Another stable group

Andreas Baudisch[1]
2. Mathematisches Institut
Freie Universität Berlin
Arnimallee 3
14195 Berlin
baudisch@math.fu-berlin.de

May 17, 1995

[1]Heisenberg-Stipendiat der Deutschen Forschungsgemeinschaft

Abstract

In [3] an uncountably categorical group is constructed that has a non-locally-modular geometry and does not allow the interpretation of a field. We consider a system Δ of elementary axioms fulfilled by some special subgroups of the group above. We show that Δ is complete, stable, but not superstable. It is even not a $\mathbf{R}$-group in the sense of [10].

1 Introduction

A structure with a stable elementary theory is called a stable group if there is a $\emptyset$-definable binary function that gives a group operation on the underlying set. If we restrict us to the pure group structure, then we have essentially the following examples. Note that direct sums of stable groups are stable again.

1. Abelian groups
   Also groups Abelian by finite are stable. Note that every stable one-based group is Abelian by finite, as proved by E.Hrushovski and A.Pillay [6].

2. Algebraic groups over algebraically closed fields
   Every superstable field is algebraically closed as proved by A.Macintyre for the ω-stable case ([7]) and by G.Cherlin and S.Shelah for the general case ([4]). Algebraic groups over algebraically closed fields are ω-stable of finite Morley rank. There is a conjecture of G.Cherlin and B.Zil'ber that the converse is true for simple groups.

3. Algebraic groups over separably closed fields
   Separably closed fields of characteristic $p \neq 0$ are the only known examples of stable fields that are not superstable (see C.Wood [11]).

4. Mekler-groups
   For every stable structure of finite signature A.Mekler [8] has constructed a group such that for the elementary theories T of the both structures the function $\kappa \to max\{|S_n(M)| \ : \ M \models T, |M| = \kappa\}$ is the same. These groups are nilpotent of class 2 and of exponent p where p is a prime greater than 2. Mekler groups cannot have finite Morley rank (U-rank).

5. Free groups $F_c(p^n, \kappa)$ in the variety of nilpotent groups of class c of exponent p^n with κ many free generators where p is a prime greater than c and κ is infinite.
   These groups are ω-stable with c dimensions. They have infinite Morley rank. See [1], [2].

6. A new uncountably categorical group
   In [3] an axiom system Σ for a new uncountably categorical group is

given. Σ has a non-locally-modular geometry and does not allow the interpretation of a field.

B.Zil'ber conjectured that every uncountably categorical theory with a non-locally-modular geometry allows the interpretation of a field. To refute this conjecture E.Hrushovski [5] constructed an uncountably categorical (even strongly minimal) relational structure with a non-locally-modular geometry that does not allow the interpretation of a group. The group in 6) above is a further counterexample to Zil'bers Conjecture in the language of group theory. It is not known whether Σ can be an algebraic group over an algebraically closed field. We conjecture that for algebraic groups over algebraically closed fields Zil'bers Conjecture is true. If this was true, it would imply that Σ is not an algebraic group over an algebraically closed field.

Let p be a prime greater than 2. Let G be a model of Σ. We use $Z(G)$ or short Z to denote the center. Let $[x, y] = x^{-1}y^{-1}xy$ and $[X, Y] = \{[x, y] : x \in X, y \in Y\}$. $\langle X\rangle$ is used to denote the subgroup generated by X in G. Then $G' = \langle [G, G]\rangle$ is the commutator subgroup of G. If $X = \{a_1, \ldots, a_n\}$ or X is a tuple $(a_1, \ldots, a_n)$, then we write $\langle a_1, \ldots, a_n\rangle$ only. The first two axioms in Σ say:

(Σ1) G is a nilpotent group of class 2. G has exponent p.

(Σ2) $\forall x \notin Z(G) \forall z \in Z(G) \exists y([x, y] = z)$.

A weaker but not elementary version of (Σ2) is the following:

$(\Sigma 2)^w$ $Z(G) = G'$.

Let $\mathcal{G}$ be the category of groups that satisfy (Σ1) and $(\Sigma 2)^w$. In [3] an auxiliary category $\mathcal{S}$ is introduced. The objects are vector spaces V over the finite field F_p with p elements together with a subspace of relations $N(V)$ of the exterior square $\wedge^2 V$. There is a functorial correspondence between $\mathcal{G}$ and $\mathcal{S}$ that is 1 - 1 at the level of isomorphism types of objects. In $\mathcal{S}$ morphisms are lost. An $\mathcal{S}$-structure corresponding to a countable ω-saturated model of Σ can be constructed by successive amalgamation of finite objects in $\mathcal{S}$ that satisfy some further conditions. This process starts with some finite $(V, N(V))$ where $N(V) = \langle 0\rangle$. We are interested in the subgroup that is constructed using the simplest steps only. That means: If $(V_n, N(V_n))$ is construced in the nth round, then $V_{n+1} = V_n \oplus \langle a\rangle$ and $N(V_{n+1}) = N(V_n)$ or

$N(V_{n+1})$ is generated by one relation $\Phi + (c \wedge a)$ where $c \in V_n$, $\Phi \in \wedge^2 V_n$, and there is no $\Phi + (c \wedge x)$ in $N(V_n)$. We assume that all possible problems as described in the step from $(V_n, N(V_n))$ to $(V_{n+1}, N(V_{n+1}))$ are solved at the end of the construction. The group obtained in this way satisfies the following set Δ of axioms:

(Δ0) There are three elements linearly independent modulo Z.

(Σ1) The group is nilpotent of class 2 and of exponent p.

(Σ2) $\forall x \notin Z \forall z \in Z \exists y([x, y] = z)$.

(Δ3) Every three elements that are linearly independent modulo the center generate a subgroup isomorphic to $F_2(p, 3)$.

(Δ4) For every finite subgroup H with $(\Sigma 2)^w$ and $dim(H/Z(H)) > 3$ there is a subgroup H_0 with $(\Sigma 2)^w$, $dim(H_0/Z(H_0)) + 1 = dim(H/Z(H))$, and $dim(Z(H)) \geq dim(Z(H_0)) + dim(H_0/Z(H_0)) - 1$.

In (Δ4) $H/Z(H)$, $H_0/Z(H_0)$, $Z(H)$, and $Z(H_0)$ are considered as vectorspaces over F_p. Furthermore note that if $(\Sigma 2)^w$ is true for H and H_0 and $dim(H/Z(H)) = dim(H_0/Z(H_0)) + 1$, then $dim(Z(H)) \leq dim(Z(H_0)) + dim(H_0/Z(H_0))$.

We show that Δ is a complete theory. It is stable but not superstable in a very strong sense. If G is a model of Δ and $\overline{G} = G/Z(G)$, then every three elements of $\overline{G}$ are in the algebraic closure of two elements of $\overline{G}$. Hence Δ is not the theory of a **R**-group in the sense of F.Wagner [10]. Δ implies all axioms of Σ except (Σ5). Hence all models G of Δ fulfil (Σ3). Therefore all $\overline{G}$ carry a closure operation cl that gives us a Hrushovski-geometry. cl is part of the algebraic closure but the converse is not true. Let $A \subseteq B$ be finite subspaces of $\overline{G}$. Then B is *part of the past of* A if there is no subspace B_0 of B such that $A \subseteq B_0$, $dim(B) = dim(B_0) + 1$, and $dim(B') \geq dim(B_0') + dim(B_0) - 1$. B' is used to denote the subspace $\langle [B, B] \rangle$ of $Z(G)$. Then $past(A)$ is the essential part of $acl(A)$. The $past(A)$ determines the type of every preimage of A in G.

The paper is organized in the following way. Section 2 is a short summary of the results concerning the new uncountably categorical group in [3]. Hrushovski geometries for groups in $\mathcal{G}$ are considered. In section 3 Δ is introduced. We construct the standard models. In section 4 we start the investigation of the algebraic closure in $\overline{H}$ for models H of Δ. In section 5 it is shown that the past is contained in the algebraic closure. Non-superstability follows. In section 7 completeness is shown. In section 8 we prove stability.

2 The new uncountably categorical group

E. Hrushovski has introduced a geometry on structures with few relations. In this section we will explain how this can be done for groups that satisfy $(\Sigma 1)$, $(\Sigma 2)^w$, and a property $(\Sigma 3)$, that guarantees that the groups have few relations only. We explain the results of [3].

Let G be a group that satisfies $(\Sigma 1)$ and $(\Sigma 2)^w$. $\mathcal{G}$ is the category of these groups. We denote $G/Z(G)$ by $\overline{G}$. Then G is uniquely determined by the alternating bilinear map $[x, y]$ of $\overline{G} \times \overline{G}$ into $Z(G)$, where the image of this map generates $Z(G)$. Every such map defines a group with $(\Sigma 1)$ and $(\Sigma 2)^w$ (see [3]). We consider the category $\mathcal{B}$ of alternating bilinear maps $\beta : V \times V \to W$, where V and W are vectorspaces over the field with p elements and the image of β generates W. The last condition corresponds to $(\Sigma 2)^w$. The morphisms from $(V_1, W_1; \beta_1)$ to $(V_2, W_2; \beta_2)$ are pairs (f,g) with $f : V_1 \to V_2$, $g : W_1 \to W_2$ such that $\beta_2 f = g\beta_1$. Working in this category we lose some morphisms but we have a 1 - 1 - correspondence between the bilinear maps and the groups. For every vectorspace V there exists a free alternating bilinear map $\wedge$ into a vectorspace $\wedge^2 V$. $\wedge^2 V$ is called the exterior square of V.

Definition 2.1 *$\wedge$ is defined by the following property: For every alternating bilinear map $\beta : V \times V \to W$ in $\mathcal{B}$ there exists some vectorspace homomorphism f_β from $\wedge^2 V$ into W, such that $\beta = f_\beta \wedge$.*

Let $N(\beta) \subseteq \wedge^2 V$ be the kernel of f_β. Often we write $N(V)$ only. Then $(V, W; \beta)$ is isomorphic to $(V, (\wedge^2 V)/N(V)); \pi)$ where π is the composition of the canonical maps $V \times V \to \wedge^2 V \to \wedge^2/N(V)$. Hence we can work in the category $\mathcal{S}$ of pairs $(V, N(V))$ where $N(V)$ is a subspace of $\wedge^2 V$. The morphisms are maps $f : (V_1, N_1) \to (V_2, N_2)$ such that the induced map $\wedge^2 f$ between the exterior squares satisfies: $\wedge^2 f(N_1) = \wedge^2 f(V_1) \cap N_2$. The categories $\mathcal{B}$ and $\mathcal{S}$ are equivalent. For an element $(M, N(M))$ in $\mathcal{S}$ we write often M only. If H is a subspace of M, then we consider $\wedge^2 H$ as canonically identified with a subspace of $\wedge^2 M$. Then $N(H) = \wedge^2 H \cap N(M)$. We need the following lemma from [3].

Lemma 2.2 *Let G and H be groups in $\mathcal{G}$. Assume there is an isomorphism f_0 of a subgroup $G_0 \in \mathcal{G}$ of G onto a subgroup $H_0 \in \mathcal{G}$ of H. Let M and N be $\mathcal{S}$-structures corresponding to G and H respectively. Let M_0 be the*

substructure of M corresponding to G_0 and let N_0 be the substructure of N corresponding to H_0. Assume there is a $\mathcal{S}$-isomorphism g of M onto N such that the restriction of g to M_0 is the $\mathcal{S}$-isomorphism induced by f_0.
Then there is an isomorphism f of G onto H that extends f_0 and induces the given $\mathcal{S}$-isomorphism g of M onto N.

An essential function in [5] is

$$\delta(A) = \text{number of points of A} - \text{number of links in A},$$

where A is a finite subset of the considered structure. Now let A be a finite subspace of M where $(M, N(M))$ is a structure in $\mathcal{S}$. We define analogously

$$\delta(A) = dim(A) - dim(N(A)),$$

where dim is the linear dimension of the corresponding vectorspaces. To obtain the desired geometries, we assume that all considered $\mathcal{S}$ - structures satisfy

($\Sigma 3$) For all finite subspaces $A \subseteq M$ we have $\delta(A) = dim(A)$, if $dim(A) \leq 3$, and $\delta(A) \geq 3$ otherwise.

We say that a group with $\Sigma 1$) and $(\Sigma 2)^w$ satisfies ($\Sigma 3$), if the corresponding $\mathcal{S}$-structure does. ($\Sigma 3$) for groups can be written down as a set of elementary sentences.

Note that $N(A) \cap N(B) = N(A \cap B)$ for A and B in M. It follows easily:

Lemma 2.3 *For all finite subspaces A and B of M*

$$\delta(A + B) \leq \delta(A) + \delta(B) - \delta(A \cap B).$$

We say, that a finite subspace A is selfsufficient in M, if $\delta(A) \leq \delta(B)$ for all finite B with $A \subseteq B \subseteq M$. By ($\Sigma 3$) every finite A is contained in a finite selfsufficient subspace. Using Lemma 2.3 it is possible to show:

Lemma 2.4 *The intersection of two selfsufficient finite subspaces is selfsufficient .*

Hence there exists a smallest selfsufficient subspace $CSS(A)$ containing A. It is the intersection of all selfsufficient finite subspaces of M that contain A. We call $\delta(CSS(A)) = d(A)$ the geometrical dimension of A and define

$$a \in cl(\langle a_1, \ldots, a_n \rangle) \quad \text{iff} \quad d(\langle a_1, \ldots, a_n \rangle) = d(\langle a, a_1, \ldots, a_n \rangle).$$

We say that B and C are independent over A if $d(B + C) - d(C) = d(B) - d(A)$.

Theorem 2.5 *cl defines a pregeometry on M.*

As a result of the construction in the new uncountably categorical group cl is the algebraic closure on $\overline{G}$. An important notion to work in a $\mathcal{S}$ - structure M with the property $(\Sigma 3)$ is minimality. Assume A and B are finite subspaces of M with $A \subseteq B$ and A is selfsufficient. B is in $cl(A)$ if and only if $\delta(A) = \delta(B)$. In this case B is selfsufficient too, and there is a sequence $A = B_0 \subseteq B_1 \subseteq \ldots \subseteq B_n = B$ with $B_i \neq B_{i+1}$ and $\delta(B_i) = \delta(B_{i+1})$, that cannot be confined. Therefore we define B is a minimal extension of A (also for non-selfsufficient A), if $A \subseteq B$, $\delta(A) = \delta(B)$, and $\delta(C) > \delta(A)$ for every C with $A \subseteq C \subseteq B$ and $A \neq C \neq B$.

A lot of efforts are necessary to obtain the desired uncountably categorical group G via a special amalgamation process of finite $\mathcal{S}$ - structures with selfsufficient embeddings (see [3]). G is a countable saturated model of the following set Σ of axioms that determine a complete theory. Σ is the union of the already known sets of Axioms $(\Sigma 1)$, $(\Sigma 2)$, and $(\Sigma 3)$ with further sets of axioms $(\Sigma\ 4)$ and $(\Sigma 5)$. $(\Sigma 4)$ is very complicated (see [3]). It is introduced because of the following important consequence:

Corollary 2.6 *Assume that G is a model of Σ and $A \subseteq B$ are finite subspaces in $\overline{G}$, such that A is selfsufficient in $\overline{G}$ and B is a minimal extension of A. Then there are at most $2(dim(B)) + 1$ many realizations of B over A, that are linearly independent over A.*

Corollary 2.6 is the key for the proof of $acl = cl$ in $\overline{G}$. Also axiom $(\Sigma 5)$ we present by a consequence:

Corollary 2.7 *Assume that G is a saturated model of Σ. Let $A \subseteq B$ be finite $\mathcal{S}$ - structures, that satisfy $(\Sigma 3)$ and $(\Sigma 4)$. A is selfsufficient in B. Then every selfsufficient embedding of A in $\overline{G}$ can be extended to a selfsufficient embedding of B in $\overline{G}$.*

($\Sigma 5$) is a set of formulas that describe the statement of Corollary 2.7 for all possible $A \subseteq B$, where the selfsufficient embeddings of A and B into $\overline{G}$ are replaced by suitable relativations. In the last theorem of this section we summarize the results of [3].

Theorem 2.8 *Σ is a consistent complete uncountably categorical theory in the elementary language of group theory. Σ is not countably categorical. It has Morley - rank 2. The geometry of Σ is not locally modular. $\overline{G}$ and $Z(G)$ are strongly minimal sets. In $\overline{G}$ acl defines exactely the introduced Hrushovski-geometry cl. Σ is not CM-trivial. Hence it is not possible to interprete an infinite field in Σ.*

3 The standard models

In the introduction the elementary theory Δ is formulated in the elementary language of group theory. It is easier to understand the axioms ($\Delta 3$) and ($\Delta 4$) if we formulate their meaning for the corresponding $\mathcal{S}$-structures M:

($\Delta 3$) For all linearly independent elements a_1, a_2, a_3 in M we have $N(\langle a_1, a_2, a_3\rangle) = 0$.

($\Delta 4$) For every finite subspace A of M with $dim(A) > 3$ we get some subspace A_0 of A such that $dim(A_0) + 1 = dim(A)$ and $dim(N(A)) \leq dim(N(A_0)) + 1$.

Note that ($\Delta 3$) is equivalent to

($\Delta 3$) If a_1 and a_2 in M are linearly independent, then $a_1 \wedge a_2 \notin N(M)$.

A $\mathcal{S}$-structure that satisfies ($\Delta 3$) and ($\Delta 4$) is called a Δ-*structure.* A $\mathcal{S}$-structure M is a Δ-structure iff every finite substructure is a Δ-structure.

Lemma 3.1 *A finite $\mathcal{S}$-structure A with $dim(A) = n + 2 > 2$ is a Δ-structure if and only if $A = \cup_{1\leq i\leq n} A_i$ where the A_i are subspaces, $A_i \subseteq A_{i+1}$, $dim(A_1) = 3$, $dim(A_{i+1}) = dim(A_i) + 1$, $N(A_1) = \langle 0\rangle$, and if $dim(N(A_i)) < dim(N(A_{i+1}))$, then there is some generator $b \wedge a + \Phi$ of $N(A_{i+1})$ over $N(A_i)$, where b generates A_{i+1} over A_i, $a \in A_i$, $\Phi \in \wedge^2 A_i \setminus N(A_i)$, and there is no $x \in A_i$ with $x \wedge a + \Phi \in N(A_i)$.*

Proof:
($\rightarrow$) Successive application of ($\Delta 4$) gives the chain $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$. ($\Delta 3$) implies $N(A_1) = \langle 0\rangle$ and the uniqueness of solutions x for $x \wedge a + \Phi \in N(A)$.
($\leftarrow$) Use induction on n. $\square$

Definition 3.2 *Let M be a Δ-structure.*

1. *If $\{a_1, \ldots, a_m\}$ is a basis of a subspace A of M, then it is called a Δ-basis if $m < 3$ or $m = n + 2$ and $A_1 = \langle a_1, a_2, a_3\rangle$ and for $1 < i \leq n$ $A_i = \langle a_1, \ldots, a_{i+2}\rangle$ fulfil the conditions in Lemma 3.1.*

2. *If I is an ordering and $X = \{x_\alpha : \alpha \in I\}$ is a basis of the subspace A of M, then X is a Δ-basis of A if every finite subset X_0 of X is a Δ-basis of $\langle X_0\rangle$.*

3. *If $X = \{x_\alpha : \alpha \in I\}$ is a Δ-basis of $A \subseteq M$, then for $a \in A$ $o_X(a) = \alpha$ if $a = \sum_{\beta \leq \alpha} r_\beta x_\beta$ where $r_\alpha \neq 0$ and $r_\beta = 0$ for almost all $\beta \leq \alpha$. $o_X(a)$ is called the order of a. Often we write $o(a)$ only. $I = o(X) = o(\langle X \rangle)$ we call the order of X or of $\langle X \rangle$*

Let X be a subset of a Δ-structure M as in Definition 3.2. Then let $X_{<\alpha}$ be $\{x_\beta : \beta < \alpha\}$ and $X_{\leq \alpha}$ be $\{x_\beta : \beta \leq \alpha\}$. Note that X is a Δ-basis of $\langle X \rangle$ iff for $\alpha \in I$ $dim(N(X_{\leq \alpha})/N(X_{<\alpha})) \leq 1$. From these remarks follows:

Lemma 3.3 *Let M be a Δ-structure and $X = \{x_\alpha : \alpha \in I\}$ be a Δ-basis of a subspace A of M. Let $Y = \{y_\alpha : \alpha \in I\}$ where $y_\alpha = r^\alpha_\alpha x_\alpha + \sum_{\beta < \alpha} r^\alpha_\beta x_\beta$ with $0 < r^\alpha_\alpha < p$ and almost all $r^\alpha_\beta = 0$.*
Then Y is a Δ-basis.

If we have the situation described in Lemma 3.3, then we say that Y is a Δ-transformation of X. In a Δ-basis X we can replace every element $x \in X$ by every $y \in \langle X \rangle$ with $o(x) = o(y)$.

Lemma 3.4 *Let M be a $\mathcal{S}$-structure and let X be a Δ-basis of a subspace of M. If A is a finite subspace of $\langle X \rangle$, then there is a Δ-transformation Y of X such that Y contains a Δ-basis for A.*

Proof: Every element of A is a linear combination of finitely many elements of X. Hence we can assume w.l.o.g. that X is finite. Assume that $|X| = m$. We prove by induction on j $(1 \leq j \leq m)$ that there is a Δ-transformation $Y_{\leq j}$ of $X_{\leq j}$ that contains a Δ-basis of $A \cap \langle X_{\leq j} \rangle$. We start with $y_1 = x_1$. If $A \cap \langle X_{\leq j+1} \rangle = A \cap \langle X_{\leq j} \rangle$, then let $Y_{\leq j+1}$ be $Y_{\leq j} \cup \{x_{j+1}\}$.Otherwise we find the desired y_{j+1} in $(A \cap \langle X_{\leq j+1} \rangle) \setminus (A \cap \langle X_{\leq j} \rangle)$. □

Such a Δ-basis $Y \cap A$ for A as obtained in Lemma 3.4 we call a Δ-basis for A *induced* by X. A Δ-basis Z of $A \subseteq M$ is induced by the Δ-basis X of M iff o_Z for A is a restriction of o_X for M. Now we show the existence of a Δ-basis for countable structures.

Lemma 3.5 *Every countable Δ-structure M has a Δ-basis.*

Proof: Since M is countable we can write M as a union of a chain of finite subspaces: $M = \cup_{1 \leq i < \omega} M_i$ where $M_i \subseteq M_{i+1}$, $dim(M_1) = 3$, and $dim(M_{i+1}) = dim(M_i) + 1$. We define a tree in the following way. The Δ-bases of M_i are the points of level i of the tree. A point X of level $i+1$ is connected with a point Y of level i iff $Y = X \cap M_i$. There are points on each level but finitely many only. By Lemma 3.4 for every point of the tree there is a path to level 1. Hence by König's Lemma there is an infinite path. It represents the desired Δ-basis of M. □

Definition 3.6 *Let $A \subseteq B$ be two $\mathcal{S}$-structures.*

1. *If $dim(B) = dim(A) + 1$ and $dim(N(B)) \leq dim(N(A)) + 1$, then B is called an one-extension of A. It is transcendental if $N(B) = N(A)$ and algebraic otherwise.*

2. *If $a \in A$ and $\Phi \in \wedge^2 A \backslash N(A)$ and there is no b in A such that $a \wedge b + \Phi \in N(A)$, then $a \wedge x + \Phi = 0$ is called a problem over A.*

If $a \wedge x + \Phi = 0$ is a problem over A, then the following algebraic one-extension B contains a solution: $B = A \oplus \langle b \rangle$ for some b and $N(B) = N(A) + \langle a \wedge b + \Phi \rangle$. If A is a Δ-structure, then B is a Δ-structure. Furthermore we have:

Lemma 3.7 *Let A and B be Δ-structures. Assume B is an algebraic one-extension of A. Then there are $a \in A$ and $\Phi \in \wedge^2 A \setminus N(A)$ such that a problem over A has a solution in B iff it is of the form $r(a \wedge x + \Phi') = 0$ where $\Phi' = \Phi$ modulo $N(A) + \langle \{a \wedge c : c \in A\} \rangle$.* □

The standard models of our theory Δ are the groups $H_{st}(n)$ that correspond to the following Δ-structures $M_{st}(n)$. Let us fix n with $3 \leq n < \omega$:

$$M_{st}(n) = \cup_{1 \leq i < \omega} M_i$$

where $dim(M_1) = n$, $N(M_1) = \langle 0 \rangle$, M_{i+1} is an algebraic one-extension of M_i, and every problem over M_i has a solution in some M_j. From Lemma 3.1 follows that ($\Delta 3$) and ($\Delta 4$) are true. ($\Sigma 2$) is true since all problems are solved. It remains to show that all Δ-structures we obtain in this way are isomorphic. It follows by successive application of the following lemma.

Lemma 3.8 *Let A be a Δ-structure and let $a \wedge x + \Phi = 0$ and $a' \wedge x + \Phi' = 0$ be two different problems over A. That means $a \neq a'$ or $\Phi \neq \Phi'$ modulo $N(A) + \langle \{a \wedge c \ : \ c \in A\} \rangle$. Assume $A \subseteq B_1 \subseteq B_2$ where B_1 is an algebraic one-extension of A given by a solution of $a \wedge x + \Phi = 0$ and B_2 is an algebraic one-extesion of B_1 given by a solution of $a' \wedge x + \Phi' = 0$. Furthermore assume* *that* C_1 *is an algebraic one-extension of* A *given by a solution of* $a' \wedge x + \Phi' = 0$ *and* C_2 *is an algebraic one-extension of* C_1 *given by a solution of* $a \wedge x + \Phi = 0$. *Then* B_2 *and* C_2 *are isomorphic over* A. □

In [3] it is proved that every finite Δ-structure satisfies the axions $(\Sigma 3)$ and $(\Sigma 4)$. Let G be a saturated model of Σ and M be the corresponding Δ-structure. Let M_1 be a subspace of M generated by n geometrically independent elements of M. M_1 is selfsufficient and can be identified with the M_1 in the definition of $M_{st}(n)$. Using axiom $(\Sigma 5)$ we obtain a selfsufficient embedding of every M_{i+1} in M extending the selfsufficient embedding of M_i. Hence $M_{st}(n)$ is a selfsufficient subspace of M. It is the intersection of all selfsufficient subspaces of M that contain M_1 and satisfy axiom $(\Sigma 2)$. We see there are several ways to embed $H_{st}(n)$ in G.

4 The past

Every Δ-structure M fulfils ($\Sigma3$) and ($\Sigma4$) as shown in [3]. By ($\Sigma3$) we have the pregeometry cl on M as mentioned in section 2. If H is a model of Δ and M is the corresponding Δ-structure, then we can consider cl also on $\overline{H}$. If we consider the algebraic closure acl on $\overline{H}$, then we mean the algebraic closure on $\overline{H}$ with respect to the full theory of H. As in [3] we get:

Lemma 4.1 *In $\overline{H}$ we have $cl \subseteq acl$.*

Proof: Let A be a finite subspace of $\overline{H}$ and let a be an element in $cl(A)$. Since we have $CSS(A) \subseteq acl(A)$ we can assume w.l.o.g. that A is selfsufficient . Then there is a sequence $B_1 = A \subseteq B_2 \subseteq \ldots \subseteq B_m$ such that B_{i+1} is a minimal extension of B_i and $a \in B_m$. Note that each B_i is selfsufficient . By ($\Sigma4$) it follows that $B_{i+1} \subseteq acl(B_i)$. See [3]. $\square$

Let $A \subseteq B$ be finite subspaces of a Δ-structure M.

Definition 4.2 *B is in the one-closure of A ($B \subseteq cl_1(A)$) if there is a chain $A = B_1 \subseteq B_2 \subseteq \ldots \subseteq B_m = B$ such that B_{i+1} is an algebraic one-extension of B_i.*

We have $cl_1 \subseteq cl$. The Δ-structure $M_{st}(n)$ corresponding to the standard model $H_{st}(n)$ is the one-closure of n elements. If we have $A \subseteq B$ finite and we cannot find the B_{m-1} in Definition 4.2, then we have the following situation.

Definition 4.3 *Let $A \subseteq B$ be finite subspaces of a Δ-structure M. Then B is part of the past of A (short $B \in pp(A)$) if for every subspace B_1 with $A \subseteq B_1 \subseteq B$ and $dim(B_1) + 1 = dim(B)$ we get $dim(N(B)) > dim(N(B_1)) + 1$. We say $C \subseteq past(A)$ if there is some $B \in pp(A)$ with $C \subseteq B$.*

Note that $B \in pp(A)$ and $A \subseteq A_1$ implies $B + A_1 \in pp(A_1)$. We will show in Lemma 4.6 that $B \in pp(A)$ and $D \in pp(A)$ implies $B + D \in pp(A)$. This justifies the definition of the past. It is a closure operation:

1. $A \subseteq past(A)$.

2. If $A \subseteq B$, then $past(A) \subseteq past(B)$.

3. $past(past(A)) = past(A)$.

The last statement follows from Lemma 4.5 below. For infinite A let $past(A)$ be the union of all $past(A_0)$ where A_0 is a finite subspace of A. Analogously we define $pp(A)$ and $past(A)$ for finite subspaces A of $\overline{H}$ for models H of Δ. In the next section we will show that $past(A) \subseteq acl(A)$ for subspaces A in $\overline{H}$ where H is a model of Δ. Now we give examples.

Lemma 4.4 *Let $a_1, \ldots, a_n, b_1, b_2$ be linearly independent elements in a Δ-structure M that corresponds to a group that is model of Δ. Assume that n is odd. Furthermore suppose if $A = \langle b_1, b_2 \rangle$ and $B = \langle a_1, a_2, \ldots, a_n, b_1, b_2 \rangle$, then $N(A) = \langle 0 \rangle$ and $N(B)$ is generated by $\Phi_1 = b_2 \wedge a_1 - \sum_{1 \leq i \leq (n-1)/2} a_{2i} \wedge a_{2i+1}$ and $\Phi_2 = b_1 \wedge a_2 - a_1 \wedge a_n - \sum_{1 \leq i < (n-1)/2} a_{2i+1} \wedge a_{2i+2}$.*
Then $B \in pp(A)$.

Proof: Assume that $A \subseteq B_1 \subseteq B$ with $dim(B_1) + 1 = dim(B)$. We have to show that $N(B_1) = \langle 0 \rangle$. We can choose a basis $\{b_1, b_2, c_1, \ldots, c_{n-1}\}$ of B_1 where $c_i = \sum_{1 \leq j \leq n} r^i_j a_j$. Using the usual basis transformations we can assume w.l.o.g. that there is some k such that $c_i = a_{j_i} + r_i a_k$ where $\{j_1, j_2, \ldots, j_{n-1}, k\} = \{1, 2, \ldots, n\}$ and $0 \leq r_i < p$. Then $\wedge^2 B_1$ has a basis $\{b_1 \wedge b_2\} \cup \{b_l \wedge (a_{j_i} + r_i a_k) : l \in \{1, 2\}, 1 \leq i < n\} \cup \{(a_{j_h} + r_h a_k) \wedge (a_{j_i} + r_i a_k) : 1 \leq h < i < n\}$. It is easily seen that it is impossible to represent any non-trivial linear combination of Φ_1 and Φ_2 with respect to this basis. $\square$

Note that for $a_1, a_2, \ldots, a_n$ geometrically independent you get A and B as in Lemma 4.4 by axiom $(\Sigma 2)$. Since $(\Sigma 3)$ is a consequence of Δ any three linearly independent elements are geometrically independent. So for any linearly independent a_1, a_2, a_3 there are b_1, b_2 such that $a_1, a_2, a_3 \in past(\{b_1, b_2\})$.

Lemma 4.5 *Let $A \subseteq B \subseteq C$ be finite subspaces of a Δ-structure M. If $B \in pp(A)$ and $C \in pp(B)$, then $C \in pp(A)$.*

Proof: Assume $A \subseteq C_1$ with $dim(C) = dim(C_1) + 1$. If $B \subseteq C_1$, then by assumption $dim(N(C_1)) + 1 < dim(N(C))$, as desired. Otherwise we consider $B_1 = C_1 \cap B$. Then $dim(B_1) + 1 = dim(B)$. Hence by assumption $dim(N(B_1)) + 1 < dim(N(B))$. Choose Φ_1 and Φ_2 in $N(B)$ linearly independent over $\wedge^2 B_1$. Then Φ_1 and Φ_2 are linearly independent over $\wedge^2 C_1$, as desired. $\square$

Lemma 4.6 *Let A be a finite subspace of a Δ-structure M. Then $CSS(A) \subseteq pp(A)$.*

Proof: Assume $A \subseteq B \subseteq CSS(A)$ with $dim(B)+1 = dim(CSS(A))$. Then $\delta(CSS(A)) < \delta(B)$. This means $\dim(N(CSS(A)) > dim(N(B)) + 1$, as desired. $\square$

Lemma 4.5 and Lemma 4.6 imply

Corollary 4.7 *If $B \in pp(A)$, then $CSS(B) \in pp(A)$.*

Lemma 4.8 *Let A, B, C be finite subspaces of a Δ-structure M such that $B \in pp(A)$ and $C \in pp(A)$.*
Then $B + C \in pp(A)$.

Proof: Assume $A \subseteq D \subseteq B + C$ with $dim(D) + 1 = dim(B + C)$. Then $dim(D \cap B) + 1 = dim(B)$ or $dim(D \cap C) + 1 = dim(C)$. Assume w.l.o.g. $dim(D \cap B) + 1 = dim(B)$. By assumption

(1) $dim(N(D \cap B)) + 1 < dim(N(B))$.

We will show:

(2) $dim(N(D)) - dim(N(D \cap B)) \leq dim(N(B + C)) - dim(N(B))$.

Then (1) and (2) imply

$$\begin{aligned} dim(N(D)) + 1 &= dim(N(D \cap B)) + 1 + dim(N(D)) - dim(N(D \cap B)) \\ &< dim(N(B)) + dim(N(B + C)) - dim(N(B)) \\ &= dim(N(B + C) \end{aligned}$$

as desired. It remains to prove (2). Let $\Phi_1, \ldots, \Phi_m$ be a maximal set in $N(D)$ linearly independent over $N(D \cap B)$. If we consider $\Phi_1, \ldots, \Phi_m$ as elements of $N(B + C)$, then they are linearly independent over $N(B)$. Otherwise a non-trivial linear combination of them would be in $\wedge^2 B$ and therefore in $\wedge^2 D \cap \wedge^2 B = \wedge^2(D \cap B)$, a contradiction. $\square$

5 The algebraic closure

H is used to denote a model of Δ and M for the corresponding Δ-structure. In this section we will show that $past(A) \subseteq acl(A)$ for subspaces A of $\overline{H}$. Let $A \subseteq B$ be finite subspaces of M with $B \in pp(A)$. We will prove that there are finitely many Δ-structures isomorphic to B over A only. First we give an example that shows that B is not uniquely determined by its isomorphism type over A:
Assume that there are geometrically independent elements $a_3, c_3, a_4, c_4, a_5, c_5$ in M. Define a_1 and a_2 as solutions of the following problems:

$$a_2 \wedge (a_3 - c_3) + a_4 \wedge a_5 - c_4 \wedge c_5 = 0 \quad a_1 \wedge (a_5 - c_5) + a_3 \wedge a_4 - c_3 \wedge c_4 = 0.$$

Then we introduce b_1 and b_2 as solutions of the following problems:

$$b_2 \wedge a_1 = a_2 \wedge a_3 + a_4 \wedge a_5 \quad b_1 \wedge a_2 = a_1 \wedge a_5 + a_3 \wedge a_4.$$

a_1, a_2, b_1, b_2 are obtained by axion ($\Sigma 2$). Let A be $\{b_1, b_2\}$, $B = \{b_1, b_2, a_1, a_2, a_3, a_4, a_5\}$, and $C = \{b_1, b_2, a_1, a_2, c_3, c_4, c_5\}$. Then $N(B)$ is freely generated by $b_2 \wedge a_1 - a_2 \wedge a_3 - a_4 \wedge a_5$ and $b_1 \wedge a_2 - a_1 \wedge a_5 - a_3 \wedge a_4$. $N(C)$ is freely generated by $b_2 \wedge a_1 - a_2 \wedge c_3 - c_4 \wedge c_5$ and $b_1 \wedge a_2 - a_1 \wedge c_5 - c_3 \wedge c_4$. Hence B and C are isomorphic over A as Δ-structures. By Lemma 4.4 $B \in pp(A)$ and $C \in pp(A)$. The next lemma is the key-lemma of this paper. We need the following definition.

Definition 5.1 *Let M_0 and B be subspaces of M. A Δ-basis Z of B respects M_0 if $Z = XY$ where X is a Δ-basis of $B \cap M_0$.*

Lemma 5.2 *Let M and M_0 be Δ-structures. Assume that M_0 is a substructure of M. $M_0 = \langle 0 \rangle$ is possible. Let XY be a Δ-basis of M such that the initial part X is a Δ-basis of M_0. Furthermore suppose that XY contains a Δ-basis of a finite subspace A. Let $B_1, \ldots, B_r$ be finite substructures of M that are linearly independent over A. $B_i \neq B_j$ modulo M_0 for $i \neq j$. Assume for each i $A \subseteq B_i$, $dim(B_i) = m$, $A + (B_i \cap M_0) \neq B_i$, and $B_i \in pp(A + (B_i \cap M_0))$. For each i let $\{z_1^i, \ldots, z_m^i\}$ be a Δ-basis of B_i induced by XY that extends $XY \cap A$ and respects M_0. We assume that $g_{il}(z_j^i) = z_j^l$ induces an isomorphism of B_i onto B_l as Δ-structures with a unary predicate for the intersection with M_0.*
Then $r \leq m + 1$.

Proof: We assume $m + 2 = r$ and show a contradiction. Let B be $B_1 + \ldots + B_r$. $B_i \neq A + (B_i \cap M_0)$ implies that there is some $z^i_j \notin A + M_0$. We have $z^i_m \in A$ since $B_i \in pp(A + (B_i \cap M_0))$. Choose k such that $z^i_j \in A$ for $j > k$ and $z^i_k \notin A$. Then $z^i_k \notin M_0$. Note $z^i_j \in A$ if and only if $z^l_j \in A$ and $g_{il}(z^i_j) = z^l_j = z^i_j$ in this case. By Lemma 3.4 there is a Δ-basis Z of B induced by XY. Then each $\{z^i_1, \ldots, z^i_m\}$ is induced by Z. $o_{\{z^i_1,\ldots,z^i_m\}}$ is a restriction of o_Z and o_Z is a restriction of o_{XY}. We write $o(x)$ only. Note $XY \cap A$ is a Δ-basis for A. We can assume that Z contains $XY \cap A$. Hence $z^1_{k+1}, \ldots, z^1_m$ are the last elements of Z. We get $B = U_{k+1} \oplus \langle z^1_{k+1}, \ldots, z^1_m\rangle$ where $U_{k+1} \subseteq \langle (XY)_{<z_{k+1}}\rangle\rangle$.

Let Φ^1_j be a generator of $N(\langle z^1_1, \ldots, z^1_j\rangle)$ over $N(\langle z^1_1, \ldots, z^1_{j-1}\rangle)$ if these two spaces are different. Otherwise let Φ^1_j be 0.
$\Phi^1_j = 0$ implies $g_{1i}(\Phi^1_j) = 0$ and $N(\langle z^i_1, \ldots, z^i_j\rangle) = N(\langle z^i_1, \ldots, z^i_{j-1}\rangle)$ for $1 \leq i \leq r$.
If $\Phi^1_j \neq 0$, then $N(\langle z^i_1, \ldots, z^i_j\rangle) \neq N(\langle z^i_1, \ldots, z^i_{j-1}\rangle)$ and $g_{1i}(\Phi^1_j)$ generates $N(\langle z^i_1, \ldots, z^i_j\rangle)$ over $N(\langle z^i_1, \ldots, z^i_{j-1}\rangle)$. We define $\Phi^i_j = g_{1i}(\Phi^1_j)$.

With respect to the basis $\{z^i_1, \ldots, z^i_m\}$ and any h with $1 \leq h < m$ we can write:

$$\Phi^i_j = (\sum_{h \leq l \leq j} z^i_l \wedge a^{ij}_l) + \Phi^i_j(h)$$

where $a^{ij}_l \in \langle z^i_1, \ldots, z^i_{l-1}\rangle$ and $\Phi^i_j(h) \in \wedge^2\langle z^i_1, \ldots, z^i_{h-1}\rangle$. Let h be $k+1$. Then

(3) For $k < l \leq j$ we have $a^{i_1 j}_l = a^{i_2 j}_l \in A$ for all $1 \leq i_1, i_2 \leq r$.

By definition $z^{i_1}_l = z^{i_2}_l \in A$ for $k < l \leq m$. If (3) was not true, then we could chose $j > k$ maximal such that there are i and l such that $a^{ij}_l \notin A$. Choose l maximal in this situation. Note $a^{ij}_l \in A$ implies $a^{i_1 j}_l = a^{ij}_l$ for all i_1. But $a^{ij}_l \notin A$ implies that $a^{1j}_l, \ldots, a^{rj}_l$ are linearly independent over A, since $B_1, \ldots, B_r$ are linearly independent over A. As mentioned above $B = U_{k+1} \oplus \langle z^1_{k+1}, \ldots, z^1_m\rangle$, $z^1_{k+1}, \ldots, z^1_m$ are the last elements of the Δ-basis Z, and $U_{k+1} = \{u \in B \ : \ o(u) < o(z^1_{k+1})\}$. Then $\Phi^1_j, \ldots, \Phi^r_j$ are linearly independent over $\wedge^2 U_{k+1}$, since $z^1_l \wedge a^{1j}_l, \ldots, z^1_l \wedge a^{rj}_l$ are linearly independent over $\wedge^2(U_{k+1} + \langle z^1_{k+1}, \ldots, z^1_{l-1}\rangle)$. We get $dim(N(B) - dim(N(U_{k+1})) \geq r = m + 2$. But this is impossible for the Δ-basis Z. End of the proof of (3) $\square$

There is some m_0 such that $z^i_1, \ldots, z^i_{m_0-1} \in M_0$ and $z^i_{m_0}, \ldots, z^i_m$ are linearly independent over M_0. As above $o(x) = o_Z(x) = o_{XY}(x)$. We claim for every h with $m_0 \leq h \leq m$:

$(4)_h$ For every i $(1 \leq i \leq r)$ $o(z^i_h) < o(z^i_{h+1}) < \ldots < o(z^i_m)$ is the end segment of $o(B)$.
For $i \neq j$ $(1 \leq i, j \leq r)$ and $h \leq l \leq m$ we have $o(z^i_l - z^j_l) < o(z^i_h)$.
If $\Phi^i_j = (\sum_{h \leq l \leq j} z^i_l \wedge a^{ij}_l) + \Phi^i_j(h)$
where $a^{ij}_l \in \langle z^i_1, \ldots, z^i_{l-1}\rangle$
and $\Phi^i_j(h) \in \wedge^2\langle z^i_1, \ldots, z^i_{h-1}\rangle$,
then $a^{ij}_l \in A$ and $o(a^{ij}_l) < o(z^i_h)$ if $z^1_l \notin A$.

Note that $(4)_{k+1}$ follows from (3). $(4)_h$ implies $o(z^i_l) = o(z^j_l)$ for $h \leq l \leq m$. Furthermore every Δ-basis Z of B induced by XY can be Δ-transformed into a Δ-basis of the form $Z'\{z^i_h, \ldots, z^i_m\}$. Let U_h be the subspace $\{u \in B : o(u) < o(z^1_h)\}$. In $(4)_h$ we can write $a^{ij}_l \in A \cap U_h$ if $z^i_l \notin A$ and $\Phi^i_j(h) \in \wedge^2 U_h$. First we show:

(5) For $1 < h$ $(4)_h$ implies that for some i $o(z^i_{h-1})$ is the immediate predecessor of $o(z^i_h)$. $m_0 = 1$ and $(4)_1$ is impossible.

Proof of (5): First of all $m_0 = 1$ and $(4)_1$ would imply $a^{ij}_k = 0$ for all i and j. Hence $\Phi^1_j \in N(\langle\{z^1_1, \ldots, z^1_m\} \setminus \{z^1_k\}\rangle)$ for $1 \leq j \leq m$. This contradicts $B_1 \in pp(A)$. Note $M_0 \cap B_1 = \langle 0\rangle$ in this case.

Now we consider the case $1 < h$. Let α be the immediate predecessor of $o(z^1_h)$. Let U_{h-1} be the subspace of all $u \in \langle Z\rangle$ with $o(u) < \alpha$.
We assume that there is some h such that $1 < h$, $(4)_h$ is true, and $o(z^i_l) < \alpha$ for all i with $1 \leq i \leq r$ and all l with $1 \leq l < h$. We show a contradiction.
As a consequence of this assumption we get in $(4)_h$ that for l with $h \leq l \leq m$ and $z^i_l \notin A$ $a^{ij}_l \in U_h \cap A \subseteq U_{h-1}$
and $\Phi^i_j(h) \in \wedge^2 U_{h-1}$.
By $(4)_h$ U_h is generated by all z^i_l with $1 \leq i \leq r$ and $1 \leq l < h$ and all $z^i_l - z^1_l$ with $1 < i \leq r$ and $h \leq l \leq m$. Hence by the assumption of the proof there is some l with $h \leq l \leq m$ and some i with $1 < i \leq r$ such that $o(z^i_l - z^1_l) = \alpha$. Now choose l_0 minimal such that $h \leq l_0 \leq m$ and $z^i_{l_0} - z^1_{l_0} \in U_h \setminus U_{h-1}$. If there is another $z^i_t - z^1_t \in U_h \setminus U_{h-1}$ for the considered i, then $z^i_t \notin A$. There is some q $(0 < q < p)$ with $q(z^i_{l_0} - z^1_{l_0}) - (z^i_t - z^1_t) \in U_{h-1}$. For t fixed as above and all $1 \leq s \leq r$ we replace z^s_t by $y^s_t = z^s_t - qz^s_{l_0}$. Then $y^i_t - y^1_t = (z^i_t - qz^i_{l_0}) - (z^1_t - qz^1_{l_0}) = (z^i_t - z^1_t) - q(z^i_{l_0} - z^1_{l_0}) \in U_{h-1}$. Note that the assumptions of the lemma and $(4)_h$ remain true after these Δ-transformations. Hence we can assume w.l.o.g. $z^i_t - z^1_t \in U_{h-1}$ for all $t \neq l_0$ with $h \leq t \leq m$. As mentioned above $z^i_{l_0} \notin A$. Since $Z = Z'\{z^1_h, \ldots, z^1_m\}$

contains a basis for A and $m_0 \leq h \leq l_0$ we get $z^i_{l_0} \notin A + (B_i \cap M_0)$. By assumption $B_i \in pp(A + (B_i \cap M_0))$. We get $\Phi^i_{j_1}$ and $\Phi^i_{j_2}$ that are linearly independent over $\wedge^2\{z^i_1, \ldots, z^i_{l_0-1}, z^i_{l_0+1}, \ldots, z^i_m\}$. $z^i_{l_0} \notin A$ implies that $z^i_{l_0}$ does not occur in any a^{ij}_l. Hence $z^i_{l_0} \wedge a^{ij_s}_{l_0}$ is the only summand in $\Phi^i_{j_s}$ where $z^i_{l_0}$ occurs. Hence $a^{ij_1}_{l_0} \neq 0$, $a^{ij_2}_{l_0} \neq 0$, and $a^{ij_1}_{l_0} \neq a^{ij_2}_{l_0}$ is necessary to realize the independence of $\Phi^i_{j_1}$ and $\Phi^i_{j_2}$ over $\wedge^2\langle z^i_1, \ldots, z^i_{l_0-1}, z^i_{l_0+1}, \ldots, z^i_m\rangle$. Since $a^{ij}_l \in A$ we have $a^{ij}_l = a^{1j}_l$. By $(4)_h$ we get

$$\Phi^i_{j_1} - \Phi^1_{j_1} = (\sum_{h \leq l \leq m} (z^i_l - z^1_l) \wedge a^{ij_1}_l) + \Phi^i_{j_1}(h) - \Phi^1_{j_1}(h).$$

By the assumption above $z^i_l - z^1_l \in U_{h-1}$ for $l \neq l_0$. As mentioned above we have $a^{ij}_l \in U_h \cap A \subseteq U_{h-1}$ for $z^i_l \neq z^1_l \notin A$ and $\Phi^i_{j_1}(h), \Phi^1_{j_1}(h) \in \wedge^2 U_{h-1}$. Hence

$$\Phi^i_{j_1} - \Phi^1_{j_1} = (z^i_{l_0} - z^1_{l_0}) \wedge a^{ij_1}_{l_0} + \Phi$$

where $\Phi \in \wedge^2 U_{h-1}$. Analogously

$$\Phi^i_{j_2} - \Phi^1_{j_2} = (z^i_{l_0} - z^1_{l_0}) \wedge a^{ij_2}_{l_0} + \Psi$$

where $\Psi \in \wedge^2 U_{h-1}$. Since $0 \neq a^{ij_1}_{l_0} \neq a^{ij_2}_{l_0}$ we have a contradiction to Lemma 3.7. End of the proof of (5) $\square$

By induction from $h+1$ to h we will show $(4)_h$ for $m_0 \leq h \leq k+1$. Then $(4)_{m_0}$ gives the desired contradiction. If $m_0 = 1$ this follows from (5). Otherwise by (5) $o_Z(z^1_{m_0-1})$ is the immediate predecessor of $o_Z(z^1_{m_0})$. $(4)_{m_0}$ implies $o(z^i_l - z^1_l) \leq o(z^1_{m_0-1})$ for $m_0 \leq l \leq m$. Hence $z^i_l - z^1_l \in M_0$ and $B_i = B_1$ modulo M_0, a contradiction. Let us assume $(4)_{h+1}$ for $m_0 < h+1 \leq k+1$. We show $(4)_h$.

Proof of $(4)_h$**:** By the induction hypothesis $(4)_{h+1}$ there is a Δ-basis $W'\{w, z^1_{h+1}, \ldots, z^1_m\}$ of B induced by XY that contains the Δ-basis $XY \cap A$ of A. $U_{h+1} = \langle W'\{w\}\rangle$ and U_h is $\langle W'\rangle$. By induction $z^i_l - z^1_l \in U_{h+1}$ for $h+1 \leq l \leq m$. A basis induced by XY is also induced by every Δ-transformation of XY. Hence we can assume w.l.o.g. that $W'\{w, z^1_{h+1}, \ldots, z^1_m\}$ is containded in XY. Since $\{z^i_1, \ldots, z^i_m\}$ is induced by XY $z^i_l \in U_{h+1}$ for $1 \leq l \leq h$ and $1 \leq i \leq r$. Hence U_{h+1} is generated by all these z^i_l with $1 \leq i \leq h$ and $1 \leq l \leq r$ and all $z^i_{h+1} - z^1_{h+1}, \ldots, z^i_m - z^1_m$ for all $1 < i \leq r$. By (5) and $(4)_{h+1}$ we get:

There is some $z_h^i \in U_{h+1} \setminus U_h$.
If $z_h^i \in A$ for some i, then $z_h^i \in A$ for all i and they are equal. For the first assertion of $(4)_h$ we have to show $z_h^i \in U_{h+1} \setminus U_h$ for all $1 \leq i \leq r$. This and the next assertions (6), (7), (8), and (9) are trivial if $z_h^i \in A$. Hence we assume $z_h^i \notin A$. Since $B_i \in pp(A + (B_i \cap M_0))$ we can choose j_1, j_2 such that $\Phi_{j_1}^i$ and $\Phi_{j_2}^i$ are linearly independent over $\wedge^2\langle z_1^i, \ldots, z_{h-1}^i, z_{h+1}^i, \ldots, z_m^i\rangle$. $z_h^i \notin A$ implies that z_h^i does not occur in a_l^{ij} for $h < l$. Hence $z_h^i \wedge a_h^{ij_s}$ is the only summand in $\Phi_{j_s}^i$ where z_h^i occurs. We have

(6) If $z_h^i \notin A$, then for j_1 and j_2 as chosen above $0 \neq a_h^{ij_1} \neq a_h^{ij_2} \neq 0$ for all i.

Furthermore we show:

(7) If $z_h^i - z_h^t \notin U_h$ for some $t \neq i$, then we can assume after some Δ-transformations that $z_l^i - z_l^t \in U_h$ for $h < l \leq m$.

Assume that this is not true. We have $z_h^i - z_h^t \in U_{h+1} \setminus U_h$. Then there is some l with $h < l \leq m$ and $z_l^i - z_l^t = q(z_h^i - z_h^t) \; mod \; U_h$ for some $0 < q < p$. We can replace every z_l^s by $z_l^s - qz_h^s$ where $1 \leq s \leq r$ and l is fixed above. This is a Δ-transformation. $(4)_{h+1}$ remains true. If we iterate this argument, then we can assume (7). Now we consider

$$\Phi_{j_1}^i - \Phi_{j_1}^t = (\sum_{h<l\leq m} (z_l^i - z_l^t) \wedge a_l^{ij_1}) + z_h^i \wedge a_h^{ij_1} - z_h^t \wedge a_h^{tj_1} + \Phi_{j_1}^i(h) - \Phi_{j_1}^t(h)$$

where for $h < l \leq m$ $a_l^{ij_1} = a_l^{tj_1} \in A \cap U_{h+1} \subseteq U_h$ by $(4)_{h+1}$ and $a_h^{ij_1}, a_h^{tj_1} \in U_h$, and $\Phi_{j_1}^i(h), \Phi_{j_1}^t(h) \in \wedge^2 U_h$. If $z_h^t \in U_h$, then by (7)

$$\Phi_{j_1}^i - \Phi_{j_1}^t = z_h^i \wedge a_h^{ij_1} + \Phi$$

where $\Phi \in \wedge^2 U_h$. Analogously

$$\Phi_{j_2}^i - \Phi_{j_2}^t = z_h^i \wedge a_h^{ij_2} + \Psi$$

where $\Psi \in \wedge^2 U_h$. By (6) we get a contradiction to Lemma 3.7. We have shown

(8) $z_h^i \in U_{h+1} \setminus U_h$ for $1 \leq i \leq r$.

Let us consider $\Phi^i_{j_1} - \Phi^t_{j_2}$ again. Now we assume that $a^{ij}_h = a^{tj}_h$ for $j \in \{j_1, j_2\}$. Then by (7)

$$\Phi^i_{j_1} - \Phi^t_{j_1} = (z^i_h - z^t_h) \wedge a^{ij_1}_h + \Phi'$$

and

$$\Phi^i_{j_2} - \Phi^t_{j_2} = (z^i_h - z^t_h) \wedge a^{ij_2}_h + \Psi'$$

where $\Phi', \Psi' \in \wedge^2 U_h$. If $z^i_h - z^t_h \notin U_h$, then we get a contradiction by (6). We have shown

(9) If $a^{1j}_h = a^{ij}_h$ for $1 < i \leq r$ and all j, then $z^i_h - z^1_h \in U_h$.

(8) and (9) are trivial if $z^i_h \in A$. Now we have to consider this case too. Next we claim

(10) $a^{ij}_h \in A$ for $1 \leq i \leq r$ and $h \leq j \leq m$.

If this was not true we could fix j such that $a^{ij}_h \notin A$ for all i. Then $a^{1j}_h, \ldots, a^{rj}_h$ are linearly independent over A. Note that by definition $a^{ij}_h \in U_h$. Hence $z^1_h \wedge a^{1j}_h, \ldots, z^1_h \wedge a^{rj}_h$ are linearly independent over $\wedge^2 U_h$. We consider

$$\Phi^i_j = (\sum_{h<l\leq m} z^i_l \wedge a^{ij}_l) + z^i_h \wedge a^{ij}_h + \Phi^i_j(h)$$

where $\Phi^i_j(h) \in \wedge^2 U_h$, and $a^{ij}_l \in \langle z^i_1, \ldots, z^i_{l-1}\rangle$. By $(4)_{h+1}$ for $h < l \leq m$ we know $z^i_l - z^1_l \in U_{h+1}$, $a^{ij}_l = a^{1j}_l \in A$, and $a^{1j}_l \in A \cap U_{h+1}$ if $z^1_l \notin A$. Furthermore by (8) $z^i_h \in U_{h+1} \setminus U_h$. Hence $z^i_h = s_i z^1_h + v_i$ with $0 < s_i < p$ and $v_i \in U_h$. Let us rewrite

$$\Phi^i_j = (\sum_{h<l\leq m} z^1_l \wedge a^{1j}_l) + (\sum_{\substack{h<l\leq m \\ z^1_l \notin A}} ((z^i_l - z^1_l) \wedge a^{1j}_l) + s_i(z^1_h \wedge a^{ij}_h) + (v_i \wedge a^{ij}_h) + \Phi^i_j(h).$$

Since $z^i_l - z^1_l \in U_{h+1}$ for $h < l \leq m$ we can write $z^i_l - z^1_l = r^i_l z^1_h + w^i_l$ with $0 \leq r^i_l < p$ for all $h < l \leq m$ and $w^i_l \in U_h$. By the linear independence of $B_1, \ldots, B_r$ over A the following is a basis of B:

$$\begin{aligned} Z^* &= \{z^1_1, \ldots, z^1_m\} \cup \{z^i_l \notin A \,:\, 1 < i \leq r, 1 \leq l < h\} \\ &\cup \{w^i_l \,:\, 1 < i \leq r, h < l \leq m, z^1_l \notin A\} \\ &\cup \{v_i \,:\, 1 < i \leq r\}. \end{aligned}$$

We do not speak about a Δ-basis. Note that $\{z_1^1,\ldots,z_m^1\}$ contains a basis for A. $Z^*\setminus\{z_h^1,\ldots z_m^1\}$ is a basis for U_h. Furthermore $a_t^{1j}\in\{z_l^1\ :\ 1\le l\le h\}\cap A$ for $h<t\le m$ and $z_t^1\notin A$. By definition $a_h^{ij}\in\{z_l^i\ :\ 1\le l<h\}$.

We use the representation of Φ_j^i above and the basis Z^* described above to show that the linear indepedence of $a_h^{2j},\ldots,a_h^{rj}$ over $A+B_1$ implies the linear independence of $\Phi_j^2,\ldots,\Phi_j^r$ over $\wedge^2 U_h$. This is a contradiction to $(\Delta 3)$ and $(\Delta 4)$ since $r-1=m+1>m-h+1=dim(B)-dim(U_h)$.

First we assume $z_h^1\notin A$. Then $U_{h+1}\cap A\subseteq U_h$. We have

$$\Phi_j^i=s_i(z_h^1\wedge a_h^{ij})+\Phi_i$$

where $\Phi_i\in\wedge^2 B_1+\wedge^2 U_h$.

Now we assume that $z_h^1\in A$. Then for $h<l\le m$ we have $a_l^{1j}=t_l^j z_h^1+b_l^j$ where $b_l\in A\cap U_h$. We get

$$\Phi_j^i=s_i(z_h^1\wedge a_h^{ij})-\sum_{\substack{h<l\le m\\ z_l^1\notin A}} t_l^j(z_h^1\wedge w_l^i)+\Psi_i$$

where $\Psi_i\in\wedge^2 B_1+\wedge^2 U_h$. Note $a_h^{ij}\in\langle z_1^i,\ldots,z_{h-1}^i\rangle\subseteq Z^*$ and $w_l^i\in Z^*$. This completes the proof of (9). $\square$

By (9) and (10) we have

(11) $z_h^i-z_h^1\in U_h$ for $1\le i\le r$.

The next step is to show:

(12) $a_l^{ij}\in A\cap U_h$ for $h\le l\le m$ with $z_l^i\notin A$.

Note $a_h^{ij}\in U_h$ by definition and $a_h^{ij}\in A$ by (10). By induction $z_l^i\notin A$ implies $a_l^{ij}\in A\cap U_{h+1}$ for $h<l\le m$. If $z_h^1\notin A$ it follows $a_l^{ij}\in A\cap U_{h+1}\subseteq U_h$ as desired. Therefore we consider the case $z_h^1\in A$. For $z_l^i\notin A$ we can write $a_l^{1j}=a_l^{ij}=t_l^j z_h^1+b_l^j$ where $b_l^j\in A\cap U_h$ and $0\le t_l^j<p$. Let I_j be

$$\{l\ :\ h<l\le m, t_l^j\ne 0, z_l^1\notin A\}.$$

If the assertion of (12) was not true, then there would be some j_0 with $I_{j_0}\ne\emptyset$. Under this assumption we show that $\Phi_{j_0}^2-\Phi_{j_0}^1,\ldots,\Phi_{j_0}^r-\Phi_{j_0}^1$ are linearly independent over $\wedge^2 U_h$. This gives the desired contradiction since

$dim(B) - dim(U_h) = m - h + 1 \leq m < r - 1$ and there are $r-1$ many relations in $N(B)$ linearly independent over $\wedge^2 U_h$. To prove the linear independence consider

$$\Phi^i_{j_0} - \Phi^1_{j_0} = (\sum_{h \leq l \leq m} (z^i_l - z^1_l) \wedge a^{ij_0}_l) + \Phi^i_{j_0}(h) - \Phi^1_{j_0}(h).$$

Note $\Phi^i_{j_0}(h) - \Phi^1_{j_0}(h) \in \wedge^2 U_h$. For $z^i_l \in A$ we have $z^i_l - z^1_l = 0$, e.g. $z^i_h - z^1_h = 0$ since we have assumed that $z^1_h \in A$. As above assume that $z^i_l - z^1_l = r^i_l z^1_h + w^i_l$ where $0 \leq r^i_l < p$ and $w^i_l \in U_h \cap Z^*$. Then $W^* = \{w^i_l \ : \ 1 < i \leq r, h < l \leq m, z^1_l \notin A\} \subseteq U_h \cap Z^*$ is a set linearly independent over B_1.
For $l \notin I_{j_0}$ we have
$(z^i_l - z^1_l) \wedge a^{ij_0}_l = r^i_l(z^1_h \wedge b^{j_0}_l) + w^i_l \wedge b^{j_0}_l \in \langle z^1_h \rangle \wedge (A \cap U_h) + \wedge^2 U_h$ and
for $l \in I_{j_0}$ we have
$(z^i_l - z^1_l) \wedge a^{ij_0}_l = r^i_l(z^1_h \wedge b^{j_0}_l) + w^i_l \wedge t^i_l z^1_h + w^i_l \wedge b^{j_0}_l$. Hence

$$\Phi^i_{j_0} - \Phi^1_{j_0} = (\sum_{\substack{h < l \leq m \\ l \in I_{j_0}}} w^i_l \wedge t^i_l z^1_h) + \Psi^i_{j_0}$$

where $\Psi^i_{j_0} \in \langle z^1_h \rangle \wedge (A \cap U_h) + \wedge^2 U_h \subseteq \wedge^2 B_1 + \wedge^2 U_h$. The linear independence of W^* over B_1 and $W^* \subseteq U_h$ implies the desired linear independence of $\Phi^2_{j_0} - \Phi^1_{j_0}, \ldots, \Phi^r_{j_0} - \Phi^r_{j_0}$ over $\wedge^2 U_h$. □

To finish the proof of $(4)_h$ we show

(13) $z^i_l - z^1_l \in U_h$ for $h \leq l \leq m$ and $1 < i \leq r$.

Let us fix some i. By (11) $z^i_h - z^1_h \in U_h$. By induction $z^i_l - z^1_l \in U_{h+1}$ for $h < l \leq m$. We assume that there is some l_0 with $h < l_0 \leq m$ and $z^i_{l_0} - z^1_{l_0} \in U_{h+1} \setminus U_h$. Choose l_0 minimal. As above for (7) we find some suitable Δ-transformations such that w.l.o.g. $z^i_l - z^1_l \in U_h$ for $l \neq l_0$ with $h \leq l \leq m$. Since $B_i \in pp(A + (B_i \cap M_0))$ there are j_1 and j_2 such that $\Phi^i_{j_1}$ and $\Phi^i_{j_2}$ are linearly independent over $\wedge^2 \langle z^i_1, \ldots, z^i_{l_0-1}, z^i_{l_0+1}, \ldots, z^i_m \rangle$. As above it follows $0 \neq a^{ij_1}_{l_0} \neq a^{ij_2}_{l_0} \neq 0$. By our assumptions and by (12) we have

$$\begin{aligned} \Phi^i_{j_1} - \Phi^1_{j_1} &= (z^i_{l_0} - z^1_{l_0}) \wedge a^{ij_1}_{l_0} + \Phi \\ \Phi^i_{j_2} - \Phi^1_{j_2} &= (z^i_{l_0} - z^1_{l_0}) \wedge a^{ij_2}_{l_0} + \Psi \end{aligned}$$

where $\Phi, \Psi \in \wedge^2 U_h$. We get the desired contradiction to Lemma 3.7. □

Corollary 5.3 *Let M be a Δ-structure. Let X be a Δ-basis of M that extends a Δ-basis of a finite substructure A. Let $B_1, \ldots, B_r$ be finite substructures of M that contain A and are linearly independent over A. Assume that $B_i \in pp(A)$ and that there are Δ-bases $\{z_1^i, \ldots, z_m^i\}$ of B_i induced by X for each i ($1 \leq i \leq r$). Furthermore suppose that these Δ-bases contain the Δ-basis of A contained in X. We assume that $g_{it}(z_l^i) = z_l^t$ induces an isomorphism of B_i onto B_t as Δ-structures.*
Then $r \leq m + 1$.

Theorem 5.4 *There is a function $f(m)$ from the naturals into the naturals such that for every Δ-structure M and all finite subspaces A of M the number of subspaces B of M with $A \subseteq B$, $B \in pp(A)$, and $dim(B) = m$ is less or equal to $f(m)$.*

Proof: First we show that there is a function $h(m)$ such that for every Δ-structure M, every finite subspace A of M, and every $B_1, \ldots, B_r$ linearly independent over A with $A \subseteq B_i$, $dim(B_i) = m$, and $B_i \in pp(A)$ we have $r \leq h(m)$. W.l.o.g. we can assume that M is countable. By Lemma 3.5 there is a Δ-basis X of M. By Lemma 3.4 we can assume that X extends a Δ-basis Z of A. Assume tha $B_1, \ldots, B_r$ are given as above. For each B_i we fix a Δ-basis $z_1^i, \ldots, z_m^i$ induced by X and containing Z. We say $B_i \sim B_t$ if $g_{it}(z_l^i) = z_l^t$ induces an isomorphism of B_i onto B_t over A as Δ-structures. There is a finite bound $h_\sim$ for the number of possible $\sim$-classes. By Corollary 5.3 we have

$$r \leq (m + 1)h_\sim = h(m).$$

Now we show by induction on i ($1 \leq i < m$) that there is a function $f(m, i)$ such that for every Δ-structure M for every subspace A of M with $m - i \leq dim(A) < m$ the number of subspaces B with $A \subseteq B$, $dim(B) = m$, and $B \in pp(A)$ is bounded by $f(m, i)$.

Choose any M and any A with $dim(A) = m - 1$. Assume that r is chosen maximal such that there are $B_1, \ldots, B_r$ that are linearly independent over A, have size m, contain A, and are parts of the past of A. As shown above $r \leq h(m)$. If B is any other subspace of M with $dim(B) = m$, $A \subseteq B$, and $B \in pp(A)$, then $B \subseteq B_1 + \ldots + B_r$ since r was chosen maximal. It follows that $f(m, 1) = 2^{(p^{h(m)m})}$ is a bound with the desired properties.

Now assume that $f(m,i)$ is given. Let $A \subseteq M$ be subspace of size $m-(i+1)$. Again we consider a maximal set $\{B_1,\ldots,B_r\}$ linearly independent over A with $dim(B_i)=m$, $A \subseteq B_i$, and $B_i \in pp(A)$. We know $r \leq h(m)$. If B is any subspace of M with $dim(B)=m$, $A \subseteq B$, and $B \in pp(A)$, then there is some $a \in (B \setminus A) \cap (B_1+\ldots+B_r)$. Define $A_1 = A + \langle a \rangle$. Then $dim(A_1) = m-i$, $A_1 \subseteq B$, and $B \in pp(A_1)$. By induction the number of these B is bounded by $f(m,i)$. Furthermore $A_1 \subseteq B_1+\ldots+B_r$. Hence

$$f(m,i+1) = f(m,i)2^{(p^{h(m)m})} + f(m,i)$$

has the desired properties. We can define $f(m) = f(m,m-1)$. □

Corollary 5.5 *Let H be a model of Δ. Let $A \subseteq B$ be finite subspaces of $\overline{H}$. If $B \in pp(A)$, then $B \subseteq acl(A)$. Hence $past(A) \subseteq acl(A)$.*

Corollary 5.6 *If A is a finite subspace of a Δ-structure M, then $past(A)$ is countable.*

6 Extending a Δ-basis

Lemma 6.1 *Let M be a Δ-structure and M_0 be a substructure with $past(M_0) \subseteq M_0$. Let A be a finite subspace of M. Then there is a Δ-basis $X = X_0X_1$ of A such that the initial part X_0 is a Δ-basis of $A \cap M_0$.*

Proof: We use induction on $dim(A) = m$. The case $m = 1$ is clear. Assume that the assertion is proved for all A with $dim(A) = m$. Now we consider A with $dim(A) = m + 1$. If $A \subseteq M_0$, then the assertion follows by Lemma 3.4. We assume $A \not\subseteq M_0$. Let A_0 be $A \cap M_0$. Then there exists some A_1 with $A_0 \subseteq A_1 \subseteq A$ such that $dim(A_1) = dim(A) - 1$ and $dim(N(A_1)) + 1 \geq dim(N(A))$. Otherwise we would get $A \subseteq past(A_0) \subseteq past(M_0) \subseteq M_0$, a contradiction to our assumption $A \not\subseteq M_0$. By induction hypothesis there is a Δ-basis X_0Y of A_1 such that X_0 is a Δ-basis of $A_1 \cap M_0 = A \cap M_0$. By the choice of A_1 for every $a \in A \setminus A_1$ $X_0Y\{a\}$ is the desired Δ-basis of A. $\square$

Note that we have defined X respects M_0 if we are in the situation described in the Lemma 6.1. If H is a model of Δ, H_0 is an elementary submodel, and M and M_0 are the Δ-structures that correspond to H and H_0 respectively, then we have $past(M_0) \subseteq M_0$ by Corollary 5.5.

Lemma 6.2 *Let M be a countable Δ-structure and M_0 be a substructure with $past(M_0) \subseteq M_0$. Then there exists a Δ-basis of M that respects M_0.*

Proof: We use a similar argument as in Lemma 3.5. Let M be $\cup_{1 \leq i < \omega} N_i$ such that $N_i \subseteq N_{i+1}$ and $dim(N_i) = i$. We consider the following tree. The set P_i of points of level i is the set of the Δ-bases of N_i that respect M_0. By Lemma 6.1 $P_i \neq \emptyset$. A Δ-basis of N_i is connected with a Δ-basis of N_{i+1} if it is a subset. Every point of the tree has finite valency and by Lemma 6.1 there are points at every level. As in Lemma 3.4 it is easily seen that from every point at level i there is a path to level 1. By König's Lemma there is an infinite path. The union of the Δ-bases of this path is the desired Δ-basis of M that respects M_0. $\square$

Corollary 6.3 *Let M_0 be a substructure of a countable Δ-structure M such that $past(M_0) \subseteq M_0$. Then every Δ-basis X of M_0 can be extended to a Δ-basis of M.*

Proof: By Lemma 6.2 there is a Δ-basis ZY of M that respects M_0 where Z is a Δ-basis of M_0. If we replace Z by X, then we get the desired Δ-basis of M. $\Box$

Lemma 6.4 *Let $B \subseteq C$ be finite subspaces of a Δ-structure M. Assume that $C \subseteq cl_1(B)$. If $B \subseteq C_1 \subseteq C$, $dim(C_1)+1 = dim(C)$, and $dim(N(C_1))+1 \geq dim(N(C))$, then $C_1 \subseteq cl_1(B)$.*

Proof: Since $C \subseteq cl_1(B)$ there are $c_1, \ldots, c_n$ such that $C = B + \langle c_1, \ldots, c_n \rangle$ and $B + \langle c_1, \ldots, c_{i+1} \rangle$ is an algebraic one-extension of $B + \langle c_1, \ldots, c_i \rangle$. Assume that Φ_{i+1} generates $N(B + \langle c_1, \ldots, c_{i+1} \rangle)$ over $N(B + \langle c_1, \ldots, c_i \rangle)$. Choose i_0 minimal such that $c_{i_0} \notin C_1$. By $dim(C_1) + 1 = dim(C)$ after some Δ-transformations we can assume that $c_{i_0+1}, \ldots, c_n \in C_1$. By the assumption Φ_{i_0} generates $N(C)$ over $N(C_1)$. Hence we can choose all Φ_i with $i \neq i_0$ in $N(C_1)$ as desired. $\Box$

Lemma 6.5 *Let $A \subseteq B \subseteq C \subseteq D$ be a finite subspaces of a Δ-structure M. If $B \in pp(A)$, $C \subseteq cl_1(B)$, and $D \in pp(C)$, then there is a $B_1 \in pp(A)$ such that $D = B_1 + C$.*

Proof: We show the assertion by induction on $dim(D) - dim(A) = m$. If $D \in pp(A)$, then we are done. Otherwise there exists D_1 such that $A \subseteq D_1 \subseteq D$, $dim(D_1) + 1 = dim(D)$, and $dim(N(D_1)) + 1 \geq dim(N(D))$. Then $B \subseteq D_1$ since otherwise $dim(B \cap D_1) + 1 = dim(B)$ and $dim(N(B \cap D_1)) + 1 \geq dim(N(B))$, a contradiction to $B \in pp(A)$. Note $A \subseteq B \cap D_1$. Furthermore $C \not\subseteq D_1$. $C \subseteq D_1$ would imply $D \in pp(D_1)$, a contradiction to the defining condition of D_1. Hence $D = D_1 + C$. Let C_1 be $D_1 \cap C$. Then $dim(C_1) + 1 = dim(C)$ and $dim(N(C_1)) + 1 \geq dim(N(C))$. Furthermore $B \subseteq D_1 \cap C = C_1$. By Lemma 6.4 $C \subseteq cl_1(B)$ implies $C_1 \subseteq cl_1(B)$. Let $c \in C \setminus D_1$. Then $C_1 \oplus \langle c \rangle = C$ is an algebraical one-extension of C_1. We have $D = D_1 \oplus \langle c \rangle$ and $c \in cl_1(D_1)$. We claim $D_1 \in pp(C_1)$. Otherwise there is some D_2 with $C_1 \subseteq D_2 \subseteq D_1$, $dim(D_2)+1 = dim(D_1)$, and $dim(N(D_2))+1 \geq dim(N(D_1))$. Since $C_1 \subseteq D_1$ and $c \in cl_1(C_1)$ we have $dim(N(D_2 \oplus \langle c \rangle)) = dim(N(D_2)) + 1$. But then $dim(N(D_2 \oplus \langle c \rangle)) + 1 = dim(N(D_2)) + 2 \geq dim(N(D_1)) + 1 \geq dim(N(D))$. This contradicts $D \in pp(C)$. Now we have $B \in pp(A)$, $B \subseteq C_1$, $C_1 \subseteq cl_1(B)$ and $D_1 \in pp(C_1)$. Since $dim(D_1) < dim(D)$ and $A \subseteq D_1 \subseteq D$ we obtain by induction $D_1 = B_1 + C_1$ where $B_1 \in pp(A)$. Hence $D = B_1 + C$, as desired. $\Box$

Corollary 6.6 *Let A be a finite subspace in a Δ-structure M. Then $past(cl_1(past(A))) = cl_1(past(A))$.*

7 Completeness

In this section we show that Δ is a complete theory. Let H be a model of Δ, M be the corresponding Δ-structure, and $\overline{a}$ be a tuple of elements linearly independent over $Z(H)$. Then we also use $\overline{a}$ to denote the corresponding tuple of images in M that are linearly independent. The elementary type of $\overline{a}$ in H is determined by the isomorphism type of $past(\overline{a})$ in the corresponding Δ-structure M. The next lemma will be used for ω-saturated H to find finite subspaces of M with no proper $past$. Note that we assume $A \subseteq B$ if we write $B \in pp(A)$. If $B \in pp(A)$ and $A \subseteq A_1$, then $B + A_1 \in pp(A_1)$. If $B \in pp(A)$, $B_0 \subseteq B$, and $N(B) = N(B_0)$, then $B_0 \in pp(A \cap B_0)$. If X is a Δ-basis of M that contains a Δ-basis Y of a finite subspace A, y is the maximal element of Y, and $B \in pp(A)$, then $B \subseteq \langle X_{\leq y} \rangle$. Furthermore for an infinite set A and a finite set C we have $C \subseteq past(A)$ if there is a finite subset A_1 of A and a finite set $B \in pp(A_1)$ such that $C \subseteq B$.

Lemma 7.1 *Let H be a model of Δ, M be the corresponding Δ-structure, and A be a finite subspace of M. Assume natural numbers m and n are given. Then there are elements $b_1, \ldots, b_n$ in M linearly independent over A such that for every $B \in pp(A \oplus \langle b_1, \ldots, b_n \rangle)$ with $dim(B) \leq m$ there is some B_A with $A \subseteq B_A$, $B = B_A \oplus \langle b_1, \ldots, b_n \rangle$, and $N(B) = N(B_A)$. Hence $B_A \in pp(A)$.*

Proof: Let $h = 3^{m+2}n+1$. Assume w.l.o.g. $dim(A) \geq 4$ and M is countable. By Lemma 3.5 there is a Δ-basis X of $past(A)$. There is some $a_3 \in A$ such that $o_X(a_3)$ is the last element of the order given by X. W.l.o.g. a_3 is the last element of X.
If $N(past(A)) \neq N(past(\langle X_{<a_3} \rangle))$, then assume that $a_3 \wedge d - \Psi$ with $d \in \langle X_{<a_3} \rangle$ and $\Psi \in \wedge^2 \langle X_{<a_3} \rangle$ is a generator of $N(past(A))$ over $N(past(\langle X_{<a_3} \rangle))$. Choose a_1, a_2 in $past(A)$ such that a_1, a_2, a_3 are linearly indpendent and $d \notin \langle a_1, a_2, a_3 \rangle$. Then the problem

$$x \wedge a_1 = a_3 \wedge a_2$$

has no solution in $past(A)$. Let a_4 be a solution in M. It exists by axiom $(\Sigma 2)$. By Lemma 6.5

$$past(A \oplus \langle a_4 \rangle) = past(A) \oplus \langle a_4 \rangle.$$

$X\{a_4\}$ is a Δ-basis of this space and the problem

$$x \wedge a_2 = a_4 \wedge a_3$$

has no solution in it. Let a_5 be a solution. Then by Lemma 6.5

$$past(A \oplus \langle a_4, a_5\rangle) = past(A) \oplus \langle a_4, a_5\rangle$$

and $X\{a_4, a_5\}$ is a Δ-basis of this subspace. We can continue in this way and obtain a set $Y = \{a_4, \ldots, a_h\}$ where a_{i+2} is a solution of

$$x \wedge a_{i-1} = a_{i+1} \wedge a_i.$$

Then by Lemma 6.5

$$past(A \oplus \langle Y\rangle) = past(A) \oplus \langle Y\rangle$$

and XY is a Δ-basis of this space. For $1 \leq i \leq 3^{m+1}n$ let a_i^{m+1} be $a_{3(i-1)+4}$. We define $A_{m+1} = \{a_1^{m+1}, \ldots, a_{3^{m+1}n}^{m+1}\}$. A_{m+1} is a subset of Y. Note that the problems

$$x \wedge a^{m+1}_{3(l-1)+1} = a^{m+1}_{3(l-1)+2} \wedge a^{m+1}_{3l)}$$

for $1 \leq l \leq 3^m_n$ have no solutions in $past(A) \oplus \langle Y\rangle$.

Using axiom ($\Sigma 2$) we define inductively $A_j = \{a_1^j, \ldots, a^j_{3^{j-1}n}\}$ where a_l^j is a solution of the following problem

$$x \wedge a^{j+1}_{3(l-1)+1} = a^{j+1}_{3(l-1)+2} \wedge a^{j+1}_{3l}.$$

and show that this problem has no solution in $past(A) \oplus \langle YA_m \ldots A_{j+1}\{a_1^j, \ldots, a_{l-1}^j\}$. Define $b_i = a_i^1$ for $1 \leq i \leq n$. Then $Z = XYA_m \ldots A_1$ is a Δ-basis of $\langle Z\rangle$. By Corollary 6.5 we have $past(\langle Z\rangle) \subseteq \langle Z\rangle$. Hence by Lemma 6.3 there is a Δ-basis W of M such that Z is an initial segment of W.

Now assume $B \in pp(A \oplus \langle b_1, \ldots, b_n\rangle)$ with $dim(B) = m$. As mentioned above $past(\langle Z\rangle) \subseteq \langle Z\rangle$. Hence $B \subseteq \langle Z\rangle$. Let B_A be $B \cap \langle XYA_m \ldots A_2\rangle$. Then $B = B_A \oplus \langle b_1, \ldots, b_n\rangle$. It is sufficient to prove $N(B) = N(B_A)$. Hence we assume that $N(B) \neq N(B_A)$ and show a contradiction. Choose i minimal such that $N(B) = N(\langle Z_{\leq b_i}\rangle \cap B)$. Assume that Φ_1 generates $N(B) \cap \langle Z_{\leq b_i}\rangle$ over $N(B \cap \langle Z_{<b_i}\rangle)$. Then Φ_1 generates $N(\langle Z_{\leq b_i}\rangle)$ over $N(\langle Z_{<b_i}\rangle)$. By Lemma 3.7

$$\Phi_1 = b_i \wedge a^2_{3(i-1)+1} - \Psi_1$$

where $\Psi_1 \in \wedge^2\langle Z_{<b_i}\rangle$. Then $a^2_{3(i-1)+1} \in B$ and $\Psi_1 \in \wedge^2 B$. Since $B \in pp(A \oplus \langle b_1, \ldots, b_n\rangle)$ there is some $\Phi_2 \in N(B)$ linearly independent from Φ_1 over $\wedge^2(B \cap \langle Z \setminus \{a^2_{3(i-1)+1}\}\rangle)$. If we write Φ_2 as a linear combination of basic commutators with respect to Z, then for summands of the form $a^2_{3(i-1)+1} \wedge d$ we get $d \in \langle a^3_{3^2(i-1)+1}, b_i\rangle$. Hence the linear independence of Φ_1 and Φ_2 over $\wedge^2(B \cap \langle Z \setminus \{a^2_{(i-1)+1}\}\rangle)$ implies $a^3_{3^2(i-1)+1} \in B$ and

$$\Phi_2 = a^2_{3(i-1)+1} \wedge a^3_{3^2(i-1)+1} - \Psi_2$$

where $\Psi_2 \in \wedge^2(\langle Z \setminus \{a^3_{3^2(i-1)+1}\}\rangle \cap B)$. Since $B \in pp(A \oplus \langle b_1, \ldots, b_n\rangle)$ there is some $\Phi_3 \in N(B)$ such that Φ_2 and Φ_3 are linearly independent over $\wedge^2(B \cap \langle Z \setminus \{a^3_{3^2(i-1)+1}\rangle)$. We continue in this way and get $a^j_{3^{j-1}(i-1)+1} \in B$ for $1 \leq j \leq m+1$. This contradicts $dim(B) = m$. □

Theorem 7.2 *1. Δ is complete.*

2. Let G and H be two models of Δ. Let $(a_1, \ldots, a_n)$ be a tuple of elements in G linearly independent over $Z(G)$. Let $(b_1, \ldots, b_n)$ be a tuple of elements in H linearly independent over $Z(H)$. Let M and N be the corresponding Δ-structures respectively. Assume there is an isomorphism f_n of $\langle a_1, \ldots, a_n\rangle$ onto $\langle b_1, \ldots, b_n\rangle$ defined by $f_n(a_i) = b_i$. Let g_n be the induced $\mathcal{S}$-isomorphism of the corresponding finite substructures of M and N. We also use a_i and b_i to denote their images in M and N respectively. If g_n can be extended to an isomorphism of $past(\langle a_1, \ldots, a_n\rangle)$ onto $past(\langle b_1, \ldots, b_n\rangle)$, then $(a_1, \ldots, a_n)$ in G and $(b_1, \ldots, b_n)$ in H have the same type.

Proof: 2) implies 1). To show 2) we can assume w.l.o.g. that G and H are ω_1-saturated. We describe a winnning strategy for player II in the Ehrenfeucht - Fraissé - Game. If the considered sequences are empty ($n = 0$), then 1) follows. We assume that $(m-n)$ rounds are played and the following situation is provided. Note that for $a \in G$ and $b \in H$ we use a and b also to denote the corresponding elements in M and N respectively.

$(EF)_m$) There are extensions $(a_1, \ldots, a_m)$ of $(a_1, \ldots, a_n)$ in G and $(b_1, \ldots, b_m)$ of $(b_1, \ldots, b_n)$ in H of elements linearly independent over $Z(G)$ and $Z(H)$ respectively and an isomorphism f_m of $\langle a_1, \ldots, a_m\rangle$ onto $\langle b_1, \ldots, b_m\rangle$ defined by $f_m(a_i) = b_i$. We assume that f_m gives the desired isomophism for the chosen elements. Let g_m be the induced isomorphism of the corresponding Δ-structures. There is an isomorphism h_m of $past(\langle a_1, \dots, a_m \rangle)$ in M onto $past(\langle b_1, \dots, b_m \rangle)$ in N that extends g_m.

Now we assume w.l.o.g. that player I has chosen an element $c \in G$. If $n = m = 0$ and $c \in Z(G)$, then player II fixes some element $c_1 \in G \setminus Z(G)$ and assumes that player I has chosen this element first. Working with c is then already the second round for him. Hence w.l.o.g. if $n = m = 0$, then first $a_1 \in G \setminus Z(G)$ is chosen. If $0 < m$ and c is linearly dependent from $a_1, \dots, a_m$ modulo $Z(G)$, then we can assume w.l.o.g. that $c \in Z(G) \setminus (\langle a_1, \dots, a_m \rangle)'$. By axiom $(\Sigma 2)$ there is some d such that $c = [a_1, d]$. Then d is linearly independent from $a_1, \dots, a_m$ modulo $Z(G)$. Hence w.l.o.g. player I has chosen an element a_{m+1} linearly independent from $a_1, \dots, a_m$ modulo $Z(G)$. Now we see by Lemma 2.2 that it is sufficient to play the game in the corresponding Δ-structures only. We have to find b_{m+1} in N linearly independent from $b_1, \dots, b_m$ such that we can extend h_m to an isomorphism h_{m+1} from $past(\langle a_1, \dots, a_{m+1} \rangle)$ onto $past(\langle b_1, \dots, b_{m+1} \rangle)$ with $h_{m+1}(a_{m+1}) = b_{m+1}$.

Let A_i be the subspace $\langle a_1, \dots, a_i \rangle$ of M. For every t let C_t the subspace of M generated by all B with $dim(B) \leq t$ and $B_t \in pp(A_{m+1})$. By Theorem 5.4 C_t is finite and $past(A_{m+1})$ is countable. Hence there is a countable elementary subgroup G_0 of G such that $past(A_{m+1}) \subseteq M_0$ where M_0 is the substructure of M corresponding to G_0. By Lemma 6.2 there is a Δ-basis XY of M_0 where the initial segment X is a Δ-basis for $past(A_m)$. Let us fix t for the next considerations. XY induces a Δ-basis $X_t Y_t$ of C_t such that the initial part X_t is a Δ-basis of $C_t \cap past(A_m)$. Using some Δ-transformations we can assume w.l.o.g. that $X_t \subseteq X$ and $Y_t \subseteq Y$. The same argument as for Lemma 3.4 works.

(14) There is some isomorphism h^t_{m+1} of C_t into N extending the restriction of h_m to $C_t \cap past(A_m)$ such that for every B with $dim(B) \leq t$ and $B \in pp(h^t_{m+1}(A_{m+1}))$ we have $B \subseteq h^t_{m+1}(C_t)$.

Proof: Since XY_t is a Δ-basis of $past(A_m) + C_t$ we can find a permutation ZW of Y_t such that $N(\langle XZ \rangle) = N(\langle X \rangle)$ and $W \subseteq cl_1(\langle XZ \rangle)$.

If $Z \neq \emptyset$, then by Lemma 7.1 and the ω_1-saturation of H there is a set Z^N with $|Z| = |Z^N|$ and $past(h_m(A_m) \oplus \langle Z^N \rangle) = past(h_m(A_m)) \oplus \langle Z^N \rangle$. For $Z = \emptyset$ let Z^N be $\emptyset$. Using a one-one correspondence between Z and Z^N

we can extend h_m to an isomorphism u of $cl_1(\langle XZ\rangle)$ onto $cl_1(\langle h_m(X)Z^N\rangle)$. Axiom $(\Sigma 2)$ is used. Then u is defined on $\langle XZW\rangle$. Let $B \in pp(u(A_{m+1}))$ with $dim(B) \leq t$. We show $B \subseteq u(C_t)$.

By construction $u(A_{m+1}) \in cl_1(past(h_m(A_m) \oplus \langle Z^N\rangle))$. By Corollary 6.6 there is some B_1 such that $B_1 \in pp(h_m(A_m) \oplus \langle Z^N\rangle)$ and $B \subseteq cl_1(B_1)$. By the properties of Z^N there are $E_2 \in pp(h_m(A_m))$ and $E_3 \subseteq \langle Z^N\rangle$ such that $B_1 = E_2 \oplus E_3$. Then $u^{-1}(E_2) \in pp(A_m)$ and $u^{-1}(E_3) \subseteq \langle Z\rangle$. Hence $F_1 = u^{-1}(B_1) \subseteq past(A_m) \oplus \langle Z\rangle$. Since $B \subseteq cl_1(B_1)$ there is a chain $B_1 \subseteq B_2 \subseteq \ldots \subseteq B_k$ such that B_{i+1} is an algebraic one-extension of B_i and $B \subseteq B_k$. Using axiom $(\Sigma 2)$ we find a chain $F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k$ in M such that F_{i+1} is an algebraic one-extension of F_i and $u(F_i) = B_i$. By the isomorphism u between B_k and F_k we obtain $u^{-1}(B) \in pp(A_{m+1})$. Hence $u^{-1}(B) \subseteq C_t$ and therefore $B \subseteq u(\langle X_t ZW\rangle) = u(C_t)$. The restriction of u on C_t is the desired isomorphism h^t_{m+1}. This proves (14). $\square$

By ω_1-saturation and (14) we get an element b_{m+1} such that for every t there is a subspace D_t in N and an isomorphism h^t_{m+1} of C_t onto D_t that is an extension of h_m, sends a_{m+1} to b_{m+1}, and D_t is generated by all subspaces B with $dim(B) \leq t$ and $B \in pp(\langle b_1, \ldots b_{m+1}\rangle)$. Now we consider the following tree. At level t we have isomorphisms of C_t onto D_t extending h_m and sending a_{m+1} to b_{m+1}. An isomorphism of level $t+1$ is connected with an isomorphism of level t if the first one extends the second one. Since the C_t are finite we can apply König's Lemma and get the desired isomorphism h_{m+1}. $\square$

Corollary 7.3 *Let H_0 and H_1 be subgroups with $(\Sigma 2)^w$ of a model G of Δ. Assume $past(\overline{H_i}) \subseteq \overline{H_i}$. Assume that there is an isomorphism f of H_0 onto H_1. Then H_0 and H_1 have the same type.*

Proof: If K is a finite subspace of $\overline{H_0}$, then $f(past(K) = past(f(K))$. $\square$

8 Stability

We need a modified version of Theorem 5.4.

Theorem 8.1 *There is a function $g(m)$ from the naturals into the naturals such that for every pair $H_0 \preceq H$ of models of Δ where H_0 is an elementary substructure of H the following holds:*
Let M_0 and M respectively be the corresponding Δ-structures. If A is a finite subspace of M, then $g(m)$ is a bound for the number of subspaces B modulo M_0 where $dim(B) = m$, $A \subseteq B$, and $B \in pp(A + (B \cap M_0))$.

Proof: It is sufficient to consider countable models H only.

First we show that there is a function $h(m)$ such that for all H, H_0, M, M_0 as above, for all finite subspaces A of M and all $B_1, \ldots, B_t$ that are linearly independent over A with $dim(B_i) = m$, $A \subseteq B_i$, $B_i \neq A + (M_0 \cap B_i)$, $B_i \neq B_j$ modulo M_0 for $i \neq j$, and $B_i \in pp(A + (B_i \cap M_0))$ we have $t \leq h(m)$. Since we can restrict us to countable M by Lemma 6.2 we can assume that we have a Δ-basis XY of M where the initial part X is a Δ-basis of M_0. W.l.o.g. we can suppose that XY contains a basis Z for A. Let $\{z^i_1, \ldots, z^i_m\}$ be a Δ-basis of B_i that contains Z and is induced by XY. Then there is some m_i such that $z^i_1, \ldots, z^i_{m_{i-1}} \in M_0$ and $z^i_{m_i}, \ldots, z^i_m$ are linearly independent over M_0. We say $B_i \sim B_j$ if $g(z^i_l) = z^j_l$ induces an isomorphism of B_i onto B_j as Δ-structures with an additional predicate for for the intersection with M_0. Let $h_\sim$ be a bound for the number of $\sim$-classes. By Lemma 5.2 there are at most $m + 1$ elements in each $\sim$-class. Hence $h(m) = (m + 1)h_\sim$.

Next we show by induction on i that there is a function $g(m, i)$ such that for all H, H_0, M, M_0 as above and every finite subspace A of M with $(m - i) \leq dim(A) < m$ the number of subspaces B of M modulo M_0 with $dim(B) = m$, $A \subseteq B$, and $B \in pp(A + (M_0 \cap B))$ is less or equal to $g(m, i)$.

First assume $dim(A) = m - 1$. We choose a maximal set $\{B_1, \ldots, B_t\}$ of subspaces B_i with $A \subseteq B_i$, $B_i \neq A + (M_0 \cap B_i)$, and $B_i \in pp(A + (M_0 \cap B_i))$. that are linearly independent over A and different modulo M_0. As shown above $t \leq h(m)$. Let B be a subspace with $A \subseteq B$, $B \in pp(A + (M_0 \cap B)$. Then either $B = A$ modulo M_0 or by the maximality of t we have $B \subseteq B_1 + \ldots + B_t$ modulo M_0. Hence $g(m, 1) = 2^{(p^{h(m)m})}$ has the desired properties.

Now assume that $g(m, i)$ is given and we want to construct $g(m, i + 1)$. Let A be a subspace of M of size $m - (i + 1)$. Again we consider a maximal

set $\{B_1, \ldots, B_t\}$ linearly independent over A with the following properties:
$B_i \neq B_j$ modulo M_0 for $i \neq j$,
$dim(B_i) = m$, $A \subseteq B_i$,
$B_i \neq A + (M_0 \cap B_i)$, and $B_i \in pp(A + (M_0 \cap B_i))$.
As shown above $t \leq h(m)$. Let B be a subspace of M with $dim(B) = m$, $A \subseteq B$, and $B \in pp(A + (M_0 \cap B))$. Then $B = A$ modulo M_0 or there is some $a \in (B \cap (B_1 + \ldots + B_t)) \setminus A$. If we define $A_1 = A + \langle a \rangle$, then $B \in pp(A_1 + (M_0 \cap B)$. The number of possible A_1 in $B_1 + \ldots + B_t$ is bounded by $2^(p^{h(m)m})$. Hence by induction

$$g(m, i+1) = g(m, i)2^{(p^{h(m)m})} + g(m, i)$$

has the desired properties. We define $g(m) = g(m, m-1)$. □

Lemma 8.2 *Let H be a model of Δ considered as an elementary submodel of the monster model $\mathbf{C}$ of Δ. Let $M \subseteq M_{\mathbf{C}}$ be the corresponding Δ-structures. Let $\overline{a} = (a_1, \ldots, a_n)$ be a tuple of elements in $M_{\mathbf{C}}$ linearly independent over M.*

1. *There is a finite subset A of M such that*
   $N(M + \langle \overline{a} \rangle) = N(M) + N(\langle A \cup \{\overline{a}\} \rangle)$.
   Then furthermore $dim(N(\langle A \cup \{\overline{a}\} \rangle)) - dim(N(\langle A \rangle)) < n$.

2. *There is a countable elementary submodel $H_{\overline{a}}$ of H such that for the corresponding Δ-substructure $M_{\overline{a}}$ of M the following holds:*
   $past(M_{\overline{a}} + \langle \overline{a} \rangle) \cap M = M_{\overline{a}}$,
   $past(M + \langle \overline{a} \rangle) = past(M_{\overline{a}} + \langle \overline{a} \rangle) + M$, *and*
   $N(past(M + \langle \overline{a} \rangle)) = N(past(M_{\overline{a}} + \langle \overline{a} \rangle)) + N(M)$.

Proof:
ad 1) If $\overline{a} \subseteq M$, then $A = \langle \overline{a} \rangle$. Otherwise let $\Phi_1, \ldots, \Phi_t$ be elements of $N(M + \langle \overline{a} \rangle)$ that are linearly independent over $N(M)$. Then $t < n$. Otherwise we use $\overline{a}$ also to denote the corresponding tuple in $\mathbf{C}/\mathbf{Z}(\mathbf{C}) = \overline{\mathbf{C}}$. Then $\langle \overline{a} \rangle \subseteq cl(\overline{H}) \subseteq acl(\overline{H}) = \overline{H}$ by Lemma 4.1, a contradiction. Choose t maximal. Then $\Phi_1, \ldots, \Phi_t$ generate $N(M + \langle \overline{a} \rangle)$ over $N(M)$. Let A be any finite subspace of M such that each Φ_j is defined over A.
ad 2) By Theorem 8.1 we can find a countable sum B of finite subspaces B_i of $M_{\mathbf{C}}$ such that $B_i \in pp(\langle \overline{a} \rangle + (M \cap B_i))$ and $past(M + \langle \overline{a} \rangle) = M + B$. By 1) there is a countable subspace C of M such that $N(M + B) = N(C + B) +$

$N(M)$. Let H_0 be a countable elementary submodel of H and let M_0 be the corresponding Δ-substructure of M such that $B \cap M$ and C are subspaces of M_0. Then $past(M_0 + \langle\overline{a}\rangle) + M = past(M + \langle\overline{a}\rangle)$ and $N(past(M + \langle\overline{a}\rangle)) = N(M) + N(past(M_0 + \langle\overline{a}\rangle))$. Now we choose an elementary chain of countable elementary submodels H_i of H and the corresponding Δ-substructures M_i of M such that $M_i + (past(M_i + \langle\overline{a}\rangle) \cap M) \subseteq M_{i+1}$. Then $H_{\overline{a}} = \cup_{0 \leq i < \omega} H_i$ has the desired properties. $\square$

Corollary 8.3 Δ *is a stable theory.*

Proof: We count the 1-types $tp(a/H)$ over a model H of Δ. As above we consider H as an elementary submodel of some monster model $\mathbf{C}$. Let $M \subseteq M_{\mathbf{C}}$ be the corresponding Δ-structures. Let us fix some $g \in G \setminus Z(G)$. If $a - c \in Z(\mathbf{C})$ where $c \in H$, then by axiom $(\Sigma 2)$ there is some $d \notin Z(\mathbf{C})$ such that $(a - c) = [g, d]$. Hence $tp(a/H)$ is algebraic over $tp(d/H)$.It is sufficient to count types that contain the formulas $x - c \notin Z$ for all $c \in H$.

Let us consider $tp(a/H)$ where a is not in $Z(\mathbf{C})$ modulo H. We choose a countable elementary submodel H_a of H and the corresponding M_a according to Lemma 8.2 2). Now let b be any element in $\mathbf{C}$ such that $tp(a/H_a) = tp(b/H_a)$, $past(M_a + \langle b\rangle) \cap M = M_a$, $past(M + \langle b\rangle) = past(M_a + \langle b\rangle) + M$, and $N(past(M + \langle b\rangle)) = N(past(M_a + \langle b\rangle)) + N(M)$. $tp(a/H_a) = tp(b/H_a)$ implies the existence of some isomorphism f of $past(M_a + \langle a\rangle)$ onto $past(M_a + \langle b\rangle)$ that is the identity on M_a and sends a to b since $past \subseteq acl$. By the properties of M_a with respect to a and b we see that we can extend f to an isomorphism g $past(M + \langle a\rangle)$ onto $past(M + \langle b\rangle)$ that is the identity on M. By Theorem 7.2 we have $tp(a/H) = tp(b/H)$.

We have shown that $|S_1(H)|$ is the sum of all $|S_1(H_0)|$ where H_0 varies over all countable elementary submodels of H. It follows stability. $\square$

References

[1] A. Baudisch, *Decidability and stability of free nilpotent Lie algebras and free nilpotent p-groups of finite exponent*, Ann. of Math. Logic 23 (1982), 1 - 25.

[2] A. Baudisch, *On Lascar rank in non-multidimensional ω-stable theories*, Logic Colloquium '85 in Paris, North-Holland, Amsterdam 1987, 33 - 51.

[3] A. Baudisch, *A new uncountably categorical group*, to appear in Transactions of the AMS.

[4] G. Cherlin and S. Shelah, *Superstable fields and groups*, Annals of Mathematical Logic 18 (1980), 227 - 270.

[5] E. Hrushovski, *A new strongly minimal set*, Ann. Pure Appl. Logic 62 (1993), 147 - 166.

[6] E. Hrushovski and A. Pillay, *Weakly normal groups*, Logic Colloquium'85 in Paris, edited by the Paris Logic Group, North - Holland, Amsterdam 1987, 233 - 244.

[7] A. Macintyre, *On ω_1-categorical theories of fields*, Fundamenta Mathematicae 71 (1971), 1 - 25.

[8] A.H. Mekler, *Stability of nilpotent groups of class 2 and prime exponent*, Journal of Symbolic Logic 46 (1981), 781 - 788.

[9] B. Poizat, *Groupes stables*, Nur al - Mantiq wal Ma'rifah , Lyon 1987.

[10] F.O. Wagner, *Small stable groups and generics*, Journal of Symbolic Logic 56 (1991), 1026 - 1037.

[11] C. Wood, *Notes on the stability of separably closed fields*, Journal of Symcolic Logic 44 (1979), 412 - 416.