From cumuli to planetary waves:
Asymptotic multiscale analysis of atmospheric motions

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Thanks to ...

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Deutsche
Forschungsgemeinschaft

MetStröm DFG

* One of our tough reviewers

Asymptotic modelling framework

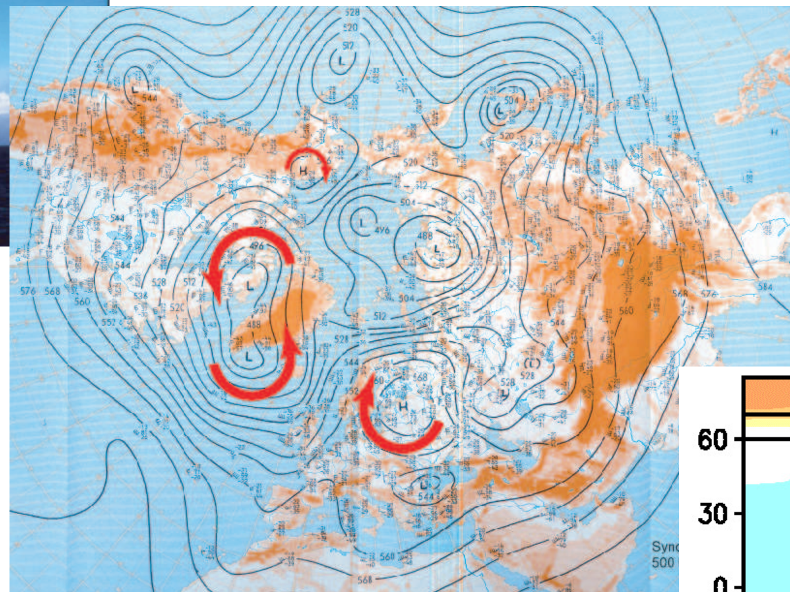
Motion and structure of atmospheric vortices

Sound-proof flow models

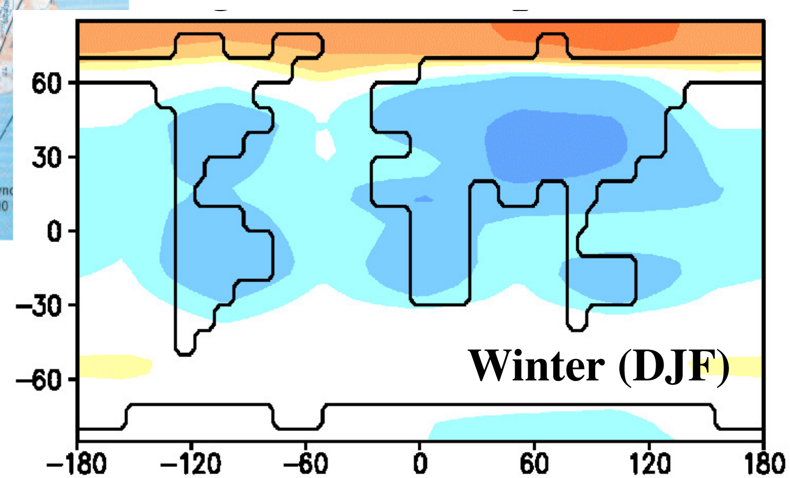
Clouds and waves

Scale-Dependent Models

10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Thanks to:

P.K. Taylor, Southampton Oceanogr. Inst.; P. Névir, Freie Universität Berlin; S. Rahmstorf, PIK, Potsdam

Scale-Dependent Models

$$
\begin{aligned}
\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi &= \boldsymbol{S}_{\boldsymbol{u}}\\
w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z &= -\theta' + S_w\\
\theta'_t + \boldsymbol{u}\cdot\nabla\theta' + w\theta'_z &= S'_\theta\\
\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z &= 0
\end{aligned}
$$

$$\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)$$

Anelastic Boussinesque Model

10 km / 20 min

$$\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla\right) q = 0$$

$$q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)$$

$$\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega_0}\boldsymbol{k}\times\nabla\pi^{(3)}$$

Quasi-geostrophic theory

1000 km / 2 days

$$
\begin{aligned}
\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T &= S_T\\
\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q &= S_q
\end{aligned}
$$

$$Q_\varphi = \int_{z_s}^{H_a}\rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a}\rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right)dz\,, \qquad \left(\varphi\in\{T, q\}\right)$$

$$T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\left(\min(z, H_T) - z_s\right), \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z - z_s}{H_q}\right)$$

$$\rho = \rho_*\exp\left(-\frac{z}{h_{\mathrm{sc}}}\right), \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{\mathrm{sc}}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z\frac{T}{T_*}\,dz'$$

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0$$

V. Petoukhov et al., *CLIMBER-2 ...*, Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Scale-Dependent Models

Earth's radius	a	$\sim$	$6 \cdot 10^6$	m
Earth's rotation rate	Ω	$\sim$	10^{-4}	s^{-1}
Acceleration of gravity	g	$\sim$	9.81	ms^{-2}
Sea level pressure	p_{ref}	$\sim$	10^5	$\mathrm{kgm}^{-1}\mathrm{s}^{-2}$
H_2O freezing temperature	T_{ref}	$\sim$	273	K
Tropospheric potential temperature variation	$\Delta\Theta$	$\sim$	40	K
Dry gas constant	R	$\sim$	287	$\mathrm{m}^2\mathrm{s}^{-2}\mathrm{K}^{-1}$
Dry isentropic exponent	γ	$\sim$	1.4	

Distinguished limit:

$$\Pi_1 = \frac{h_{\mathrm{sc}}}{a} \sim 1.6 \cdot 10^{-3} \sim \varepsilon^3$$

$$\Pi_2 = \frac{\Delta\Theta}{T_{\mathrm{ref}}} \sim 1.5 \cdot 10^{-1} \sim \varepsilon$$

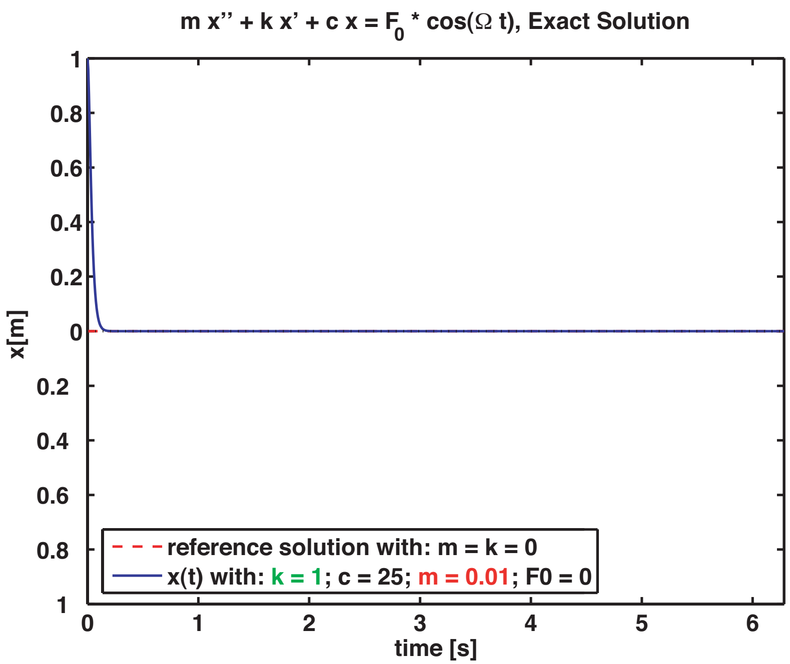
$$\Pi_3 = \frac{c_{\mathrm{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1} \sim \sqrt{\varepsilon}$$

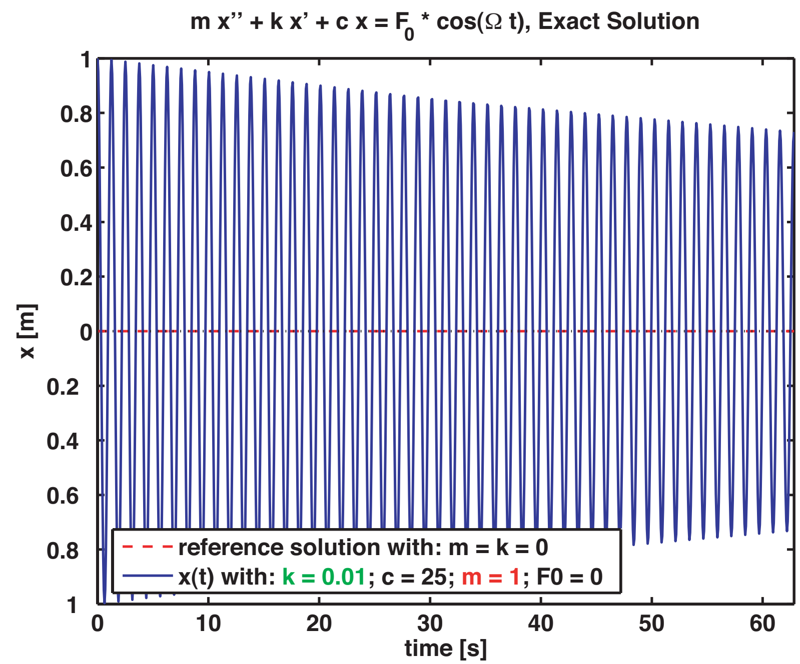
where

$$h_{\mathrm{sc}} = \frac{RT_{\mathrm{ref}}}{g} = \frac{p_{\mathrm{ref}}}{\rho_{\mathrm{ref}}\, g} \sim 8.5\,\mathrm{km}$$

$$c_{\mathrm{ref}} = \sqrt{RT_{\mathrm{ref}}} = \sqrt{g h_{\mathrm{sc}}} \sim 300\ \mathrm{m/s}$$

$$\varepsilon\ddot{y} + \delta\dot{y} + y = 0; \qquad y(0) = 1; \quad \dot{y}(0) = 0$$





$\varepsilon = 0.0004$

$\delta = 0.04$

$\varepsilon = 0.04$

$\delta = 0.0004$

The limit is path-dependent!

Scale-Dependent Models

Nondimensionalization

$$(\boldsymbol{x}, z) = \frac{1}{h_{\mathrm{sc}}} (\boldsymbol{x}', z') , \qquad t = \frac{u_{\mathrm{ref}}}{h_{\mathrm{sc}}} t'$$

$$(\boldsymbol{u}, w) = \frac{1}{u_{\mathrm{ref}}} (\boldsymbol{u}', w') , \qquad (p, T, \rho) = \left(\frac{p'}{p_{\mathrm{ref}}}, \frac{T'}{T_{\mathrm{ref}}}, \frac{\rho' R T_{\mathrm{ref}}}{p_{\mathrm{ref}}} \right)$$

where

$$u_{\mathrm{ref}} = \frac{2}{\pi} \frac{g h_{\mathrm{sc}}}{\Omega a} \frac{\Delta\Theta}{T_{\mathrm{ref}}} \qquad \text{(thermal wind scaling)}$$

Length scales, dimensionless numbers, and distinguished limits

$$
\begin{aligned}
L_{\mathrm{mes}} &= \varepsilon^{-1} \; h_{\mathrm{sc}} \\
L_{\mathrm{syn}} &= \varepsilon^{-2} \; h_{\mathrm{sc}} \\
L_{\mathrm{Ob}} &= \varepsilon^{-5/2} h_{\mathrm{sc}} \\
L_{\mathrm{p}} &= \varepsilon^{-3} \; h_{\mathrm{sc}}
\end{aligned}
\qquad\qquad
\begin{aligned}
\mathrm{Fr}_{\mathrm{int}} &\sim \varepsilon \\
\mathrm{Ro}_{h_{\mathrm{sc}}} &\sim \varepsilon^{-1} \\
\mathrm{Ro}_{L_{\mathrm{Ro}}} &\sim \varepsilon \\
\mathrm{Ma} &\sim \varepsilon^{3/2}
\end{aligned}
$$

Compressible flow equations with general source terms

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \boldsymbol{v}_{\shortparallel} \; + \; \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\shortparallel} + \frac{1}{\varepsilon^3 \rho} \; \nabla_{\shortparallel} p \; &= \; \boldsymbol{S}_{\boldsymbol{v}_{\shortparallel}} \, , \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) w \; + \; \varepsilon \left(2\boldsymbol{\Omega} \times \boldsymbol{v}\right)_{\perp} + \frac{1}{\varepsilon^3 \rho} \frac{\partial p}{\partial z} \; &= \; S_w - \frac{1}{\varepsilon^3} \, , \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \rho \; + \; \rho \nabla \cdot \boldsymbol{v} \; &= \; 0 \, , \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \Theta \; &= \; S_{\Theta} \, .
\end{aligned}
$$

Recovered classical single-scale models:

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$ Linear small scale internal gravity waves

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$ Anelastic & pseudo-incompressible models

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$ Linear large scale internal gravity waves

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$ Mid-latitude Quasi-Geostrophic Flow

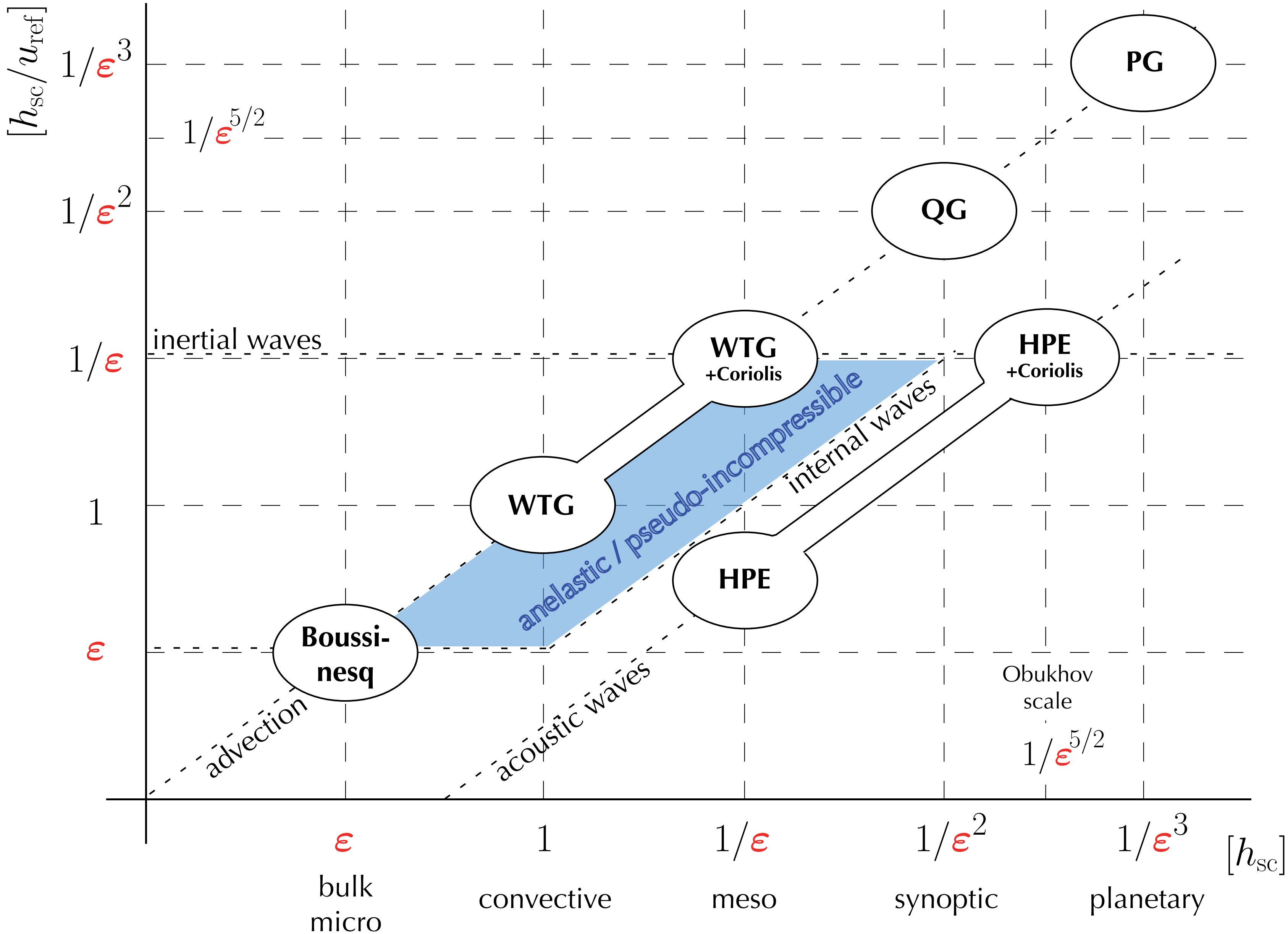
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$ Equatorial Weak Temperature Gradients

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1} \xi(\varepsilon^2 \boldsymbol{x}), z)$ Semi-geostrophic flow

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\underline{\varepsilon^{3/2}} t, \underline{\varepsilon^{5/2}} x, \underline{\varepsilon^{5/2}} y, z)$ Kelvin, Yanai, Rossby, and gravity Waves

... and many more

Scale-Dependent Models



R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

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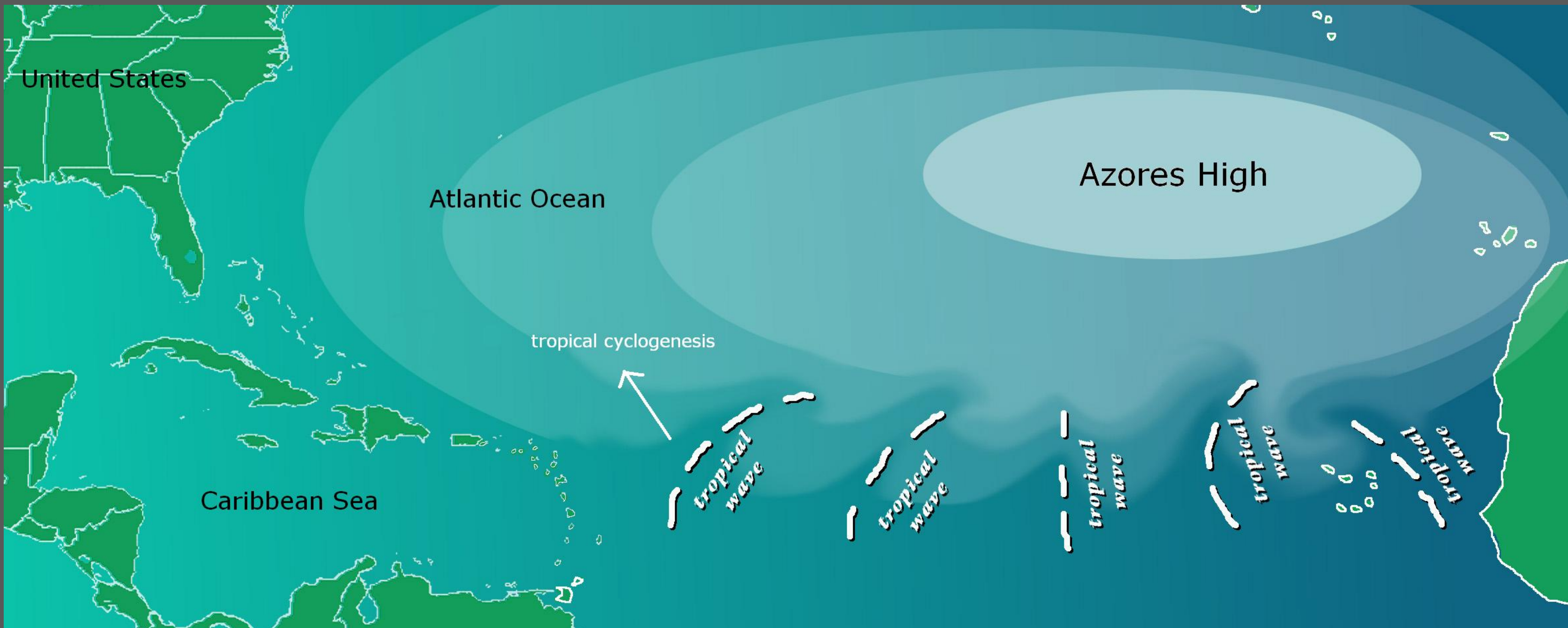
Päschke, Marschalik, Owinoh, K., JFM, **701**, 137–170 (2012)

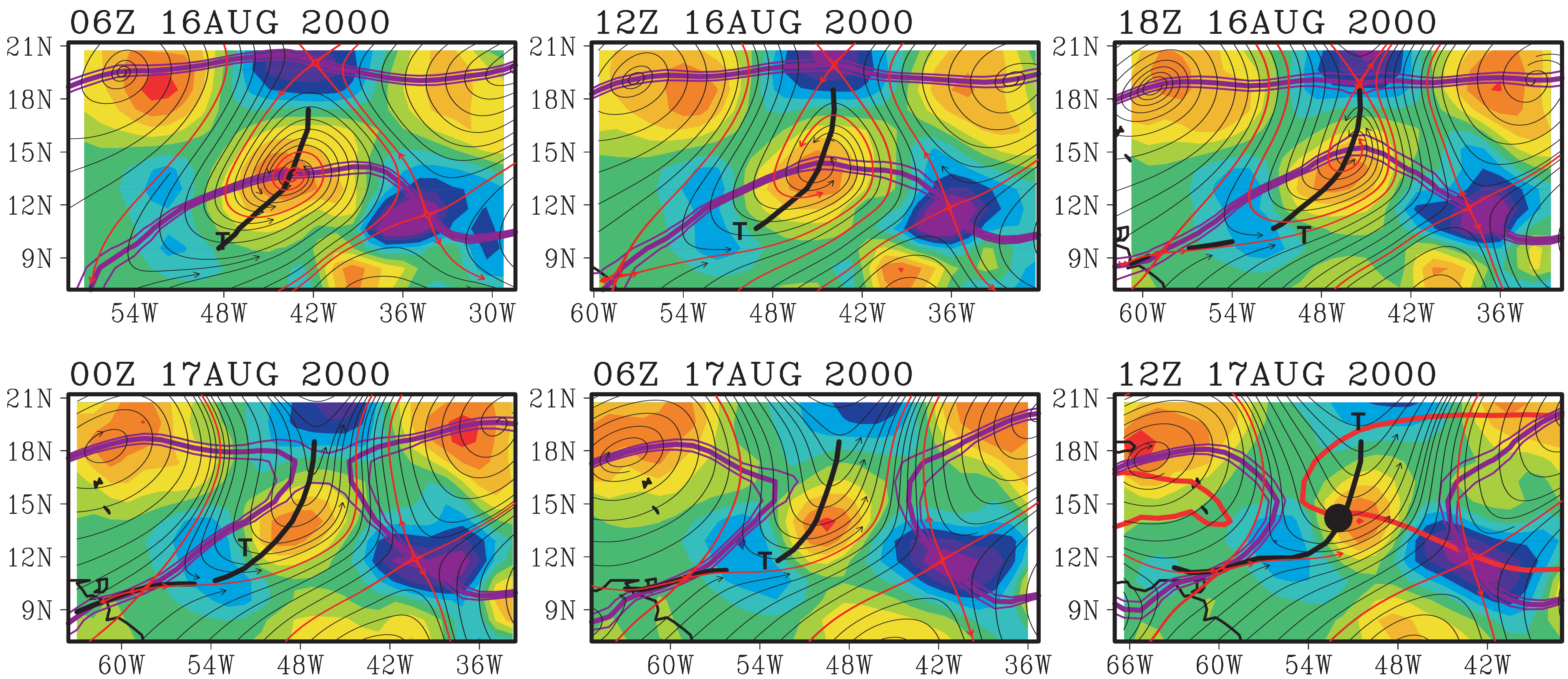
Tropical easterly african waves



http://www.aoml.noaa.gov/hrd/tcfaq/A4.html

Developing tropical storm

(streamlines in co-moving frame and Okubo-Weiss-parameter (color))



$$\mathrm{Ro} = \frac{|\boldsymbol{v}|}{fL} \sim \frac{1}{\mathbf{10}}$$

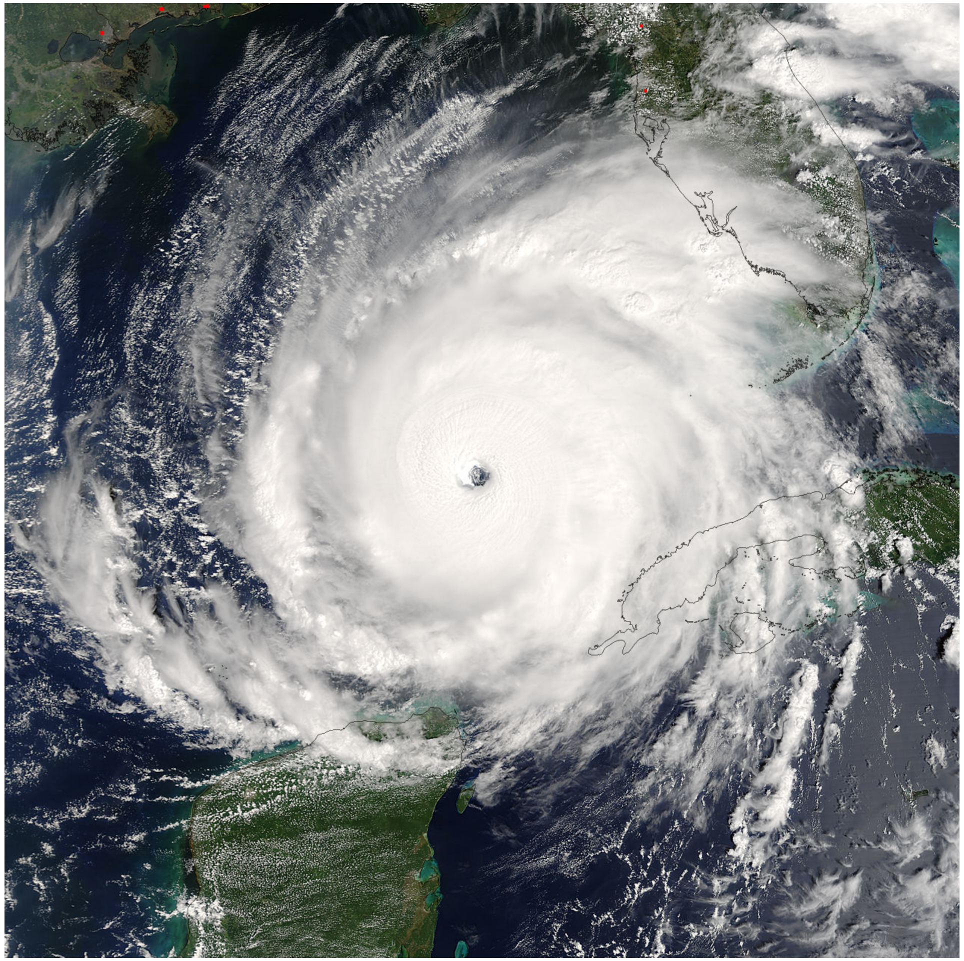
Dunkerton et al., Atmos. Chem. Phys., **9**, 5587–5646 (2009)

Developed hurricane

$$R^{*}_{\mathrm{mw}} \approx 50 \dots 200 \text{ km}$$

$$u_\theta \approx 30 \dots 60 \text{ m/s}$$

R_{mw}: radius of max. wind



Hurricane "Rita"

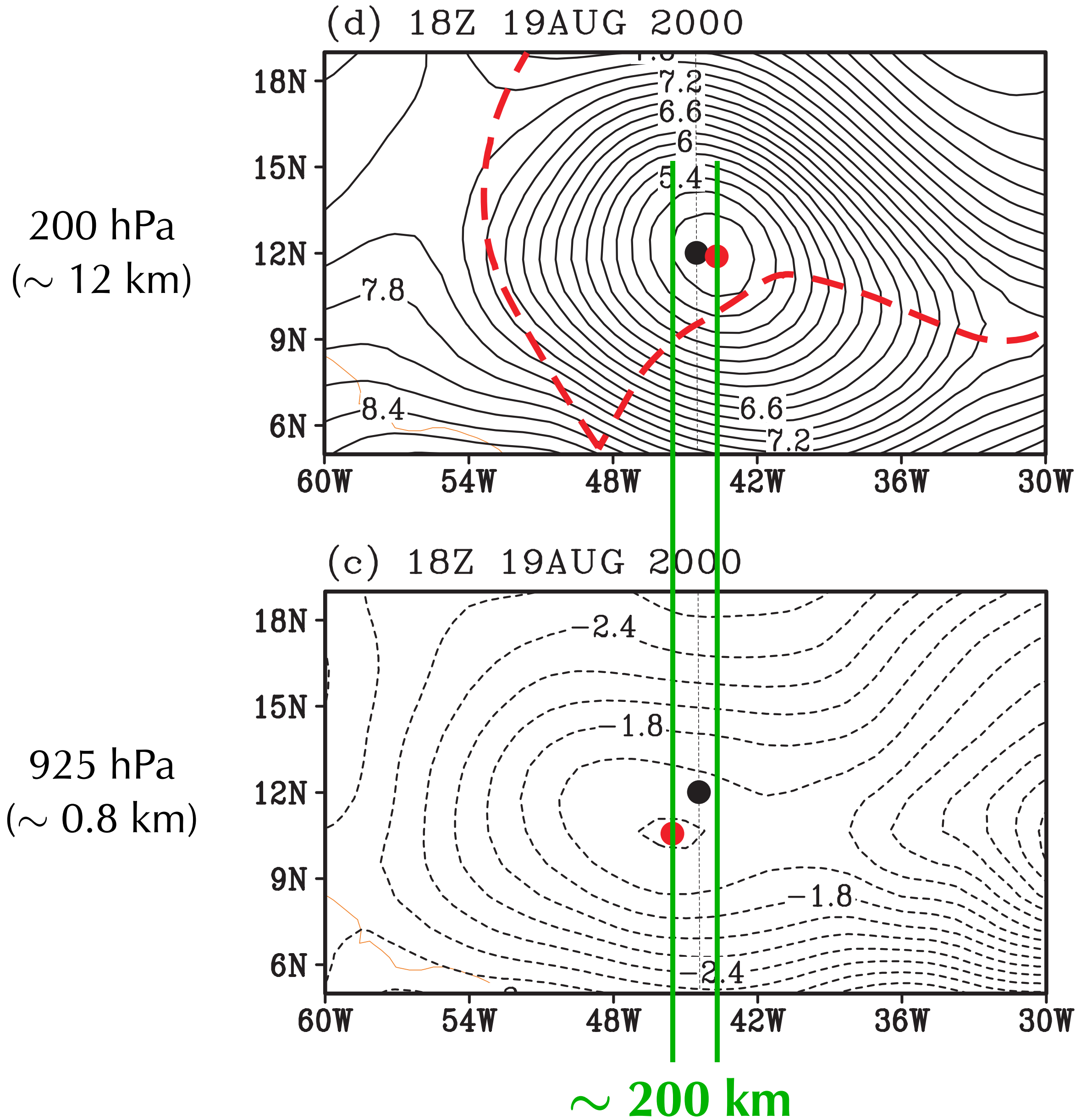
$$\mathrm{Ro} = \frac{u_{\theta,\max}}{f R_{\mathrm{mw}}} \sim \mathbf{10}$$

Radial momentum balance regimes

$- \frac{1}{\rho}\frac{\partial p}{\partial r} + f u_\theta = o(1)$	**geostrophic**	$\mathrm{Ro} \ll 1$	**typical "weather"**
$\frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + f u_\theta = o(1)$	**gradient wind**	$\mathrm{Ro} = \mathcal{O}(1)$	**tropical storm** **incipient hurricane**
$\frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} = o(1)$	**cyclostrophic**	$\mathrm{Ro} \gg 1$	**hurricane**

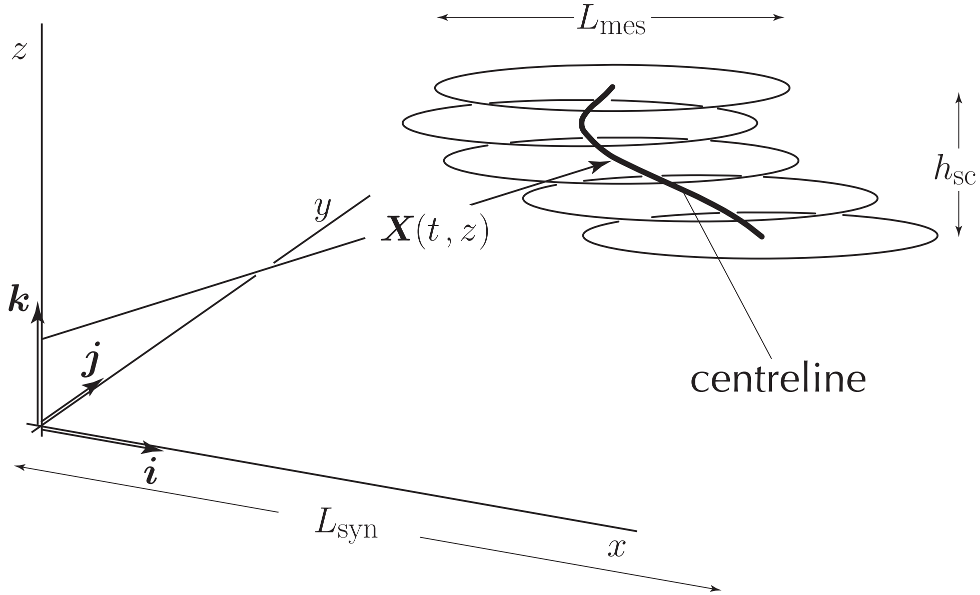
Vortex tilt in the incipient hurricane stage

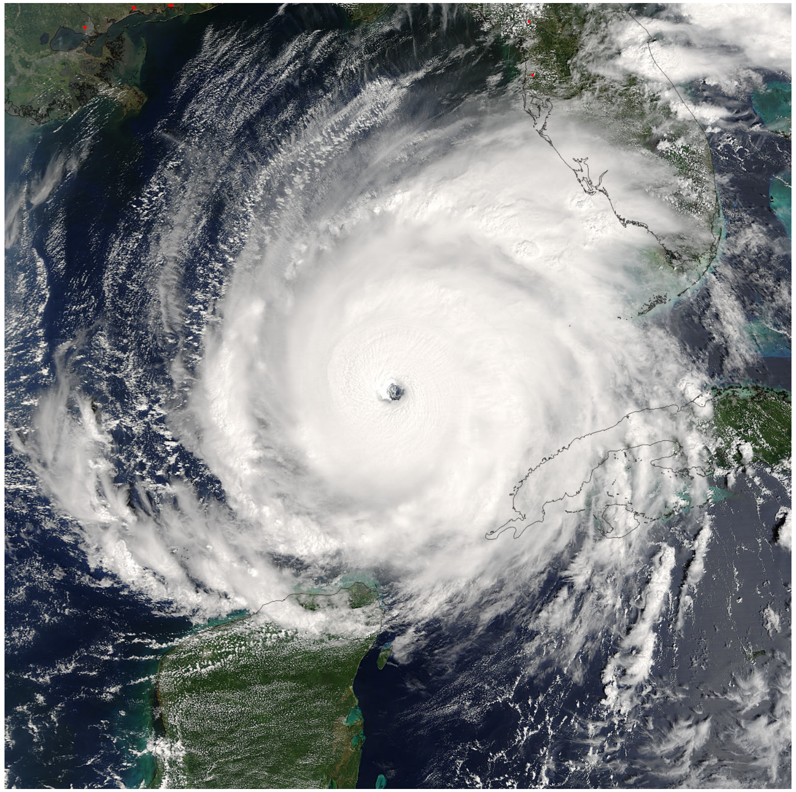
(Velocity potential)



Dunkerton et al., Atmos. Chem. Phys., **9**, 5587–5646 (2009)

Asymptotic scaling regime





$$t_{\text{syn}} = \frac{h_{\text{sc}}/u_{\text{ref}}}{\varepsilon^2}; \qquad L_{\text{syn}} = \frac{h_{\text{sc}}}{\varepsilon^2}; \qquad |\boldsymbol{v}_{\shortparallel}| = \mathcal{O}(1);$$

farfield: classical QG theory

$$|\boldsymbol{v}_{\shortparallel}|\, L = \mathcal{O}\left(\varepsilon^{-2}\right); \quad |\boldsymbol{v}_{\shortparallel}|/fL = \mathcal{O}(\varepsilon)$$

$$L_{\text{mes}} = \frac{h_{\text{sc}}}{\varepsilon^{3/2}}; \qquad |\boldsymbol{v}_{\shortparallel}| = \mathcal{O}\left(\frac{1}{\varepsilon^{1/2}}\right)$$

core: gradient wind scaling

$$|\boldsymbol{v}_{\shortparallel}|\, L = \mathcal{O}\left(\varepsilon^{-2}\right); \quad |\boldsymbol{v}_{\shortparallel}|/fL = \mathcal{O}(1)$$

Result of matched asymptotic expansion analysis:

3D Theory for

vortex motion, vortex core dynamics*,

and the role of subscale moist processes*

*** Includes strong vortex tilt**

*** Modelled by prescribed heating patterns here**

The adiabatic lifting mechanism*

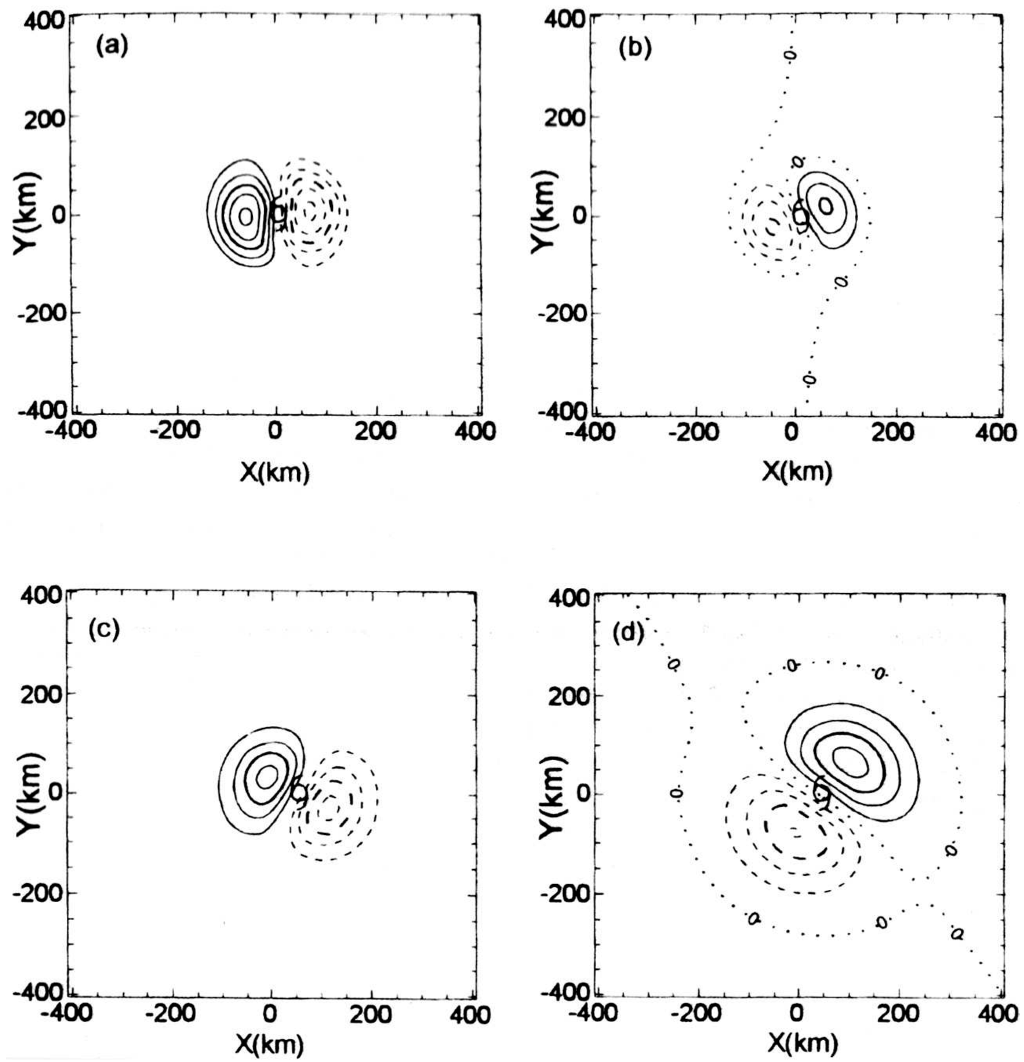


Figure 4.4: Horizontal cross-sections showing wavenumber-one vertical velocity (a) and potential temperature fields (b) after 30 min simulation; (c), (d) show the same after 6 h simulation (taken from Jones (1994), Fig. 3)

* Frank & Ritchie, Mon. Wea. Rev., **127**, 2044–2061 (1999)

The adiabatic lifting mechanism

(0th & 1st circumferential Fourier modes: $w = w_0 + w_{1\boldsymbol{1}} \cos\theta + w_{1\boldsymbol{2}} \sin\theta + \ldots$)

gradient wind balance (0th) and hydrostatics (1st) in the tilted vortex

$$\frac{1}{\overline{\rho}} \frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} + f\, u_\theta \,, \qquad \Theta_{1\boldsymbol{k}} = -\frac{1}{\overline{\rho}} \frac{\partial p}{\partial r} \frac{\partial}{\partial z} \left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} \right)_{1\boldsymbol{k}}$$

potential temperature transport (1st)

$$-(-1)^k \frac{u_\theta}{r} \Theta_{1\boldsymbol{k}^*} + w_{1k} \frac{d\overline{\Theta}}{dz} = Q_{\Theta,1\boldsymbol{k}} \qquad (\boldsymbol{k}^* = 3 - k)$$

1st-mode phase relation: vertical velocity – diabatic sources & vortex tilt

$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz} \left[Q_{\Theta,1\boldsymbol{k}} + (-1)^k \left(\boldsymbol{e}_r \cdot \frac{\partial \widehat{\boldsymbol{X}}}{\partial z} \right)_{1\boldsymbol{k}^*} \frac{u_\theta}{r} \left(\frac{u_\theta^2}{r} + f\, u_\theta \right) \right]$$

Spin-up by asynchronous heating

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = - \boldsymbol{u_{r,*}} \left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle w \frac{\partial}{\partial z} \left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right) \right\rangle_\theta$$

$$\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} \;=\; \widehat{X} \cos\theta + \widehat{Y} \sin\theta$$

$$w_{1\boldsymbol{k}} \;=\; \frac{1}{d\overline{\Theta}/dz} \left[Q_{\Theta,1\boldsymbol{k}} + \frac{\partial}{\partial z} \left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right) \frac{u_\theta}{r} \left(\frac{u_\theta^2}{r} + f\, u_\theta\right)\right]$$

Spin-up by asynchronous heating

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\,\boldsymbol{u_{r,*}}\left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle w\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\frac{1}{d\overline{\Theta}/dz}\left(Q_{\Theta,1\mathbf{1}}\frac{\partial \widehat{X}}{\partial z} + Q_{\Theta,1\mathbf{2}}\frac{\partial \widehat{Y}}{\partial z}\right)} \qquad \textbf{!!}$$

$$\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} = \widehat{X}\cos\theta + \widehat{Y}\sin\theta$$

$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[Q_{\Theta,1\boldsymbol{k}} + \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)\frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

Vortex theory

Achieved:

Large displacement, nonlinear theory

for core dynamics and motion of

concentrated atmospheric vortices

with

simple spin-up criterion w.r.t. asymmetric heating

Todo: $\Rightarrow$ **CRC 1114**

- corroboration against 3D simulations
- realistic Q_Θ via moist-air thermodynamics
- boundary layer analysis
- large vortex Rossby number theory for full-fledged hurricanes

Päschke, Marschalik, Owinoh, K., JFM, **701**, 137–170 (2012)

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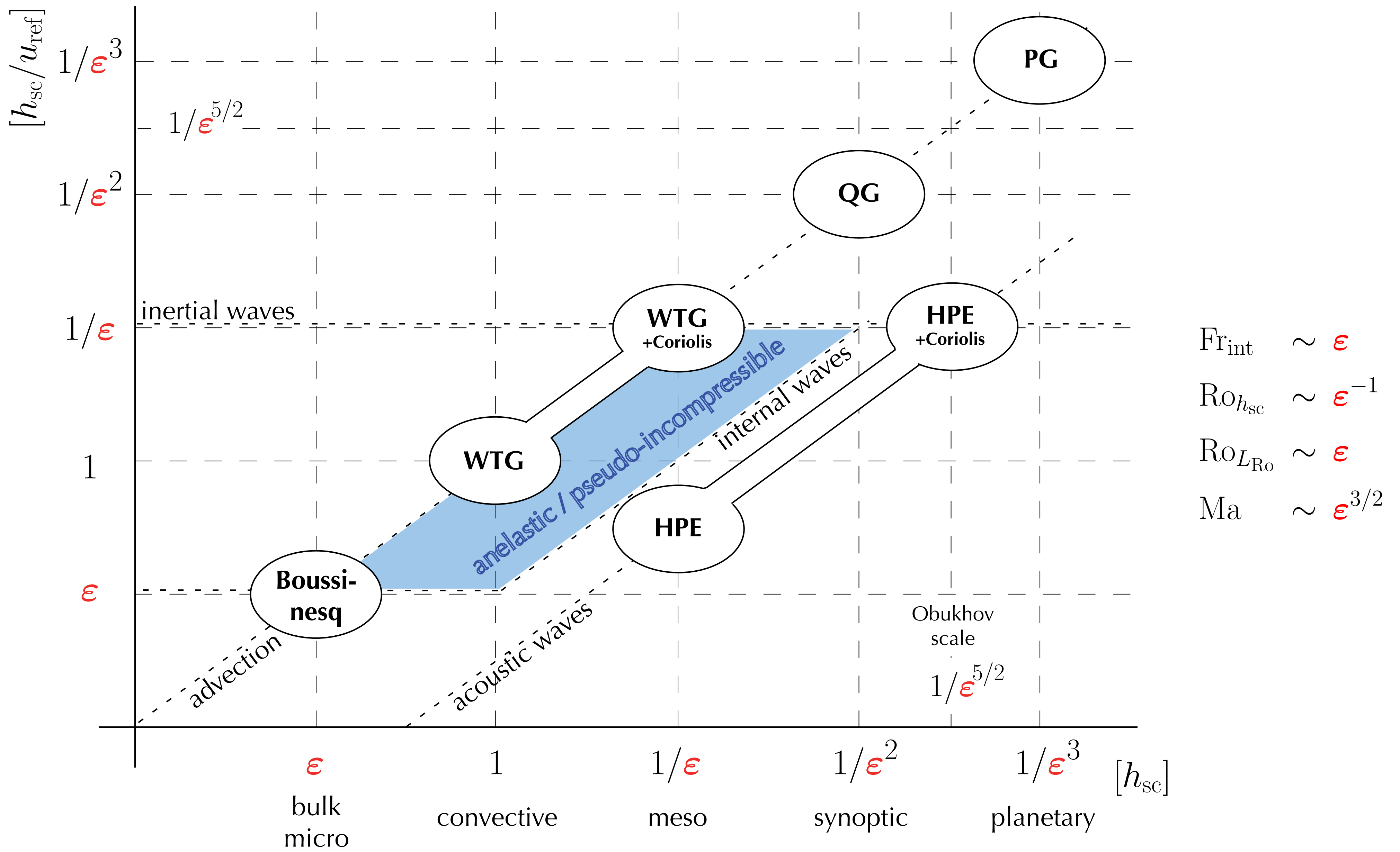
Atmospheric Flow Regimes



R.K., Ann. Rev. Fluid Mech, **42**, 249–274 (2010)

Sound-Proof Models

Compressible & sound-proof flow equations

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi = 0$$

$$(\rho w)_t + \nabla \cdot (\rho \boldsymbol{v} w) + P \pi_z = -\rho g$$

$$P_t + \nabla \cdot (P \boldsymbol{v}) = 0$$

drop term for:

anelastic[†] (approx.)

pseudo-incompressible*

hydrostatic-primitive

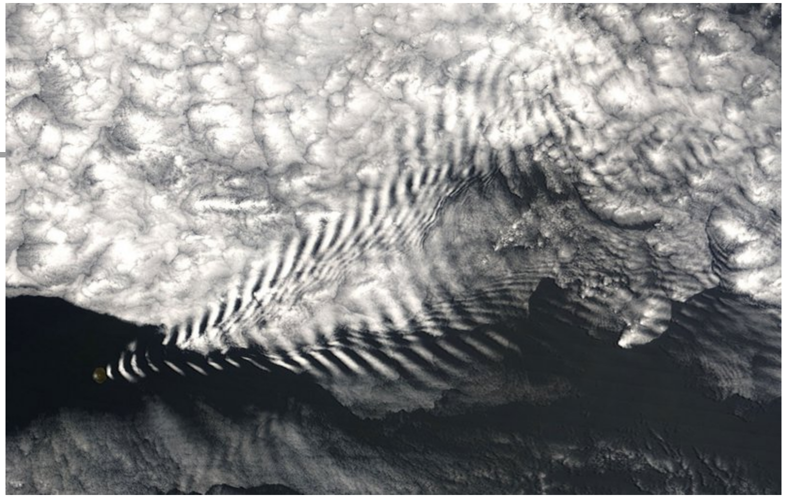
$$P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)$$

Parameter range & length and time scales of asymptotic validity ?

[†] e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

* Durran, JAS, **46**, 1453–1461 (1988)

Sound-Proof Models



Compressible & sound-proof flow equations

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drop term for:

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Parameter range & length and time scales of asymptotic validity ?

† e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

* Durran, JAS, **46**, 1453–1461 (1988)

From here on: ε is the Mach number

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon} \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}} \dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\mathrm{ref}}}{h_{\mathrm{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{u_{\mathrm{ref}}}\sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}}{h_{\mathrm{sc}}} = \dfrac{\sqrt{g h_{\mathrm{sc}}}}{h_{\mathrm{sc}}}$	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{u_{\mathrm{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\mathrm{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips, JAS, **19**, 173–179 (1962)

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
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sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \qquad \Rightarrow \qquad \boldsymbol{\Delta\overline{\theta}\,\big|_{z=0}^{h_{\text{sc}}} < 1\ \mathrm{K}}$$

* Ogura & Phillips, JAS, **19**, 173–179 (1962)

Design Regime (10 km / 20 min)

Characteristic (inverse) time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\mathrm{ref}}}{h_{\mathrm{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{gh_{\mathrm{sc}}}}{u_{\mathrm{ref}}}\sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}}{h_{\mathrm{sc}}} = \dfrac{\sqrt{gh_{\mathrm{sc}}}}{h_{\mathrm{sc}}}$	$\dfrac{\sqrt{gh_{\mathrm{sc}}}}{u_{\mathrm{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\mathrm{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Design Regime (10 km / 20 min)

$$
\begin{aligned}
\tilde{\theta}_\tau \; + \; \frac{1}{\varepsilon^{\nu}}\, \tilde{w}\, \frac{d\widehat{\theta}}{dz} \qquad\qquad\qquad\qquad\qquad\qquad & = \; -\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau \; + \; \frac{1}{\varepsilon^{\nu}}\, \frac{\tilde{\theta}}{\overline{\overline{\theta}}}\, \boldsymbol{k} \quad + \; \frac{1}{\varepsilon}\, \overline{\theta} \nabla \tilde{\pi} \qquad\qquad\qquad & = \; -\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\boldsymbol{v}} - \varepsilon^{1-\nu} \tilde{\theta} \nabla \tilde{\pi} \;\; \cdot \\
\tilde{\pi}_\tau \qquad\qquad\qquad\quad + \; \frac{1}{\varepsilon} \left(\gamma \Gamma \overline{\pi} \nabla \cdot \tilde{\boldsymbol{v}} + \tilde{w} \frac{d\overline{\pi}}{dz} \right) & = \; -\tilde{\boldsymbol{v}} \cdot \nabla \tilde{\pi} - \gamma \Gamma \tilde{\pi} \nabla \cdot \tilde{\boldsymbol{v}}
\end{aligned}
$$

Key question:

Time scale of validity of sound-proof models
for internal waves ?

Design Regime (10 km / 20 min)

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_\vartheta + \tilde{w}\,\frac{d\overline{\theta}}{dz} = 0$$

$$\tilde{\boldsymbol{v}}_\vartheta + \frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \overline{\theta}\nabla\pi^* = 0$$

$$\varepsilon^\mu\,\pi^*_\vartheta + \left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^* \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^* \\ \boldsymbol{U}^* \\ W^* \\ \Pi^* \end{pmatrix}(z)\ \exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Design Regime (10 km / 20 min)

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\overline{\theta}\,\overline{P}}\frac{dW^*}{dz}\right)+\frac{\lambda^2}{\overline{\theta}\,\overline{P}}W^* = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}\,\overline{P}}W^*$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)\right)$

- pseudo-incompressible modes/EVals $=$ compressible modes/EVals $+\ O(\varepsilon^{\mu})$ †
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

The anelastic and pseudo-incompressible models remain relevant for stratifications

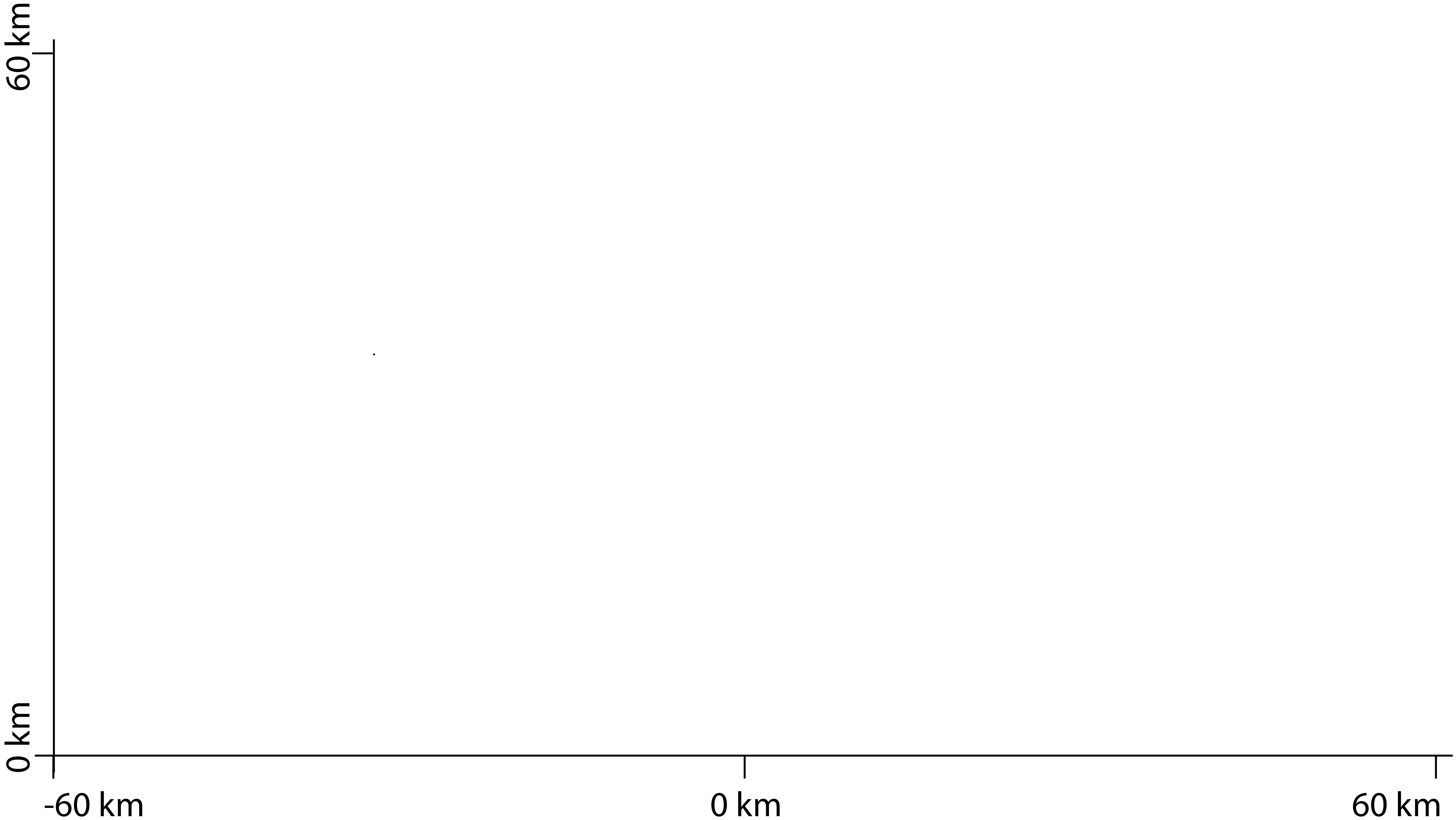
$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{sc}} \lesssim 40\ \mathrm{K}$$

not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

† rigorous proof with D. Bresch

K., Achatz, Bresch, Knio, Smolarkiewicz, JAS, **67**, 3226–3237 (2010)

Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))

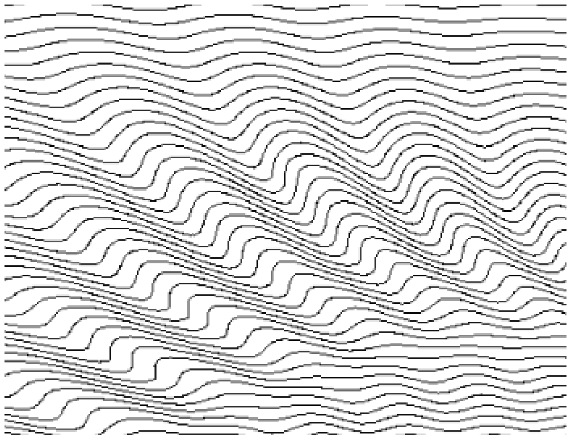


Potential temperature contours

Wave breaking regime

Large amplitude WKB theory:

- short wavelength wave packets
- modulated over $\sim 10 \text{ km}$ distances
- ***θ*–stratification of order** $O(1)$
- scalings allow for overturning of θ-contours



Expansion scheme:

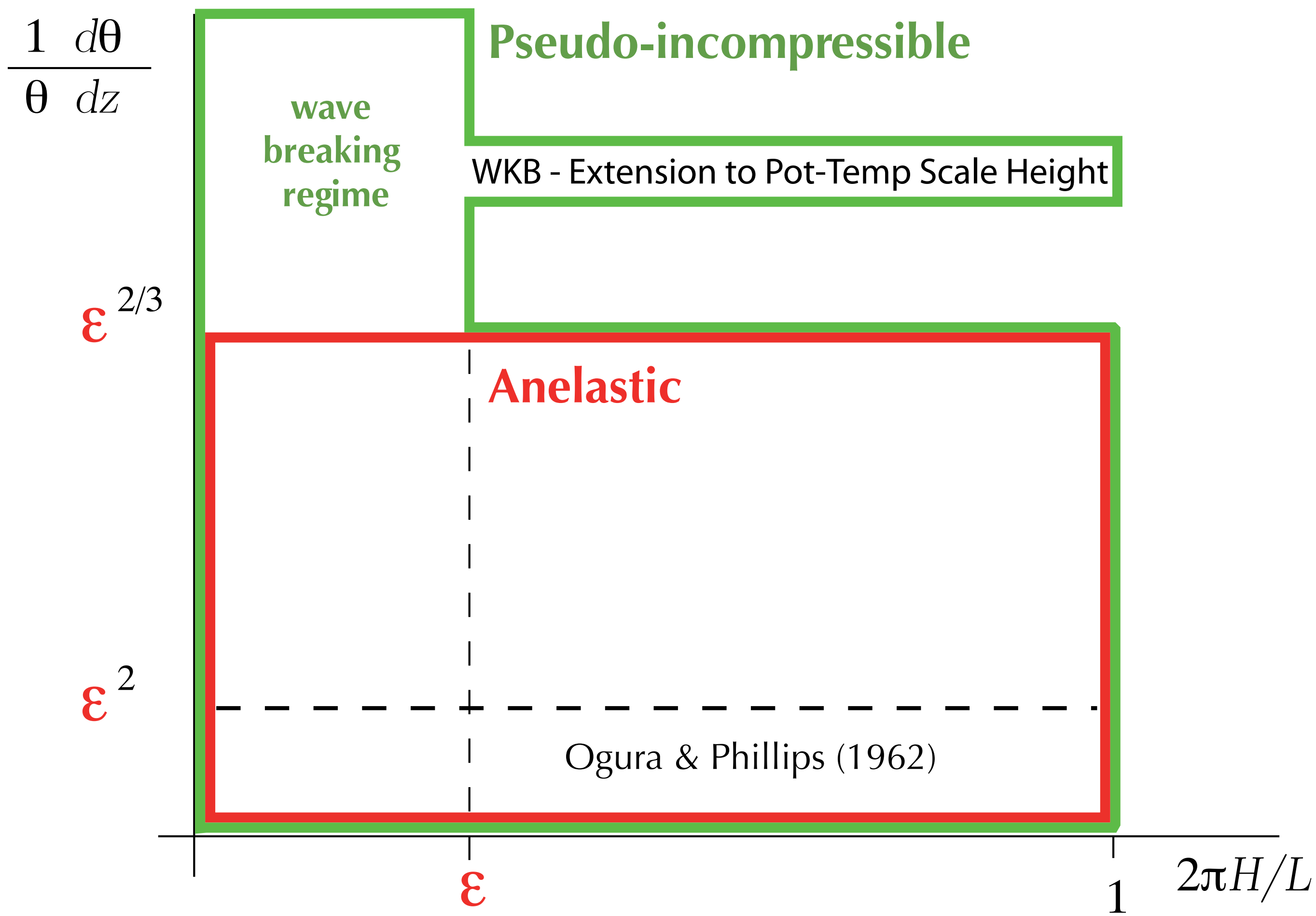
$$U(t, \boldsymbol{x}, z; \varepsilon) = \underline{\overline{U}(z) + U_1^{(0)} \exp\left(i \frac{\varphi^\varepsilon}{\varepsilon}\right)} + \varepsilon \sum_{n=0}^{2} U_n^{(1)} \exp\left(in \frac{\varphi^\varepsilon}{\varepsilon}\right)$$

$$\varphi^\varepsilon = \varphi^{(0)} + \varepsilon \varphi^{(1)} + o(\varepsilon)$$

$$\left(U_n^{(i)}, \varphi^{(i)}\right) \equiv \left(U_n^{(i)}, \varphi^{(i)}\right)(t, \boldsymbol{x}, z)$$

Achatz, Senf, K., JFM, **663**, 120?-147, 2010

Regimes ... Summary



The pseudo-incompressible model wins marginally

Asymptotic modelling framework

Motion and structure of atmospheric vortices

Sound-proof flow models

Clouds and waves

Bulk microphysics closure

Mass, momentum, energy equations

$$
\begin{aligned}
\rho_t &\;+\; \nabla_\parallel \cdot (\rho \boldsymbol{u}) + (\rho w)_z && = \; 0 \\
\boldsymbol{u}_t &\;+\; \boldsymbol{u} \cdot \nabla_\parallel \boldsymbol{u} + w \boldsymbol{u}_z + \varepsilon\, (\boldsymbol{\Omega} \times \boldsymbol{v})_\parallel + \frac{1}{\varepsilon^4 \rho} \nabla_\parallel p && = \; \boldsymbol{D}_{\boldsymbol{u}} \\
w_t &\;+\; \boldsymbol{u} \cdot \nabla_\parallel w + w w_z + \varepsilon\, (\boldsymbol{\Omega} \times \boldsymbol{v})_\perp + \frac{1}{\varepsilon^4 \rho} p_z && = \; D_w \; - \; \frac{1}{\varepsilon^4} \\
\theta_t &\;+\; \boldsymbol{u} \cdot \nabla_\parallel \theta + w \theta_z && = \; D_\theta \; + \; \boldsymbol{S}_{\boldsymbol{\theta}}
\end{aligned}
$$

$$
\theta = \frac{p^{1/\gamma}}{\rho}\,, \qquad \frac{\gamma - 1}{\gamma} = \varepsilon \Gamma^*
$$

$$
\boldsymbol{S}_{\boldsymbol{\theta}} = \hat{S}_\theta + S_\theta^q\,, \qquad S_\theta^q = \varepsilon^2\, \frac{\theta}{p}\, \Gamma^* L^* \left(C_{\mathrm{d}} - C_{\mathrm{ev}}\right).
$$

K., Majda, TCFD, **20**, 525–552, (2006); K., Ann. Rev. Fluid Mech., **42** (2010); Hittmeir, K., TCFD, in press (2017)

Bulk microphysics closure (Kessler, ...)

Moisture balances

$$
\begin{aligned}
q_{\mathrm{v},t} &+ \boldsymbol{u}\cdot\nabla_{\parallel} q_{\mathrm{v}} + w q_{\mathrm{v},z} && = (C_{\mathrm{ev}} - C_{\mathrm{d}}) && + D_{q_{\mathrm{v}}} \\
q_{\mathrm{c},t} &+ \boldsymbol{u}\cdot\nabla_{\parallel} q_{\mathrm{c}} + w q_{\mathrm{c},z} && = (C_{\mathrm{d}} - C_{\mathrm{cr}} - C_{\mathrm{ac}}) && + D_{q_{\mathrm{c}}} \\
q_{\mathrm{r},t} &+ \boldsymbol{u}\cdot\nabla_{\parallel} q_{\mathrm{r}} + w q_{\mathrm{r},z} + \tfrac{1}{\rho}(\rho q_{\mathrm{r}} V_{\mathrm{T}})_z && = (C_{\mathrm{ac}} + C_{\mathrm{cr}} - C_{\mathrm{ev}}) && + D_{q_{\mathrm{r}}}
\end{aligned}
$$

$$
\begin{aligned}
C_{\mathrm{ev}} &= C_{\mathrm{ev}}^{*}\,\frac{p}{\rho}\,(\boldsymbol{q}_{\mathbf{vs}} - q_{\mathrm{v}})\; q_{\mathrm{r}}^{1/2+\delta^{*}}\; H_{>}(q_{\mathrm{r}}) \\
C_{\mathrm{d}} &= \frac{C_{\mathrm{d}}^{*}}{\varepsilon^{\boldsymbol{n}}}\,(q_{\mathrm{v}} - \boldsymbol{q}_{\mathbf{vs}})\,(q_{\mathrm{c}} + q_{\mathrm{c}n}^{*})\; H_{*}(q_{\mathrm{c}}, q_{\mathrm{v}}, q_{\mathrm{vs}}) \qquad (\boldsymbol{n} \gg 1) \\
C_{\mathrm{cr}} &= \frac{C_{\mathrm{cr}}^{*}}{\varepsilon}\, q_{\mathrm{c}} q_{\mathrm{r}}^{(1+\alpha^{*})} \\
C_{\mathrm{ac}} &= C_{\mathrm{ac}}^{*}\, \max\,(0, q_{\mathrm{c}} - q_{\mathrm{c}}^{*})
\end{aligned}
$$

Bulk microphysics closure

Saturation vapor mixing ratio*

$$q_{\mathrm{vs}}(\theta, p) = q_{\mathrm{vs}}^{*} \; \exp\left(\frac{A^{*}}{\varepsilon} \, \frac{T(\theta, p) - 1}{T(\theta, p) - \varepsilon T_1^{*}}\right) .$$

Temperature

$$T(\theta, p) = \theta \, p^{\varepsilon \Gamma^{*}} .$$

* K. Emanuel, *Atmospheric Convection*, Oxford University Press, (1994),
slightly modified and scaled in terms of ε

Clouds and internal waves

Columnar clouds / internal wave time scales*

general expansion scheme

$$\mathbf{U}(\boldsymbol{x}, z, t; \varepsilon) = \sum_i \varepsilon^i \mathbf{U}^{(i)} (\boldsymbol{\eta}, \boldsymbol{x}, z, \tau)$$

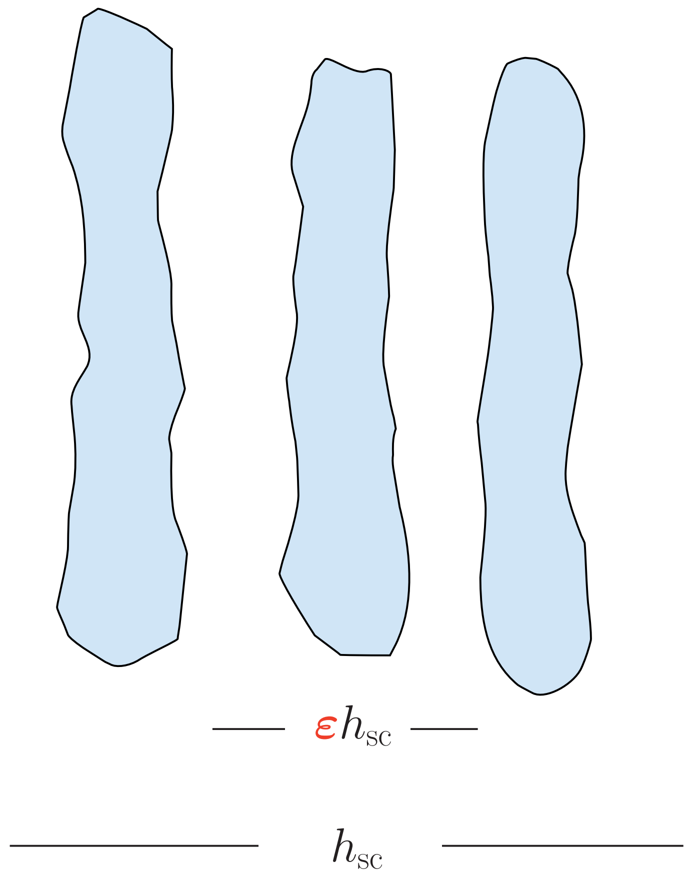
horizontal velocity scaling

$$\boldsymbol{u}^{(0)}(\boldsymbol{\eta}, \boldsymbol{x}, z, \tau) \equiv \boldsymbol{u}(\boldsymbol{x}, z, \tau)$$

$$\boldsymbol{\eta} = \boldsymbol{x}/\varepsilon$$

$$\tau = t/\varepsilon$$

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{h_{\rm sc}}, \qquad t = \frac{t' u_{\rm ref}}{h_{\rm sc}}$$



*Klein & Majda, TCFD, **20**, 525–552, (2006)

Clouds and internal waves

Convective scale

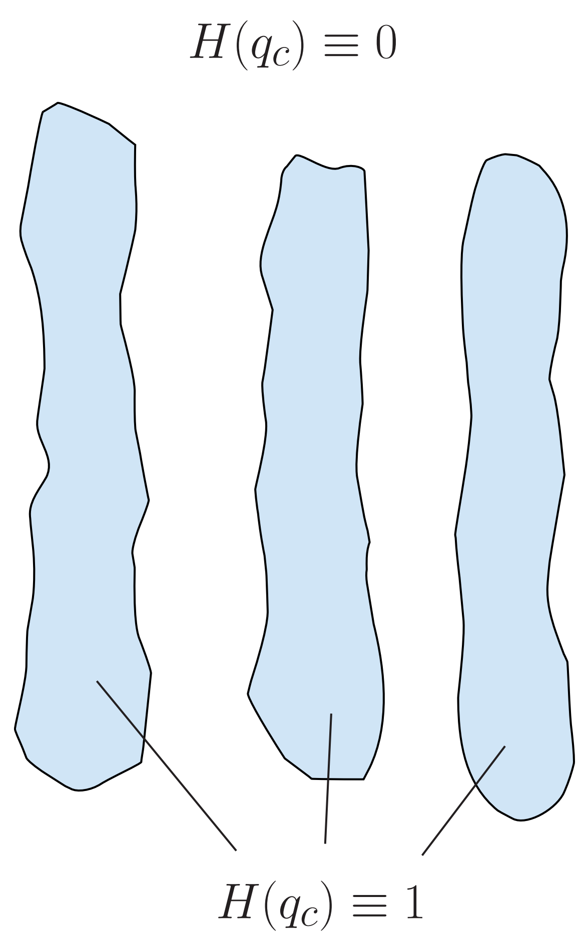
$$
\begin{aligned}
\boldsymbol{u}_\tau + \nabla_{\boldsymbol{x}} \pi &= 0 \\
\overline{w}_\tau + \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau + \overline{w} N^2 &= \frac{\Gamma L^{**}}{p_0} \overline{\boldsymbol{C}} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + \left(\rho_0 \overline{w}\right)_z &= 0
\end{aligned}
$$

Cloud column scale

$$
\begin{aligned}
\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) \widetilde{w} &= \widetilde{\theta} \\
\left(\partial_\tau + \boldsymbol{u}^{(0)} \cdot \nabla_{\boldsymbol{\eta}}\right) \theta + \widetilde{w} N^2 &= \frac{\Gamma L^{**}}{p_0} \widetilde{\boldsymbol{C}} \,.
\end{aligned}
$$

Moisture coupling

$$
\boldsymbol{C} = H(q_\mathrm{c})\, \boldsymbol{C}_\mathrm{d} + \left[1 - H(q_\mathrm{c})\right] \boldsymbol{C}_\mathrm{ev}
$$



After averaging over the small scales ...

Clouds and internal waves

Closed coupled micro-macro dynamics on convective scales

$$
\begin{aligned}
\boldsymbol{u}_\tau \;+ \nabla_{\boldsymbol{x}} \pi &= 0 \\
\overline{w}_\tau \;+ \pi_z &= \overline{\theta} \\
\overline{\theta}_\tau \;+ (1 - \boldsymbol{\sigma}) \overline{w} N^2 &= \boldsymbol{w'} N^2 - \overline{C} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + (\rho_0 \overline{w})_z &= 0
\end{aligned}
$$

$$
\begin{aligned}
w'_\tau &= \theta' \\
\theta'_\tau \;+ \sigma w' N^2 &= \sigma (1 - \sigma) \overline{\boldsymbol{w}} N^2 + \sigma \overline{C} \,.
\end{aligned}
$$

where

$\sigma(\boldsymbol{x}, z), \overline{C}(\boldsymbol{x}, z), N(z)$ are prescribed

Closed coupled micro-macro dynamics on convective scales (with mean advection)

$$
\begin{aligned}
D_\tau \, \boldsymbol{u} + \nabla_{\boldsymbol{x}} \pi &= 0 \\
D_\tau \, \overline{w} + \pi_z &= \overline{\theta} \\
D_\tau \, \overline{\theta} + (1 - \boldsymbol{\sigma}) \overline{w} N^2 &= \boldsymbol{w'} N^2 - \overline{C} \\
\rho_0 \nabla_{\boldsymbol{x}} \cdot \boldsymbol{u} + (\rho_0 \overline{w})_z &= 0
\end{aligned}
$$

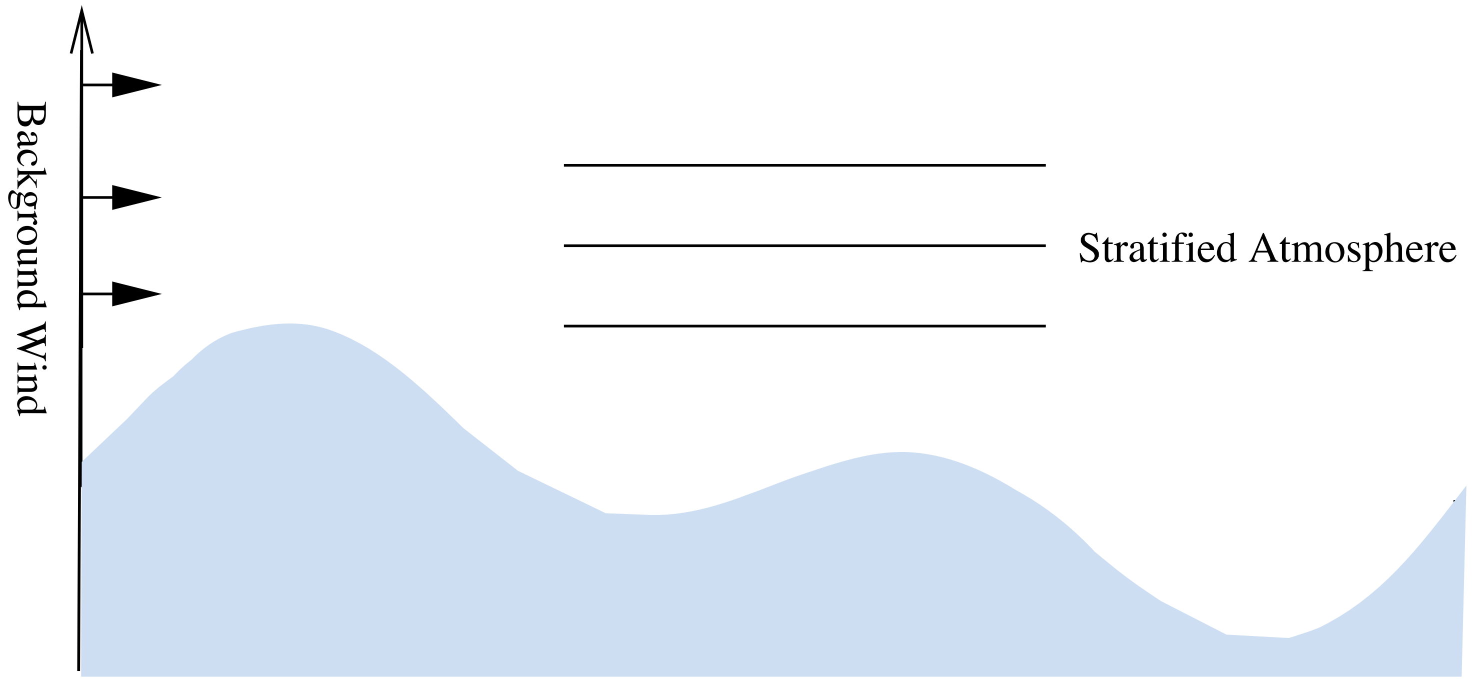
$$
\begin{aligned}
D_\tau \, w' &= \theta' \\
D_\tau \, \theta' + \sigma w' \, N^2 &= \sigma (1 - \sigma) \overline{\boldsymbol{w}} N^2 + \sigma \overline{C} \, .
\end{aligned}
$$

where

$D_\tau = \partial_\tau + \boldsymbol{u}^\infty \cdot \nabla_{\boldsymbol{x}}$ and $\sigma(\boldsymbol{x}, z), \overline{C}(\boldsymbol{x}, z), N(z)$ are prescribed

Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

Clouds may narrow the spectrum of lee waves



without cloud

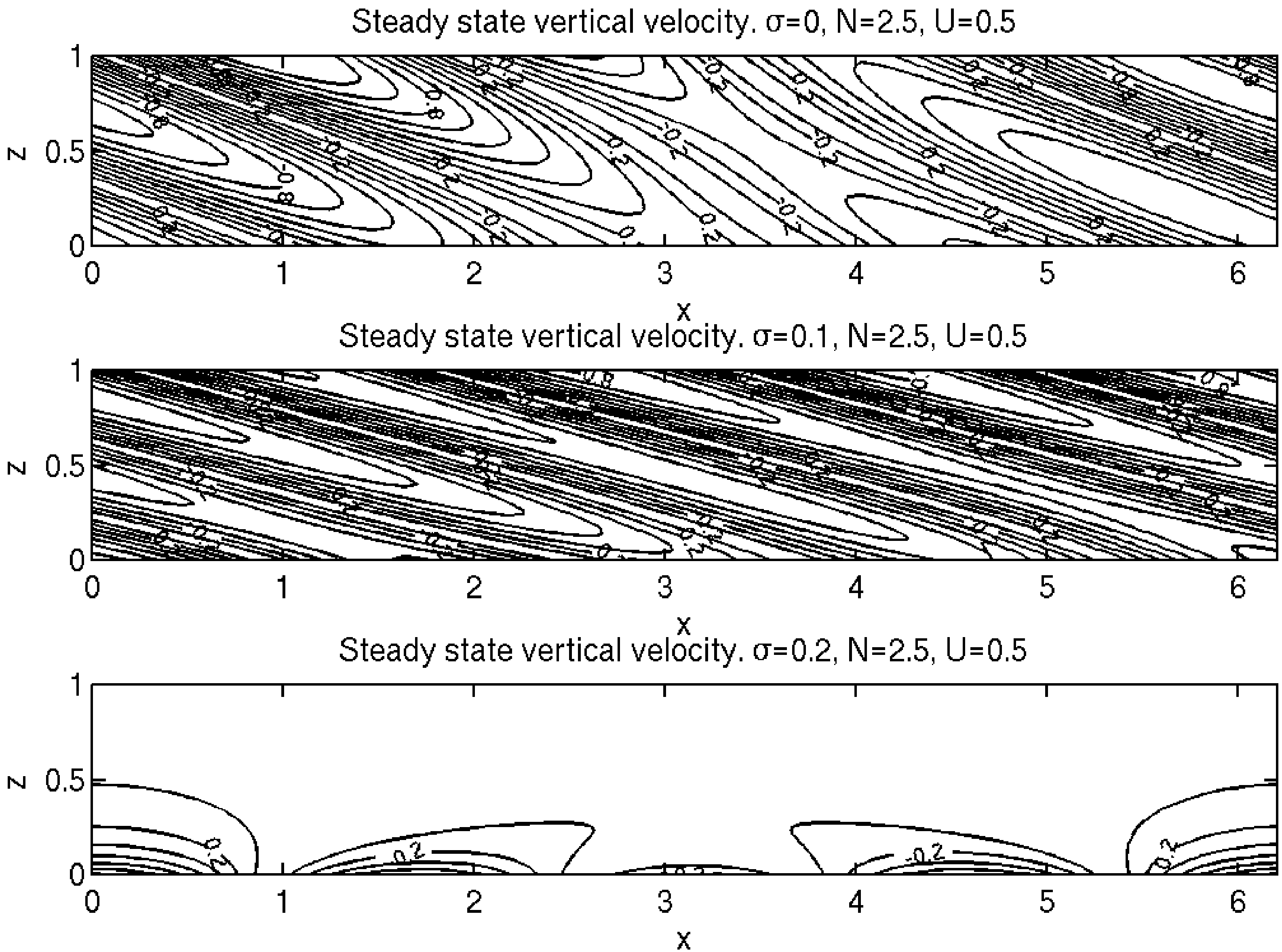
$$k_{\mathrm{up}} = \frac{N}{u^{\infty}}$$

with cloud

$$k_{\mathrm{up}} = \frac{N}{u^{\infty}} \quad \text{and} \quad \boldsymbol{k_{\mathrm{low}} = \sqrt{\sigma}\,\frac{N}{u^{\infty}}}$$

Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

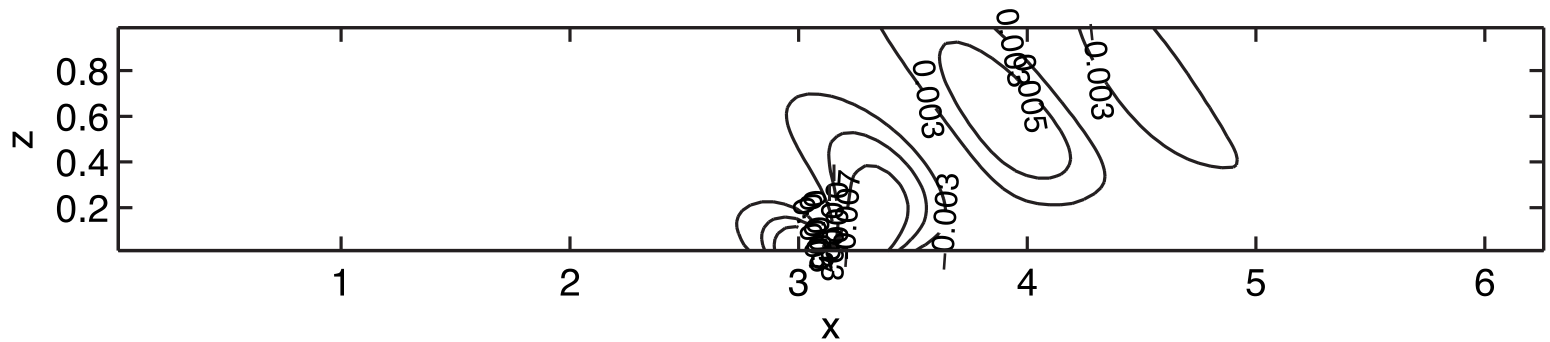
Lee waves over $sin(x) + sin(2x)$–topography



Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

Cloud meets lee wave

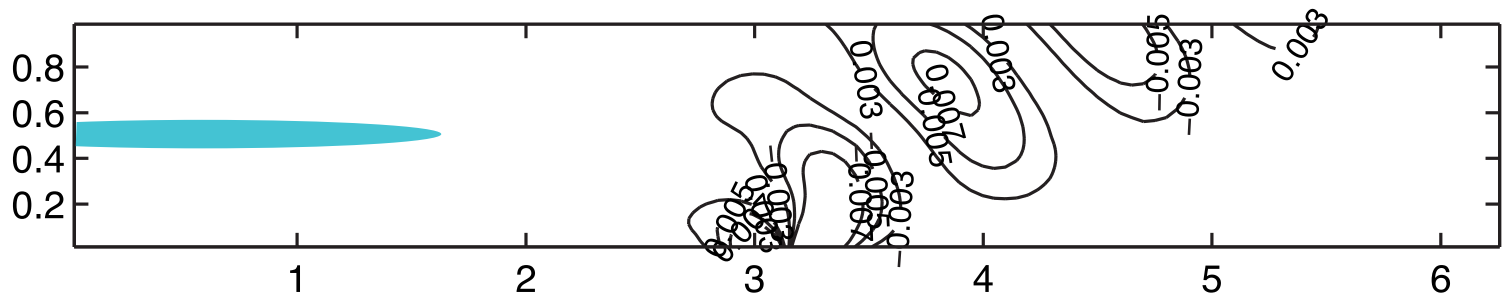
Vertical velocity at t = 5.0, U = 0.5



Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

Cloud meets lee wave

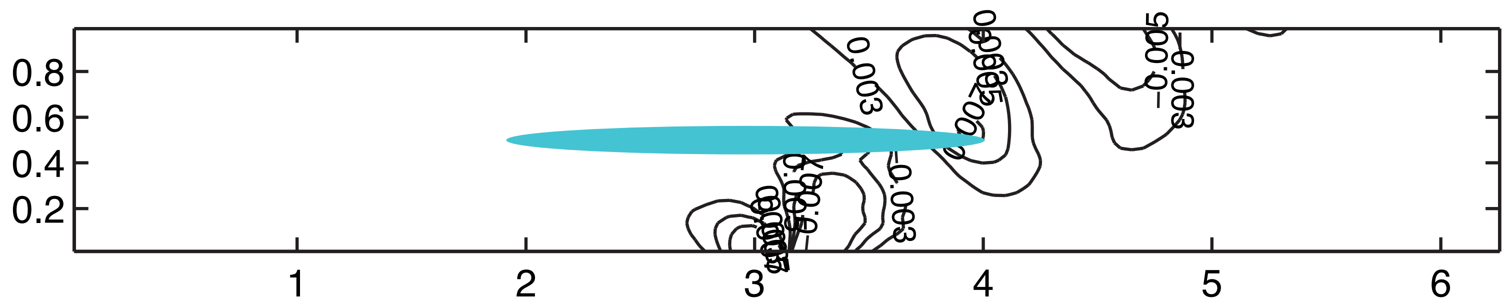
Vertical velocity at t = 10.0, U = 0.5



Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

Cloud meets lee wave

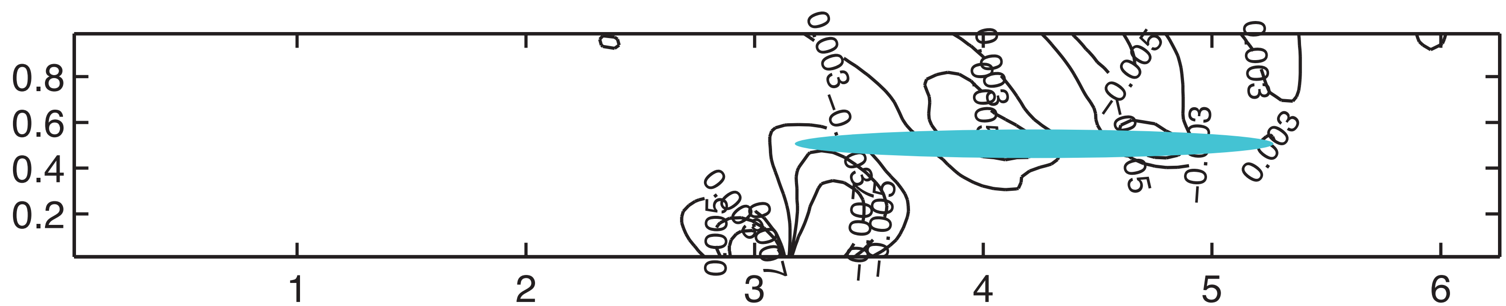
Vertical velocity at t = 15.0, U = 0.5



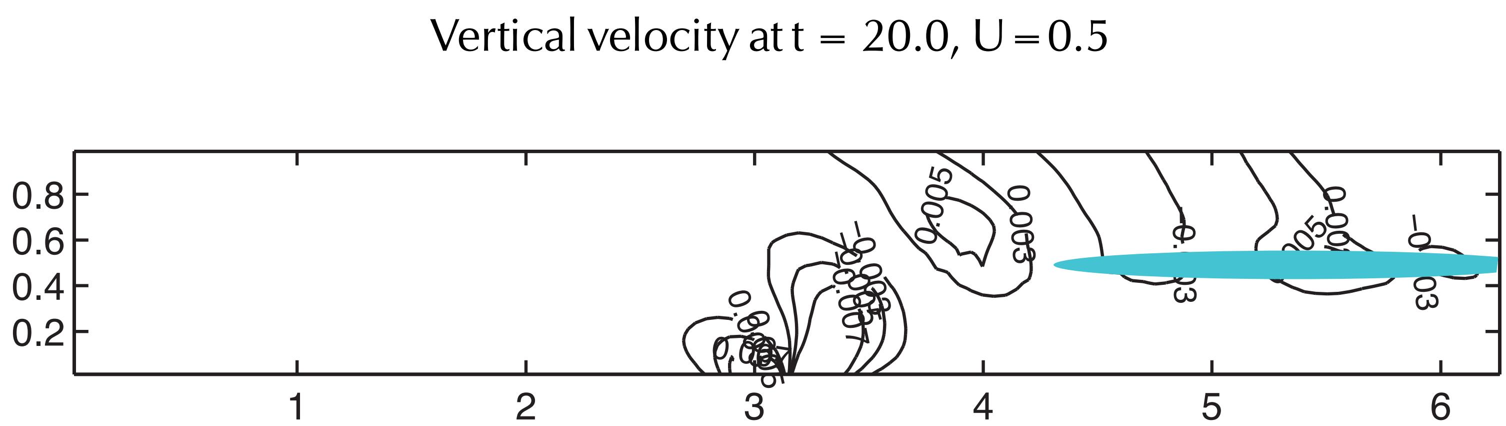
Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

Cloud meets lee wave

Vertical velocity at t = 17.5, U = 0.5

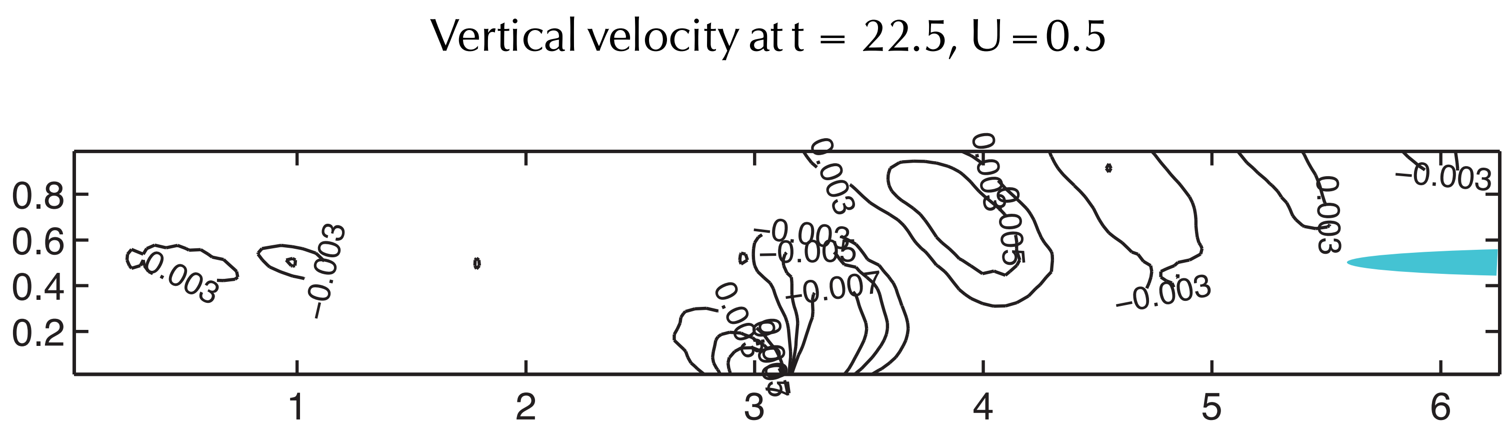


Cloud meets lee wave



Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

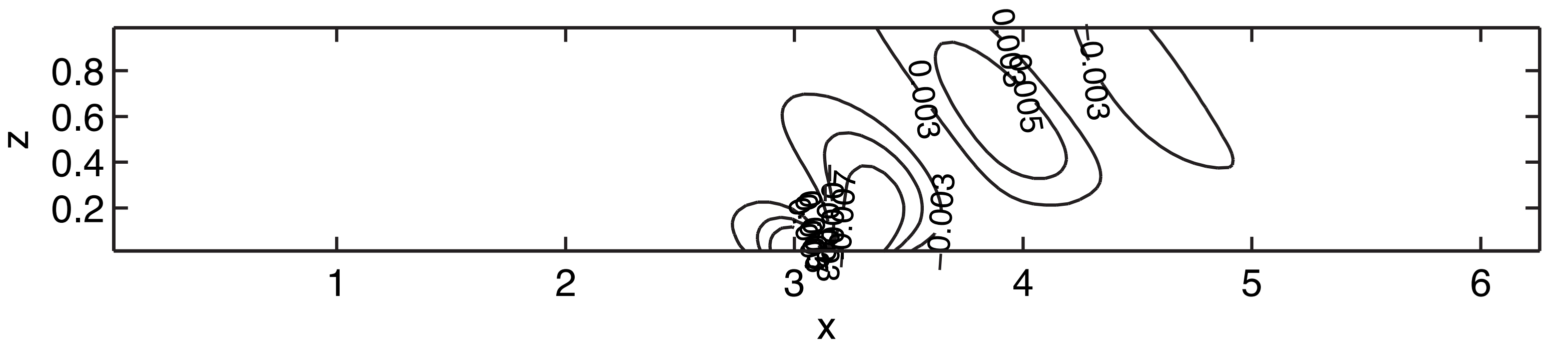
Cloud meets lee wave



Ruprecht et al., J. Atmos. Sci., **67**, 2504–2519, (2010)

Cloud meets lee wave

Vertical velocity at t = 5.0, U = 0.5



From cumuli zu planetary waves:
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Conclusions

Asymptotic modelling framework

Motion and structure of atmospheric vortices

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