2. Lecture II: Constructing hyperbolic symmetric attractors

In this lecture we want to describe some recent work by Field, Melbourne & Nicol [24] on the construction of symmetric hyperbolic expanding attractors.

Suppose that $\Gamma \subset \mathrm{O}(n)$ is finite. Let H be a subgroup of Γ. We ask for conditions on $(\mathbb{R}^n, \Gamma)$ that yield the existence of a Γ-equivariant diffeomorphism or flow on $\mathbb{R}^n$ which has a hyperbolic attractor A with $\Sigma_A = H$. (Recall that we always assume that A is topologically transitive.)

As we indicated in Lecture I, if Γ is finite it is no loss of generality to assume that principal isotropy groups are trivial. Granted this, we will attempt to construct A so that it consists of points of trivial isotropy.

An involution $\gamma \in \Gamma$ is called a *reflection* if the fixed point space of γ is $(n-1)$-dimensional. We call an $(n-1)$-dimensional linear subspace V of $\mathbb{R}^n$ a *reflection hyperplane* of Γ contains a reflection with fixed point space V. Reflections provide the only obstruction to constructing attractors with specified symmetry group.

Let $\mathcal{L}$ denote the union of all the reflection hyperplanes of Γ. The connected components of $\mathbb{R}^n \setminus \mathcal{L}$ are permuted by Γ. Obviously, an equivariant flow on $\mathbb{R}^n$ fixes the components of $\mathbb{R}^n \setminus \mathcal{L}$ (as the reflection hyperplanes are invariant by the flow). Consequently, if A is the attractor of a flow on $\mathbb{R}^n$ then A must lie in (the closure of) a single connected component C of $\mathbb{R}^n \setminus \mathcal{L}$. In particular, C is fixed by the subgroup Σ_A. Consequently, if H is a subgroup of Γ it follows that a necessary condition for there to exist a Γ-equivariant flow with an H-symmetric attractor is that H fixes a connected component of $\mathbb{R}^n \setminus \mathcal{L}$. It turns out that this condition is also sufficient. Furthermore, if $n \geq 5$, then given any subgroup $H \subset \Gamma$, there exists a Γ-equivariant flow on $\mathbb{R}^n$ which has a hyperbolic attractor A with $\Sigma_A = H$.

The situation for equivariant diffeomorphisms is a little more complicated. To simplify matters, we shall assume that $(\mathbb{R}^n, \Gamma)$ has no reflections (for the general case, see [24]). Under these conditions we have

Theorem 2.1. *Let H be a subgroup of Γ and suppose $n \geq 4$. There exists a smooth Γ-equivariant diffeomorphism of $\mathbb{R}^n$ which has a hyperbolic attractor A with $\Sigma_A = H$.*

Rather than give the full proof of Theorem 2.1, we shall describe one example that indicates all the techniques needed to prove the general case[3].

2.1. An example in $\mathbb{R}^3$.

We suppose that $\Gamma = \mathbb{Z}_2 = \langle \kappa \rangle$ acts on $\mathbb{R}^3$ by

$$\kappa(x, y, z) = (-x, -y, z), \quad ((x, y, z) \in \mathbb{R}^3).$$

Note that the fixed point set of the action is the z-axis.

We construct a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$ which has a $\mathbb{Z}_2$-symmetric hyperbolic attractor.

Our strategy will be to construct a smooth $\mathbb{Z}_2$-symmetric graph $\mathcal{G}$ and $\mathbb{Z}_2$-equivariant expanding map $f : \mathcal{G} \to \mathcal{G}$ and then attempt to mimic the proof we gave for the solenoid.

[3] In fact, this example is, in some ways, harder than the general case in dimensions greater than 3.

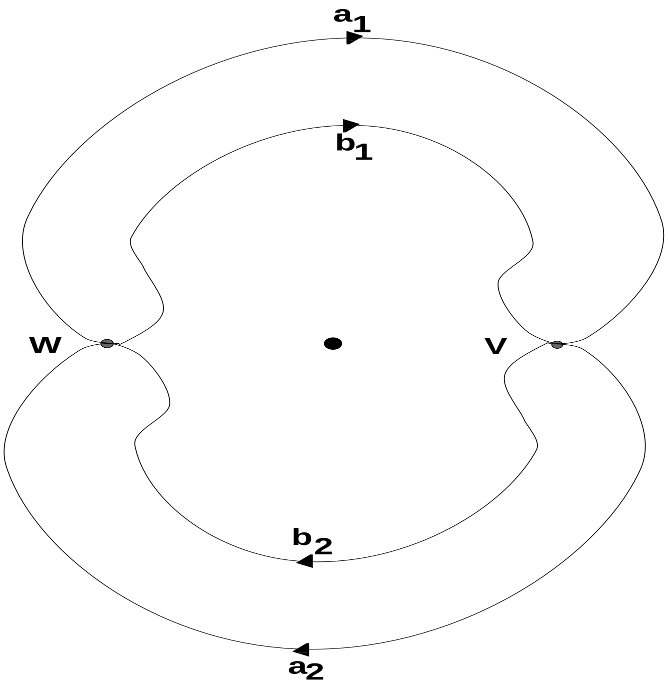

FIGURE 11. Smooth $\mathbb{Z}_2$-invariant graph $\mathcal{G} \subset \mathbb{R}^2$

The main problems we encounter will be to define the $\mathbb{Z}_2$-symmetric graph $\mathcal{G}$ and expanding map f in such a way that we can use the $\mathbb{Z}_2$-equivariant version of the isotopy theorem to obtain a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$.

Example 2.2. The obvious way to attempt to construct a $\mathbb{Z}_2$-equivariant hyperbolic attractor in $\mathbb{R}^3$ is to mimic the construction of the classical solenoid given in the last lecture. Regard $\mathbb{C} \subset \mathbb{R}^3$. Although the map $u \mapsto u^2$ is not $\mathbb{Z}_2$-equivariant, the map $f(u) = u^3$ is odd and so $\mathbb{Z}_2$-equivariant. We can thicken the unit circle in $\mathbb{C}$ to a solid torus $\mathbf{T}$, extend f $\mathbb{Z}_2$-equivariantly to $f : \mathbf{T} \to \mathbf{T}$ and then perturb f to a $\mathbb{Z}_2$-equivariant embedding $\tilde{f} : \mathbf{T} \to \mathbf{T}$. It is easy to see that we can do this construction so that $\tilde{f}$ has a $\mathbb{Z}_2$-invariant hyperbolic attractor $\Lambda \subset \mathbf{T}$. The problem lies in finding an extension of $\tilde{f}$ to a $\mathbb{Z}_2$ equivariant diffeomorphism of $\mathbb{R}^3$. In fact, no such extension exists. Rather than proving this, we just observe that the isotopy theorem cannot be applied to the map $\tilde{f}$ as $\tilde{f}$ is not *equivariantly* isotopic to the identity map of $\mathbf{T}$: The image of any isotopy joining $\tilde{f}$ to $I_{\mathbf{T}}$ would have to cross the z-axis and this would violate the preservation of isotopy type by embeddings. ♡

Construction of the graph. Regard $\mathbb{Z}_2$ as acting on the (x, y)-plane $\mathbb{R}^2 \subset \mathbb{R}^3$ as multiplication by $\pm I$.

In Figure 11 we show a $\mathbb{Z}_2$-invariant graph $\mathcal{G} \subset \mathbb{R}^2$. Following the terminology of Williams [45], $\mathcal{G}$ is a *branched manifold*. The two branches (both diffeomorphic to circles) have infinite order contact at the vertices V, W. Note that $\mathbb{Z}_2$ maps vertices to vertices, edges to edges. A map $f : \mathcal{G} \to \mathcal{G}$ will be smooth if (and only if), f extends to a smooth map

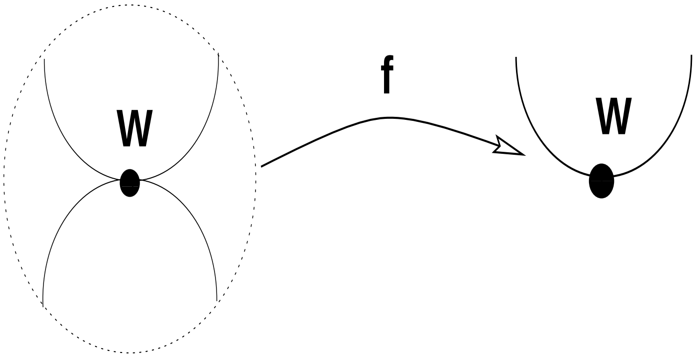

FIGURE 12. Image of a neighborhood of W

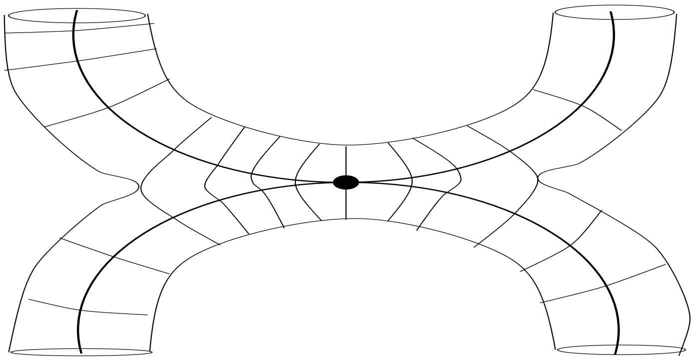

FIGURE 13. Fibers near a branch point

of $\mathbb{R}^2$.

We may define a smooth $\mathbb{Z}_2$-equivariant expanding map $f : \mathcal{G} \rightarrow \mathcal{G}$ by the rules:

$$\begin{aligned} a_1 &\mapsto a_2^{-1} b_2 a_1 \\ b_1 &\mapsto b_2^{-1} a_2 b_1 \\ a_2 &\mapsto a_1^{-1} b_1 a_2 \\ b_2 &\mapsto b_1^{-1} a_1 b_2 \end{aligned}$$

Observe that the f-image of a small connected neighborhood of V (respectively, W) is an arc through V (respectively W). That is, the image of a neighborhood of a branch point is non-singular. We refer to Figure 12.

We now construct a $\mathbb{Z}_2$-invariant (smooth) tubular neighborhood $\mathbf{G}$ of $\mathcal{G} \subset \mathbb{R}^3$. The only potential difficulty here lies with the fibers (2-disks) of the tubular neighborhood near the branch points W, V. However, our flatness condition on the branches at W, V enables us to construct $\mathbf{G}$ so that all fibers are transverse to the core $\mathcal{G}$. In Figure 13, we show the intersection of the transverse discs with the x, y-plane near a branch point.

Given $u \in \mathcal{G}$, let D_u denote the transverse disk through u. Since the f-image of a small

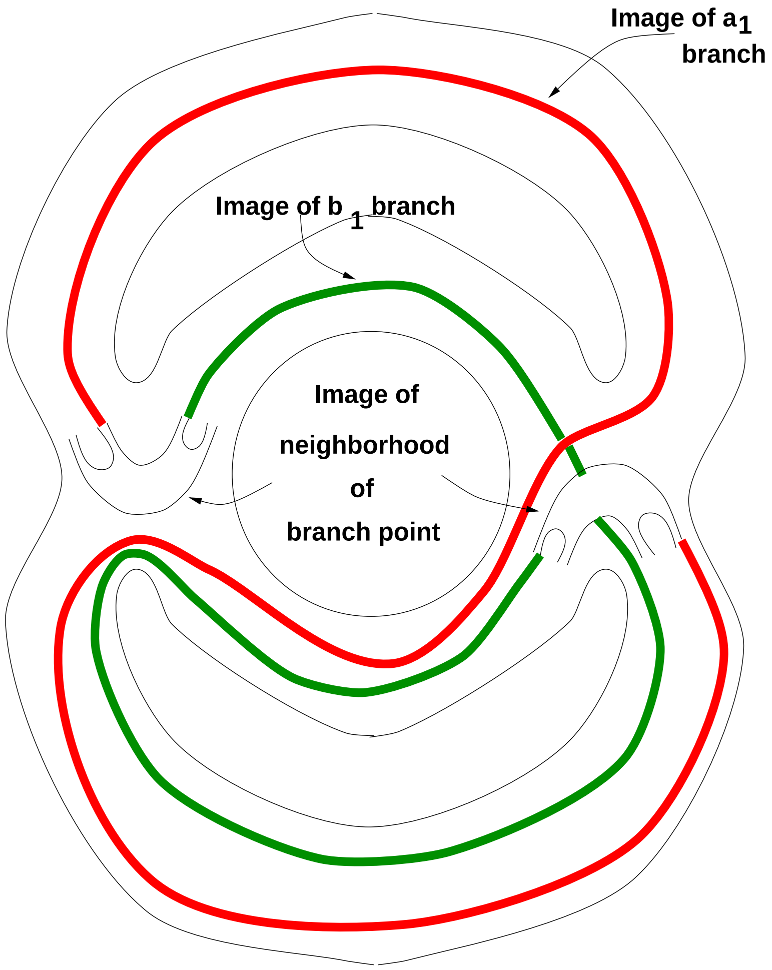

FIGURE 14. The image of F in $\mathbf{G}$

connected neighborhood of a branch point is an arc, it follows that we can choose $\mathbf{G}$ so that

$$\text{If } v \in D_u \cap \mathcal{G}, \quad \text{then } f(u) = f(v). \tag{2.6}$$

(Alternatively, we can fix $\mathbf{G}$ and then choose f.) Extend f smoothly and $\mathbb{Z}_2$-equivariantly to a map $\tilde{f} : \mathbf{G} \to \mathbf{G}$ such that

(a) $\tilde{f}(\mathbf{G}) = \mathcal{G}$.
(b) $\tilde{f}(D_u) = \{\tilde{f}(u)\}$, all $u \in \mathcal{G}$.

Note that we can satisfy (b) because of (2.6).

We deform $\tilde{f} : \mathbf{G} \to \mathbf{G}$ to a $\mathbb{Z}_2$-equivariant embedding $F : \mathbf{G} \to \mathbf{G}$ which preserves transverse disks (that is, for all $u \in \mathcal{G}$, $F(D_u) \subset D_{\tilde{f}(u)}$). Furthermore, we can suppose that F contract disks and expands in a complementary direction. More precisely, we may require that there exists a TF-invariant continuous splitting $T_{\mathbf{G}}\mathbb{R}^3 = \mathbb{E}^s \oplus \mathbb{E}^u$ and constants $C > 0$, $\lambda \in (0, 1)$ such that if $X \in \mathbf{G}$, $n \in \mathbb{N}$ and $F^n(X) \in \mathbf{G}$ then

(u) $\|TF^{-n}(v)\| \leq C\lambda^n\|v\|$, $v \in \mathbf{E}^u_X$,
(s) $\|TF^n(v)\| \leq C\lambda^n\|v\|$, $v \in \mathbf{E}^s_X$.

The contracting spaces $\mathbf{E}^s$ will be tangent to the disk foliation of $\mathbf{G}$.

In Figure 14, we show the image of the branches associated to the edges a_1 and b_1 in $\mathbf{G}$ (the remainder of the image is obtained by reflection in the z-axis). We may choose F so that the image of the a_1-branch lies below the (x, y)-plane in the upper half of the picture and above the (x, y)-plane in the lower half. Similarly, we may suppose that the image of the b_1-branch lies above the (x, y)-plane in the upper half of the picture and below (as shown) in the lower half. With this configuration, we can ensure that no knots or links are created between the images of the branches associated to the four edges.

We define $\Lambda = \cap_{n\geq 0}F^n(\mathbf{G})$. Just as we showed for the classical solenoid, $F : \Lambda \to \Lambda$ is ($\mathbb{Z}_2$-equivariantly) conjugate to the solenoid defined by the inverse limit of $f : \mathcal{G} \to \mathcal{G}$. Since Λ is a compact F-invariant subset of $\mathbf{G}$, it follows from estimates (u,s) that Λ is a hyperbolic attractor.

Finally, it remains to extend F to a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$. For this it suffices to note that that the F-images of the branches of $\mathbf{G}$ corresponding to the edges $a_1, \ldots, b_2$ do not wind round the z-axis. Since we constructed F so that there were no knots or links between the images of the edges, it follows easily that F is $\mathbb{Z}_2$-equivariant to the identity map of $\mathbf{G}$.

Remark 2.3. In dimensions greater than three, we no longer have to worry about the possibility of the embedded image of a tubular neighborhood having knots or links. If the codimension of the singular set[4] is greater than two, then the image of the tubular neighborhood cannot form a non-trivial link with the singular set. In particular, there will be no obstructions to applying the equivariant isotopy theorem. $\diamondsuit$

Notes on Lecture II

For foundational results on smooth equivariant dynamical systems see [15, 16]. In Lecture II, we ignored the simplest examples of attractors: Invariant Γ-orbits (or relative equilibria) and their periodic generalizations.

Suppose first that Γ is finite. Let x be a point of prime period p for $f : M \to M$. For simplicity, assume x has trivial isotropy. Let Σ denote the isotropy group of $o(x)$ (the f-orbit of x):

$$\Sigma = \{\gamma \in \Gamma \mid \gamma o(x) = o(x)\}.$$

It is easy to verify that $\Sigma \cong \mathbb{Z}_p$ for some $p \geq 1$. If $p = 1$, we say the orbit is *asymmetric*. Otherwise, we say it is *symmetric*. Whether symmetric or asymmetric, there is no obstruction to perturbing f so that $o(x)$ is hyperbolic [16].

If A is an attractor or ω-limit point set of an equivariant diffeomorphism, and periodic points are dense in A, one can often show that asymmetric and symmetric period points are dense. Results along these lines are given in [13]. If A is an equivariant subshift of

[4]The set of points which are not of principal isotropy type.

finite type [17], or admits an equivariant Markov partition [22], it is easy to prove density of symmetric and asymmetric periodic points directly.

The case when Γ is compact, non-finite, is more interesting. The generalization of a fixed point is a 'relatively fixed point' ('relative equilibrium' for a vector field). That is, a Γ-orbit which is left invariant by f. The natural hyperbolicity concept is then that of *normal hyperbolicity* – hyperbolicity transverse to the group orbit. Suppose then that $\alpha = \Gamma x$ is an invariant f-orbit, $f : M \to M$. Using the differentiable slice theorem, it is easy to show that we can always perturb f to require that α is normally hyperbolic [16]. It remains to understand the dynamics of $f|\alpha$. If α consists of points of trivial isotropy and Γ is connected, one may show that α is foliated by f-invariant tori [16]. For generic f, these tori will be of dimension equal to the rank of Γ. Similar results hold for flows and periodic points. If Γ is not connected or α does not consist of points of trivial isotropy, one has to examine maximal Abelian subgroups which are not tori. For more details on all of this, we refer to the articles by Krupa [31] and Field [16, 19].

In [17] constructions are given for equivariant diffeomorphisms which have subshifts of finite type with specified symmetry groups. For example, it is shown that every equivariant diffeomorphism can be C^0-approximated by an equivariantly structurally stable diffeomorphism with Ω-set consisting of subshifts of finite type. We give one simple example in the next lecture.