whenever $A \subseteq \omega$ and $0 < \mathrm{card} A < \omega$, and $uA = \omega + 1$ otherwise. Then $(\omega + 1, u) \in F_\omega(Seq_\omega)$ but $(\omega + 1, u) \notin F_\alpha(Seq_\alpha)$ for any $\alpha < \omega$. Clearly, the closure space (X, u) is additive.

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