3. Lecture III: Stable ergodicity of skew extensions

Let f be a diffeomorphism of the compact Riemannian manifold M. We say f is *partially hyperbolic* [9] if there is a continuous Tf-invariant splitting

$$TM = \mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s$$

such that Tf expands $\mathbb{E}^u$, contracts $\mathbb{E}^s$ and

$$\sup \|T_p^s f\| < \inf m(T_p^c f), \ \ \sup \|T_p^c f\| < \inf m(T_p^u f).$$

(Here $m(A) = \|A^{-1}\|^{-1}$.)

If the *center bundle* $\mathbb{E}^c$ is tangent to a C^1-foliation $\mathcal{F}$ of M, then partial hyperbolicity can be thought of as hyperbolicity *transverse* to the foliation $\mathcal{F}$.

Grayson, Pugh & Shub [27] have suggested that partial hyperbolicity provides a natural setting for *stable ergodicity*. For example, they proved that the time one-map of the geodesic flow on the unit tangent bundle of surface of constant negative curvature is stably ergodic within the class of volume preserving diffeomorphisms. Thus ergodicity holds on an open subset of volume preserving diffeomorphisms even though these diffeomorphisms will typically not be structurally stable. More recently, Pugh & Shub [42] extended this result to general manifolds of constant negative curvature and Wilkinson [44] has proved stable ergodicity of the time one map for all negatively curved surfaces.

All of these results are difficult to prove on account of the fact that leaves of the center foliation are typically non compact and so verification of ergodicity under perturbation requires delicate estimates.

3.1. Skew extensions. Partially hyperbolic sets arise naturally in the study of diffeomorphisms equivariant with respect to a compact non-finite Lie group Γ.

The set of all Γ-orbits determines a (singular) foliation $\mathcal{G}$ of M. If $f : M \to M$ is Γ-equivariant, then $\mathcal{G}$ is f-invariant.

Example 3.1. Suppose that $f(\Gamma x) = \Gamma x$. It is easy to show that one can choose a Γ-invariant Riemannian metric on M with respect to which $Tf : T\Gamma x \to T\Gamma x$ is an isometry. In this situation, it natural to require hyperbolicity transverse to the Γ-orbit (*normal hyperbolicity*) – see Figure 15. ♡

More generally, if all Γ-orbits have the same dimension – and so determine a non-singular foliation – it is natural to ask about hyperbolicity transverse to Γ-orbits on all of M.

We start by reviewing some recent results that apply when the action of Γ is *free* and Γ is *Abelian*.

3.2. Result of Adler-Kitchens-Shub. Let $\mathbf{T}^n$ denote the n-dimensional torus (no group structure). Suppose that $\phi : \mathbf{T}^n \to \mathbf{T}^n$ is *Anosov*. Let λ denote the associated (unique) C^1-invariant ergodic measure [29, §19.2].

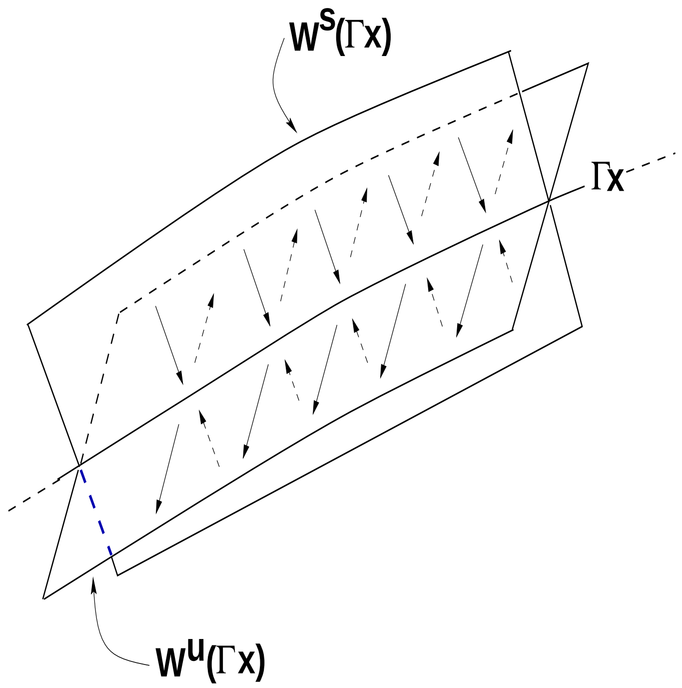

FIGURE 15. A normally hyperbolic invariant group orbit

Let $C^\infty(\mathbf{T}^n, \mathbb{K})$ denote the space of smooth $\mathbb{K}$-valued maps on $\mathbf{T}^n$. Let h denote Haar measure on $\mathbb{K}$. Given $f \in C^\infty(\mathbf{T}^n, \mathbb{K})$, define

$$\phi_f : \mathbb{K} \times \mathbf{T}^n \rightarrow \mathbb{K} \times \mathbf{T}^n$$

by

$$\phi_f(k, t) = (kf(t), \phi(t)).$$

(We call ϕ_f a $\mathbb{K}$-extension or a *skew extension* of ϕ by f.) Observe that ϕ_f is measure preserving, relative to the product measure $m = h \times \lambda$ on $\mathbb{K} \times \mathbf{T}^n$. However, ϕ_f may not be ergodic.

Theorem 3.2 ([2]). *There is an open (C^0-topology) and dense (C^∞-topology) subset $\mathcal{U}$ of $C^\infty(\mathbf{T}^n, \mathbb{K})$ such that for all $f \in \mathcal{U}$, ϕ_f is (stably) ergodic.*

Proof. We sketch the main ideas used in the proof. Let $f \in C^\infty(\mathbf{T}^n, \mathbb{K})$. It follows from an old result of Brin [8], that there exists a closed subgroup H of $\mathbb{K}$ such that the ergodic components of ϕ_f define a *partition* of $\mathbb{K} \times \mathbf{T}^n$ into closed C^1 H-principal subbundles. The ergodic components are permuted by the group action. In particular, if $E \subset \mathbb{K} \times \mathbf{T}^n$ is an ergodic component then

(a) $\Sigma_E = H$.
(b) $\phi_f|E : E \rightarrow E$ is ergodic.

If $H = \mathbb{K}$, then there is just one ergodic component and so ϕ_f is ergodic. If $H \neq \mathbb{K}$, then $H \cong \mathbb{Z}_r$, some $r \geq 1$. Using the known result that if ϕ is a toral Anosov map then

$1-\phi^\star : H^1(\mathbf{T},\mathbb{Z})\to H^1(\mathbf{T},\mathbb{Z})$ is an isomorphism, Adler et al. show that if $H \cong \mathbb{Z}_r$, then f is cohomologous to a constant function $\exp(2\pi\imath\frac{s}{r})$, $(s,r)=1$. That is, there exists a continuous function $h : \mathbf{T}^n\to\mathbb{K}$ such that

$$f(t) = \exp(2\pi\imath\frac{s}{r})h(\phi(t))/h(t), \quad (t \in \mathbf{T}^n).$$

(It follows that ϕ_f is conjugate to the transformation $\Phi(k,t) = (k\exp(2\pi\imath\frac{s}{r}),\phi(t))$ and that ergodic components are always trivial H-principal subbundles. But not necessarily connected!)

Suppose that $x \in \mathbf{T}^n$ is a point of prime period p for T. Since f is cohomologous to the constant function $\exp(2\pi\imath\frac{s}{r})$ it follows easily that

$$\Pi_{j=0}^{p-1} f(\phi^j(x)) = \exp(2\pi\imath\frac{ps}{r}). \tag{3.7}$$

Observe that this product does not depend on x - only on the period of x.

Now choose a pair of periodic points of same prime period but lying on different ϕ-orbits. It follows from (3.7) that a necessary condition for $H \cong \mathbb{Z}_r$ is

$$\Pi_{j=0}^{p-1} f(\phi^j(x)) = \Pi_{j=0}^{p-1} f(\phi^j(y)), \tag{3.8}$$

This condition does not depend on r. Hence if (3.8) fails then ϕ_f is ergodic. But now the set of cocycles f satisfying

$$\Pi_{j=0}^{p-1} f(\phi^j(x)) \neq \Pi_{j=0}^{p-1} f(\phi^j(y)),$$

clearly defines a C^0-open and C^∞-dense subset of $C^\infty(\mathbf{T}^n,\mathbb{K})$. □

3.3. Results of Parry, Parry-Pollicott. Using results of Parry [39], Parry and Pollicott [40] have recently extended the result of Adler et al. [2] to a larger class of $\mathbb{K}^m$-extensions.

Definition 3.3. Let ϕ be a diffeomorphism of M. An infinite ϕ-invariant closed subset Λ of M is *hyperbolic* if $T_\Lambda M = \mathbb{E}^u \oplus \mathbb{E}^s$ (with usual $T\phi$-invariance and estimates) and Λ has a local product structure (equivalently, Λ is maximal and isolated).

In what follows, we assume that Λ is hyperbolic and $\phi : \Lambda\to\Lambda$ is topologically mixing. We fix a Hölder equilibrium measure λ on Λ. It then follows that ϕ is measure theoretically mixing. As usual, we let h denote Haar measure on $\mathbb{K}^m$.

Let $m \geq 1$. Given the cocycle $f : \Lambda\to\mathbb{K}^m$, let $\phi_f : \mathbb{K}^m \times \Lambda\to\mathbb{K}^m \times \Lambda$ denote the skew extension defined by $\phi_f(\gamma,x) = (\gamma f(x),\phi(x))$. Note that ϕ_f is $\mathbb{K}^m$-equivariant and preserves the product measure $\lambda \times h$.

Theorem 3.4 ([40]). *Suppose that Λ is a hyperbolic set. Assume that* either *Λ is a subshift of finite type* or *Λ is connected and $\phi^\star : H^1(\Lambda,\mathbb{Z})\to H^1(\Lambda,\mathbb{Z})$ does not have one as an eigenvalue. Then there is an open (C^0-topology) and dense (C^∞-topology) subset $\mathcal{U}$ of $C^\infty(\Lambda,\mathbb{K}^m)$ such that for all $f \in \mathcal{U}$, ϕ_f is ergodic and mixing.*

The proof of this result depends crucially on Livšic regularity results proved by Parry [39].

3.4. Skew Extensions by general compact connected Lie groups. Suppose that Γ is a compact connected Lie group. Let $\phi : \Lambda \to \Lambda$ and suppose that Λ is a hyperbolic set with equilibrium measure λ.

Let $f : \Lambda \to \Gamma$ and let $\phi_f : \Gamma \times \Lambda \to \Gamma \times \Lambda$ be the associated skew extension.

More generally, we may consider a principal Γ-bundle $\pi : E \to \Lambda$ and Γ-equivariant map $\Phi : E \to E$ covering ϕ.

Given $\phi : \Lambda \to \Lambda$, let $\phi^\star : H^1(\Lambda, \mathbb{Z}) \to H^1(\Lambda, \mathbb{Z})$ denote the induced map on first cohomology.

Consider the following conditions on Λ and Γ.

(H) Λ is connected, $\dim(\Lambda) \neq 0$ and $\mathrm{rank}(\ker(I - \phi^\star)) < \infty$.
(Z) $\dim(\Lambda) = 0$ ($\phi : \Lambda \to \Lambda$ is a subshift of finite type).
(S) Γ is semisimple.

Remark 3.5. If Λ is an *attractor* then it can be shown [25] that $\mathrm{rank}(\ker(I - \phi^\star)) < \infty$. $\diamondsuit$

Theorem 3.6 (Field & Parry [25]). *Let Γ be a compact connected Lie group. Suppose that $\phi : \Lambda \to \Lambda$ is hyperbolic. Assume that one of the conditions* **(H,Z,S)** *holds. Then ϕ_f will be ergodic and mixing for f in an open and dense subset of $C^\infty(\Lambda, \Gamma)$. The same result holds for principal bundle extensions of ϕ.*

The proof of this result breaks into four steps: Proof of stable ergodicity when (1) $\Gamma = \mathbb{K}^m$ and either condition **(H)** or **(Z)** holds; (2) Γ is semisimple; and (3) Case of general Γ. (4) Stable ergodicity $\Longrightarrow$ Stable mixing.

Step (3) follows easily from (1,2). Step (4) is straightforward (trivial if Γ is semisimple!).

Surprisingly, perhaps, step (2) is much easier than step (1).

The proof of (2) depends on the following results

Theorem 3.7 ([21]). *Let Γ be a compact connected semisimple Lie group. Then the set of pairs $(g, h) \in \Gamma^2$ which topologically generate Γ form a non-empty (*Zariski*) open subset of Γ^2.*

Remark 3.8. We note that the *density* of generating pairs is a relatively old result proved by Kuranishi [32]. The openness does not seem to have been noticed before. $\diamondsuit$

Theorem 3.9 ([40, 25]). *The ergodic components of ϕ_f are* closed *principal subbundles of $\Gamma \times \Lambda$.*

Remark 3.10. Results along the lines of Theorem 3.9 were first obtained by Brin [7] in the context of principal bundle extensions of Anosov diffeomorphisms. $\diamondsuit$

Sketch proof of Theorem 3.6 - Γ semisimple. Fix $\alpha \in (0, 1)$. Generally, we work with the C^α-topology on cocycles.

Replacing ϕ by a power of ϕ, it is no loss of generality to suppose that ϕ has a fixed point, say x_0. Set $z_0 = (e, x_0) \in \Gamma \times \Lambda$. It follows from Theorem 3.9 that for each $u \in \Gamma z_0$ there is a unique closed ergodic component $\mathcal{E}_u$ of ϕ_f containing the point u and that $\gamma \mathcal{E}_u = \mathcal{E}_{\gamma u}$ for all $\gamma \in \Gamma$. Let $\Sigma_0 \subset \Gamma$ denote the isotropy group of the ergodic component $\mathcal{E}_{z_0}$. Then ϕ_f is ergodic if and only if $\Sigma_0 = \Gamma$.

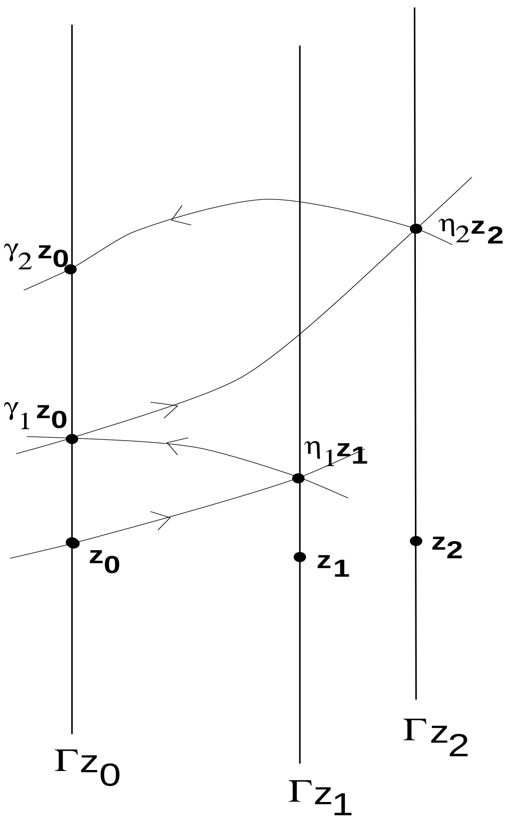

FIGURE 16. Intersections of strong stable and unstable manifolds

Since the ergodic components are closed ϕ_f-invariant sets,

$$W^{ss}(u), W^{uu}(u) \subset \mathcal{E}_u, \quad (u \in \Gamma z_0).$$

Choose $x_1, x_2 \in W^u(x_0) \cap W^s(x_0) \setminus \{x_0\}$ with distinct ϕ-orbits. Set $z_i = (e, x_i)$ and $\gamma_0 = e$. For $i = 1, 2$, there exist unique $\eta_i, \gamma_i \in \Gamma$ such that

$$\eta_i z_i \in W^{uu}(\gamma_{i-1} z_0) \cap W^{ss}(\gamma_i z_0). \tag{3.9}$$

We refer to Figure 16.

Since the ergodic components $\mathcal{E}_{\gamma z_0}$ define a partition of $\Gamma \times \Lambda$, it follows from (3.9) that $\gamma_i z_0 \in \mathcal{E}_{z_0}$, $i = 0, 1, 2$. Hence $\Sigma_0 \supset \langle \gamma_1, \gamma_2 \rangle$. In particular, if $\langle \gamma_1, \gamma_2 \rangle = \Gamma$, then $\Sigma_0 = \Gamma$ and ϕ_f is ergodic. It may be shown that η_i, γ_i depend continuously on f, C^α-topology. Since the set of pairs of topological generators is open in Γ^2, it follows that if $\langle \gamma_1, \gamma_2 \rangle = \Gamma$ the same will be true for $f' \in C^\infty(\Lambda, \Gamma)$ sufficiently C^α-close to f. Conversely, we can always make an arbitrarily C^∞-small perturbation f' of f, supported on small neighborhoods of x_1, x_2, so that the corresponding pair (γ_1', γ_2') topologically generates Γ. Hence the set of $f \in C^\infty(\Lambda, \Gamma)$ for which ϕ_f is ergodic contains an open and dense set. Finally, since Γ is semisimple, it follows that if ϕ is mixing and ϕ_f is ergodic, then ϕ_f is automatically weak

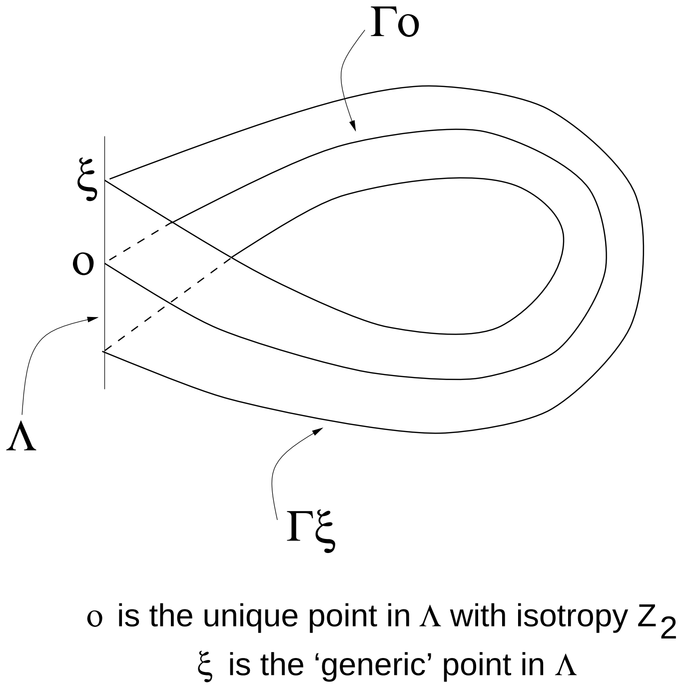

FIGURE 17. The twisted product $\mathbb{K} \times_{\mathbb{Z}_2} 3^{\mathbb{Z}}$

mixing and therefore mixing by Rudolf's theorem [43]. □

Hyperbolicity for equivariant diffeomorphisms. Assume that M is a Γ-manifold. Let $\mathbb{E}^c \subset TM$ be the Tf-invariant (singular) subbundle of TM defined by

$$\mathbb{E}^c = \cup_{x \in M} T_x \Gamma x.$$

Suppose that Λ be a compact f- and Γ-invariant subset of M. We say that Λ is *transversally hyperbolic for f* if

(a) All Γ-orbits in Λ have the same dimension.
(b) There exists a continuous Tf-invariant splitting

$$T_\Lambda M = \mathbb{E}^u \oplus \mathbb{E}^c_\Lambda \oplus \mathbb{E}^s,$$

and constants $C > 0$, $\lambda > 1$, such that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|T_x\phi^n(v)\| &\leq C\lambda^{-n}\|v\|, \ (v \in \mathbb{E}^s_x, \ x \in \Lambda), \\ \|T_x\phi^n(v)\| &\leq C\lambda^{n}\|v\|, \ (v \in \mathbb{E}^u_x, \ x \in \Lambda). \end{aligned}$$

Just as for hyperbolic sets, we can require a local product structure on a transversally hyperbolic set.

A transversally hyperbolic set Λ is Γ-*hyperbolic* if Λ has a local product structure and the induced map $\tilde{f}$ on the orbit space Λ/Γ is topologically transitive.

If Λ is Γ-hyperbolic, then $\tilde{f} : \Lambda/\Gamma \to \Lambda/\Gamma$ admits a Markov partition [22].

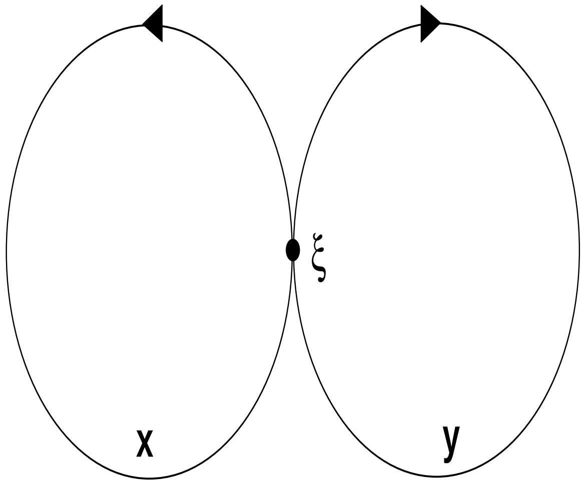

FIGURE 18. The space X

Example 3.11. Let $\Lambda = 3^{\mathbb{Z}}$ denote the full shift on three symbols $\{0, 1, 2\}$. We have a non-free action of $\mathbb{Z}_2 = \langle\kappa\rangle$ on Λ defined by $\kappa(1) = 2$, $\kappa(0) = 0$. Embed $\mathbb{Z}_2$ in $\mathbb{K}$ by $\kappa \mapsto \exp(\imath\pi)$.

Let $\mathbf{X} = \mathbb{K} \times_{\mathbb{Z}_2} \Lambda$ (that is, the twisted product – see Figure 17 and notes for Lecture I).

If $f : \Lambda \to \mathbb{K}$, then we get an induced map $\phi_f : \mathbb{K} \times_{\mathbb{Z}_2} \Lambda \to \mathbb{K} \times_{\mathbb{Z}_2} \Lambda$. In this case, it is easily seen that ϕ_f is generically stably ergodic. ♡

Generally, all Γ-hyperbolic sets are locally twisted products. This follows easily from the differentiable slice theorem.

Conjecture

If $\phi : \Lambda \to \Lambda$ is Γ-hyperbolic, then there exist C^∞-small Γ-equivariant perturbations of ϕ to a stably ergodic diffeomorphism.

3.5. A non-uniformly hyperbolic base. We conclude by describing a simple example based on a non-uniformly hyperbolic attractor constructed by Coelho et al. [12]. Let X denote the unit interval with $0, \frac{1}{2}, 1$ identified, say to $\xi \in X$. Let x and y respectively denote the oriented circles defined as the images of $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ in X. We may regard X as the 'figure of eight' consisting of the x, y with infinite order contact at ξ. Regard the circle group K as $[0, 1]$, with $0, 1$ identified. See Figure 18. Define the smooth map $\phi : X \to X$ by $\phi(x) = 3x$, mod 1. Let $f : X \to K$ be the smooth cocycle defined by $f(x) = 2x$ and $g : X \to K$ be the measurable cocycle defined by $g(x) = x$, $x \neq \xi$. Since

$$g(\phi x) - g(x) = 3x - x = 2x = f(x), \quad x \neq \xi,$$

f is a coboundary (in the class of measurable cycles). On the other hand, there is obviously no continuous solution to the cohomology equation and so f is not a continuous (let alone smooth) coboundary.

Let $\phi_f : \mathbb{K} \times X \to \mathbb{K} \times X$ denote the $\mathbb{K}$-extension of ϕ. Given $\theta \in \mathbb{K}$, define $E_\theta = \{(x+\theta, x) \mid x \in X\}$. Clearly, $\mathbf{E} = \{E_\theta \mid \theta \in \mathbb{K}\}$ is a family of closed ϕ_f-invariant subsets of $\mathbb{K} \times X$ permuted by the action of $\mathbb{K}$. Although $\cup_{\theta \in \mathbb{K}} E_\theta = \mathbb{K} \times X$, $\mathbf{E}$ does not define a partition of $\mathbb{K} \times X$: $E_\theta \cap E_{\theta+\pi} = \{(\theta, \xi), (\theta+\pi, \xi)\}$. For all $\theta \in \mathbb{K}$, $E_\theta \cong \mathbb{K}$ and $\phi_f | E_\theta$ is smoothly conjugate to the expanding map $z \mapsto z^3$.

Let $\Phi : \Lambda \to \Lambda$ denote the non-uniformly hyperbolic attractor obtained as the inverse limit of $\phi : X \to X$ and $\tilde{f}$ denote the 'lift' of f to Λ. The skew extension $\Phi_{\tilde{f}} : \mathbb{K} \times \Lambda \to \mathbb{K} \times \Lambda$ is measure theoretically trivial but not continuously trivial. We describe closed ergodic components of Φ_f. For $\theta \in \mathbb{K}$, let $\tilde{E}_\theta \subset \mathbb{K} \times \Lambda$ denote the classical solenoid defined as the inverse limit of $\phi_f : E_\theta \to E_\theta$. Up to sets of measure zero, the set of ergodic components of Φ_f is given by $\tilde{\mathbf{E}} = \{\tilde{E}_\theta \mid \theta \in \mathbb{K}\}$. Note that although the components are closed and permuted by the action of $\mathbb{K}$, $\tilde{\mathbf{E}}$ does not define a *partition* of $\mathbb{K} \times \Lambda$. In particular, not only do we have $\tilde{E}_\theta \cap \tilde{E}_{\theta+\pi} \neq \emptyset$, but also $\cup_{\theta \in \mathbb{K}} \tilde{E}_\theta \neq \mathbb{K} \times \Lambda$. It can be shown for this example that there is no partition of $\mathbb{K} \times \Lambda$ into closed ergodic components.

Finally, it is worth noting that there is an action of $\mathbb{Z}_2$ on X defined by reflection in a line tangent to X at ξ. In particular, the action interchanges x and y, preserving orientations. The action of $\mathbb{Z}_2$ extends to Λ and the fixed point set of the action is given by $\Lambda^{\mathbb{Z}_2} = (\xi)$. If we form the twisted product $\mathbb{K} \times_{\mathbb{Z}_2} \Lambda$, we find that the quotients of the components $\tilde{E}_\theta$ give the closed ergodic components for the induced map on the twisted product. Of course, the ergodic components still do not give a *partition* of $\mathbb{K} \times_{\mathbb{Z}_2} \Lambda$ into closed sets.

Notes on Lecture III

There is an extensive and varied literature on skew extensions and the problem of lifting ergodicity and mixing. From our perspective, the most interesting results develop from the work of Brin [8], on extensions over Anosov systems, and that of Livšic [33, 34]. The results of Livšic give precise criteria for extensions to be trivial. In addition, there are Livšic regularity theorems. These results allow one to make contact with work on measurable cocycles (see, for example, [30]). This type of analysis is basic to the proof of Theorem 3.6.

It is not yet clear what happens when the base is not hyperbolic. However, Melbourne has recently proved some results on lifting transitivity and weak mixing [35]. If Γ is semisimple, it is possible to prove genericity of stable transitivity without having to assume hyperbolicity of the base. For example, if the base is non-uniformly hyperbolic and Γ is semisimple then extensions are generically stably transitive and weak mixing (up to a cycle).

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