LECTURES ON SYMMETRIC ATTRACTORS

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SUMMARY OF LECTURES

- Review of equivariant dynamics and symmetric attractors.
- An exotic symmetric flow on $\mathbb{R}^4$.
- The classical solenoid.
- Constructing hyperbolic attractors with given symmetry.
- The theorem of Adler-Kitchens-Shub.
- The results of Parry and Parry & Pollicott.
- Stability of ergodicity and mixing for skew extensions.

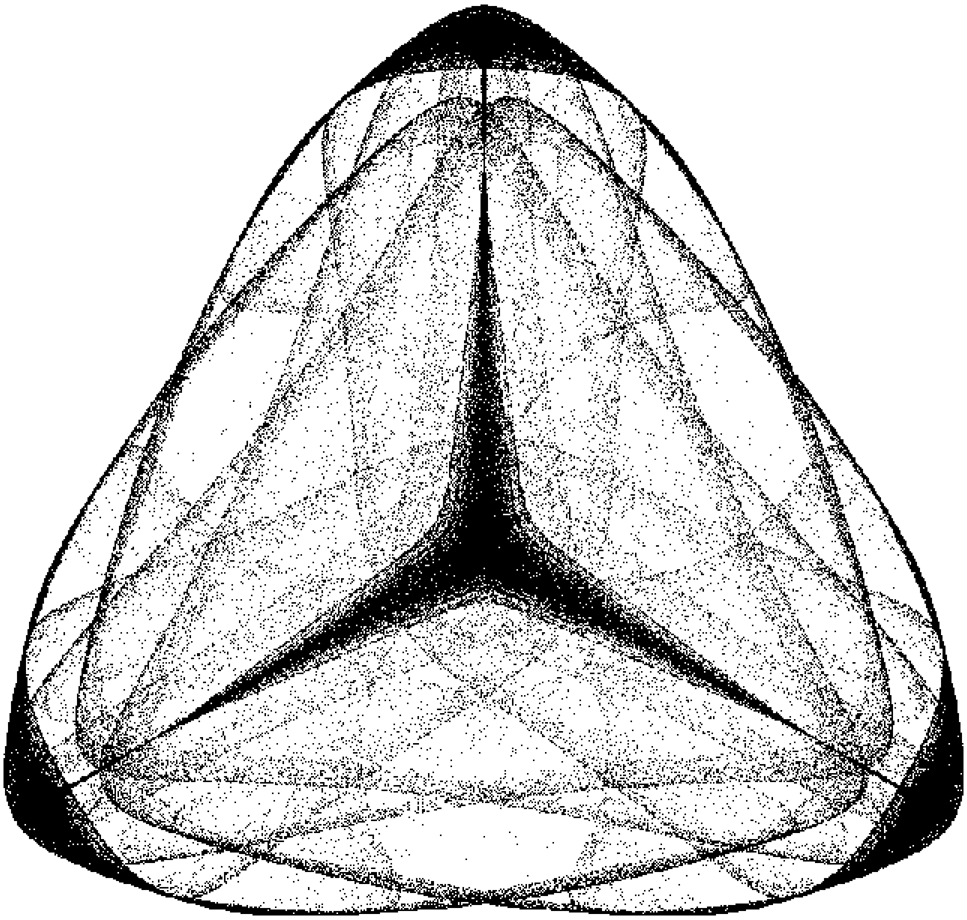

FIGURE 1. Attractor of a planar map with 3-fold symmetry

INTRODUCTION

These notes are intended to fill in some details not given in the lectures as well as to provide a source of references for further reading.

In general terms, Lecture I is a fairly gentle introduction to equivariant dynamics with a focus on attractors. We describe one highly exotic example that displays, in a persistent fashion, a variety of features not seen in non-equivariant dynamics. In Lecture II, we show how to construct symmetric hyperbolic attractors with specified finite symmetry group. In lecture III, we describe recent work, some joint with Parry (Warwick), on ergodic properties of attractors for equivariant dynamical systems equivariant by a general compact connected Lie group.

At the end of each section of the notes, we provide additional comments and suggestions for further reading.

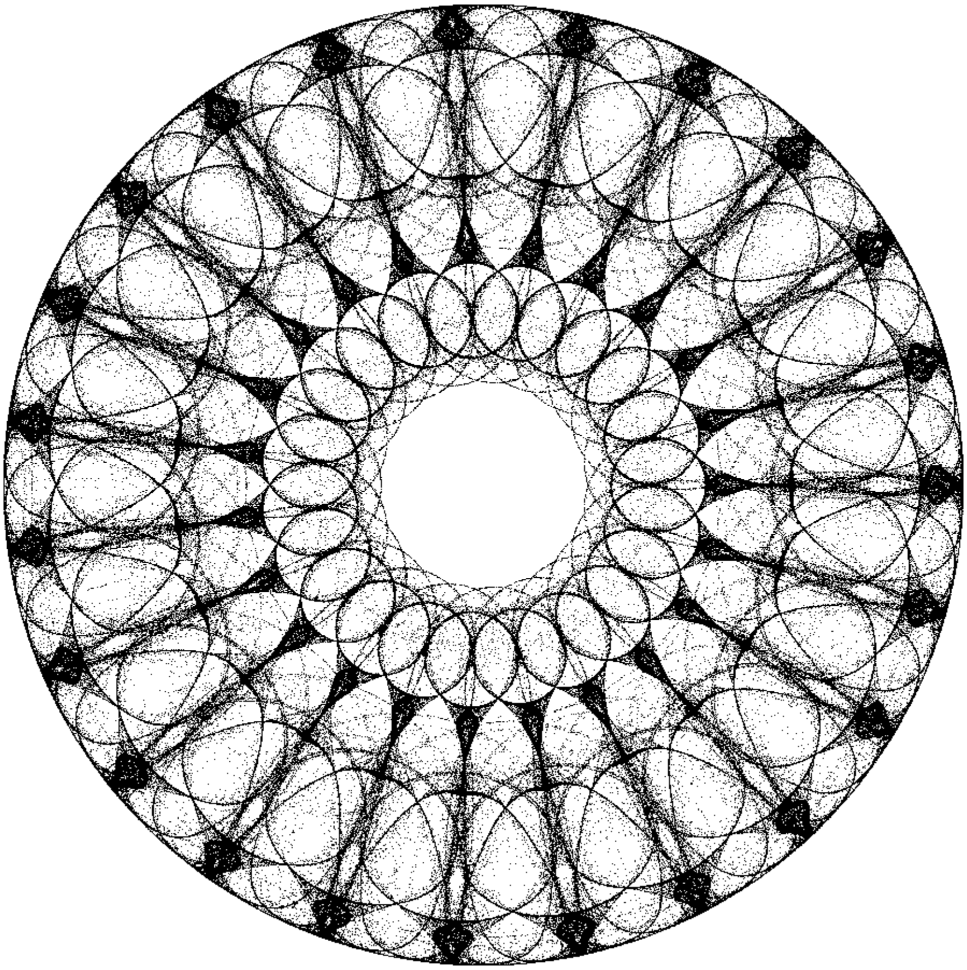

FIGURE 2. Attractor of a planar map with 23-fold symmetry

1. LECTURE I: EQUIVARIANT DYNAMICS

1.1. Lie groups. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth (that is C^∞) mapping. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation of $\mathbb{R}^n$, we say that T is a (linear) *symmetry* of f if

$$f(Tx) = T(f(x)), \quad (x \in \mathbb{R}^n).$$

Obviously, if T, S are symmetries of f so are $T \circ S$ and T^{-1}. It follows that the set of linear symmetries of f forms a group, say $\mathcal{S}(f)$. Clearly $\mathcal{S}(f)$ is a subgroup of the general linear group $\mathrm{GL}(n, \mathbb{R})$. Since every element in the closure of $\mathcal{S}(f)$ is obviously also a symmetry of f, it follows that $\mathcal{S}(f)$ is a *closed* subgroup of $\mathrm{GL}(n, \mathbb{R})$.

The group $\mathcal{S}(f)$ is an example of a *Lie group*. That is, a group which has the structure of a differential manifold and for which composition and inversion are smooth maps. A more 'nonlinear' example can be constructed by looking at the group of all smooth diffeomorphisms

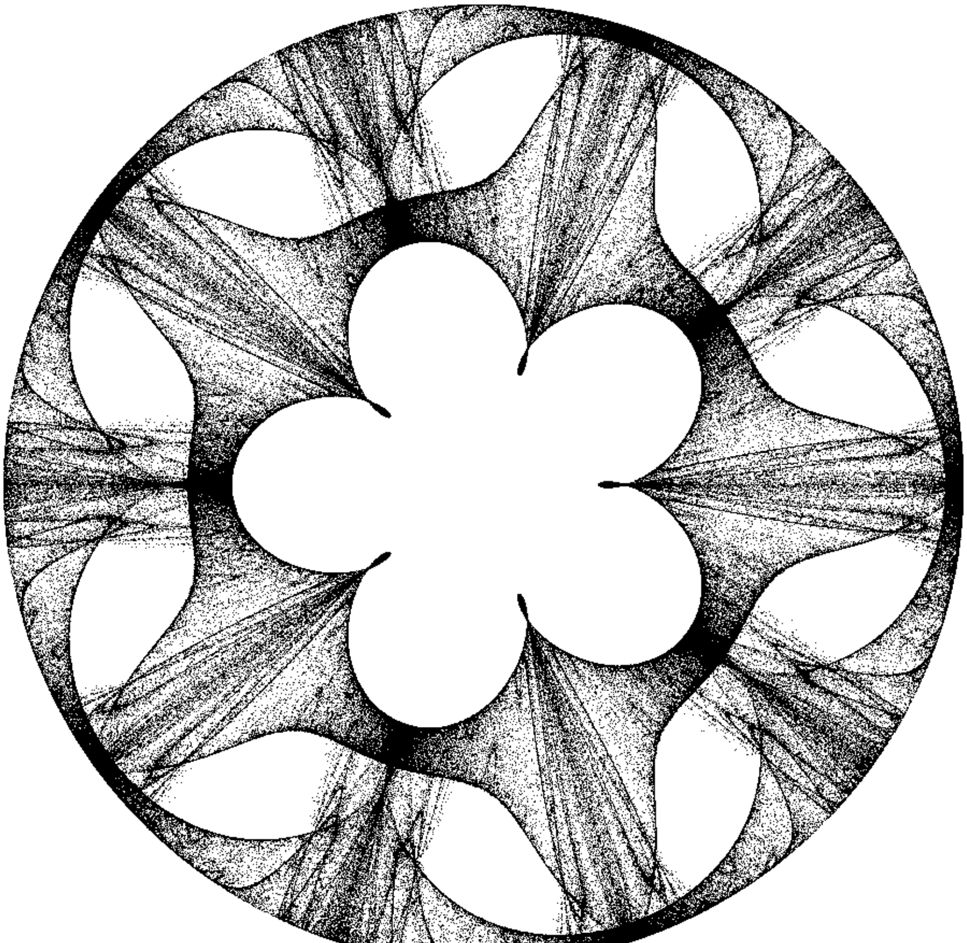

FIGURE 3. Attractor of a planar map with 5-fold symmetry

which commute with a given smooth mapping f of a (compact) manifold M. In this case, $\mathcal{S}(f)$ will be a closed subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ of M.

For our purposes, we shall restrict attention to *compact* Lie groups. This class includes all *finite* groups. It is well-known that every compact Lie group can be represented as a (closed) subgroup of an orthogonal group $\mathrm{O}(n)$.

Rather than looking at the symmetries of a specific map f, our approach will be to specify a compact Lie group Γ, an *action* of Γ on a vector space or differential manifold, and then investigate a class of dynamical systems which has (at least) Γ symmetry.

1.2. Linear group actions – representations. Suppose that Γ is a subgroup of $\mathrm{O}(n)$. We have an associated (orthogonal) action of Γ on $\mathbb{R}^n$ defined by evaluation:

$$\mathbb{R}^n \times \Gamma \to \mathbb{R}^n, \quad (x, \gamma) \mapsto \gamma x.$$

We refer to $(\mathbb{R}^n, \Gamma)$ as a *representation* of Γ (on $\mathbb{R}^n$).

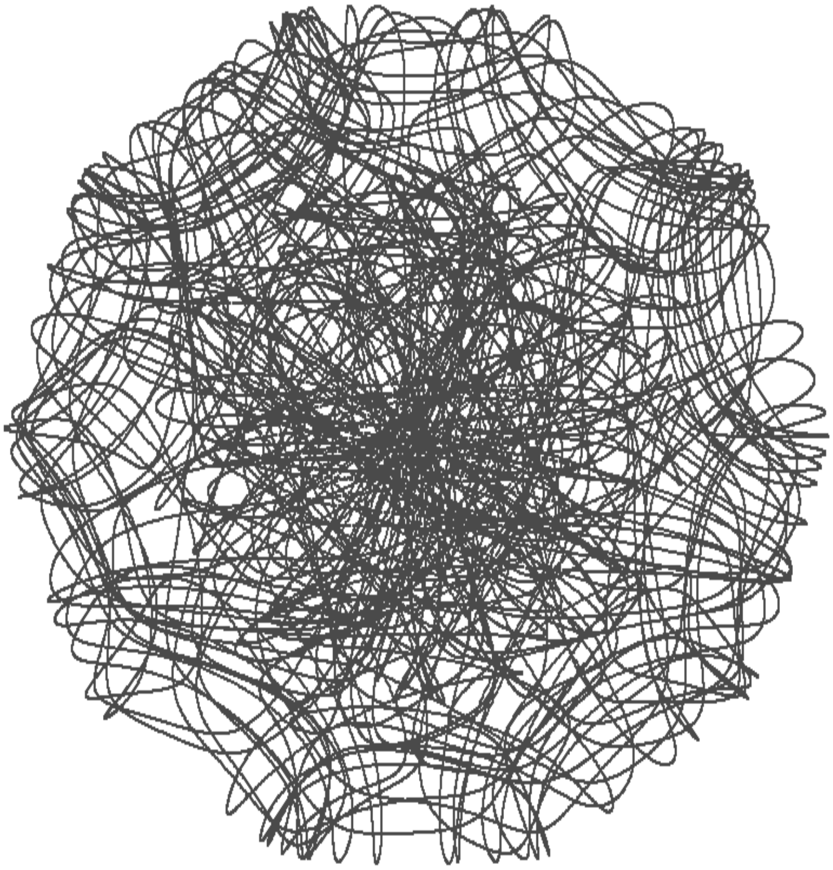

FIGURE 4. Attractor of a symmetric flow on $\mathbb{R}^4$

Remark 1.1. More generally, we can look at homomorphisms $\rho : \Gamma \to \mathrm{O}(n)$ and the associated action of Γ on $\mathbb{R}^n$. Provided that that we study dynamics on $\mathbb{R}^n$ (as opposed to manifolds), all that will matter is the image $\rho(\Gamma) \subset \mathrm{O}(n)$ and so it is no loss of generality to regard Γ as a matrix group. $\diamondsuit$

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$. We say f is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x), \quad (x \in \mathbb{R}^n, \gamma \in \Gamma).$$

Example 1.2. Let $\mathbf{D}_n$ denote dihedral group of order $2n$. That is, $\mathbf{D}_n$ is the full symmetry group of the regular n-gon. Regard $\mathbf{D}_n$ as a subgroup of $\mathrm{O}(2)$ and identify $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way. Then $\mathbf{D}_n = \langle \rho, \kappa \rangle$, where

$$\rho(z) = \exp(2\pi\imath/n)z, \quad \kappa(z) = \bar{z}.$$

Suppose that $p, q : \mathbb{C} \to \mathbb{R}$ are smooth maps. Then it is easy to verify that the map $f : \mathbb{C} \to \mathbb{C}$

defined by

$$f(z) = p(|z|^2, \mathrm{Re}(z^n))z + q(|z|^2, \mathrm{Re}(z^n))\bar{z}^{n-1},$$

is $\mathbf{D}_n$-equivariant. The converse is also true. If f is a polynomial (or analytic) this follows straightforwardly from classical invariant theory (the maps p, q will then be polynomials or analytic). If f is smooth, the result is a consequence of Schwarz' theorem on smooth invariants. For more details on these matters see [26, Chapter XII, §4]. (The slides were produced by taking $p(|z|^2, \mathrm{Re}(z^n)) = \lambda + \alpha|z|^2 + \gamma\mathrm{Re}(z^n)$ and $q(|z|^2, \mathrm{Re}(z^n)) = \delta$, where $\lambda, \dots, \delta \in \mathbb{R}$.) ♡

1.3. Dynamics. We continue to suppose that we are given a linear action of a compact Lie group Γ on $\mathbb{R}^n$ (everything we say holds just as well for smooth actions on manifolds).

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth Γ-equivariant map. Suppose that $A \subset \mathbb{R}^n$ is a compact f-invariant set.

We shall assume that $f : A \to A$ is topologically transitive. That is, there exists $x_0 \in A$ such that the ω-limit point set $\omega(x_0) = A$.

Since f is Γ-invariant, it is natural to ask about the symmetry of the set A and to this end, we define the *symmetry group of* A by

$$\Sigma_A = \{\gamma \in \Gamma \mid \gamma(A) = A\}.$$

Obviously, Σ_A is a *closed* subgroup of Γ.

Example 1.3. Let $\mathbb{K}$ denote the group of complex numbers of unit modulus (also commonly denoted by $\mathrm{SO}(2)$ and S^1). Take the standard action of $\mathbb{K}$ on $\mathbb{C}$ defined by complex multiplication. Define $f(z) = c(1 + 0.5(1 - |z|^2))z$, where $|c| = 1$. Clearly $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{K}$-equivariant. Note that the unit circle $C \subset \mathbb{C}$ is f-invariant. For all non-zero $z \in \mathbb{C}$, $\omega(z) \subset C$. If $c = 1$, then every point $u \in C$ is fixed by f and so if we take $A = \{u\}$, $f : A \to A$ is trivially topologically transitive and $\Sigma_A = \{e\}$. Suppose $c \neq 1$. Fix $u \in C$ and let A denote the closure of the f-orbit of u. There are two possibilities. First suppose $c = \exp(2\pi\imath p/q)$, where $(p, q) = 1$, $p \neq 0$. Then A is a periodic orbit of f, prime period q and $\Sigma_A \cong \mathbb{Z}_q$ (cyclic group of order q). If instead $c = \exp(2\pi\imath\alpha)$, where α is irrational, then $A = C$ and $\Sigma_A = \mathbb{K}$. ♡

Lemma 1.4 (cf. [11]). *Suppose that $A = \omega(x_0)$ as above. Then*

1. $\omega(\gamma x_0) = \gamma\omega(x_0)$, $\gamma \in \Gamma$.
2. $\{\omega(\gamma x_0) \mid \gamma \in \Gamma\}$ *defines a partition of* $\Gamma(A)$ *into compact (disjoint) sets permuted by the action of* Γ.
3. $\gamma \in \Sigma_A$ *if and only if* $\gamma x_0 \in A$.
4. $\Sigma_{\gamma A} = \gamma\Sigma_A\gamma^{-1}$.

Proof. Statement (1) follows by Γ-equivariance of f. The remaining statements follow immediately from (1). □

We say a compact f-invariant set $A \subset \mathbb{R}^n$ is an *attractor* if there exists an open neighborhood U of A such that

- $f(U) \subset U$.
- $\cap_{n\geq 0} f^n(U) = A$.

Some basic questions

(1) Suppose that (V, Γ) is a representation of a finite group Γ. What subgroups of Γ can occur as symmetry groups of attractors for Γ-equivariant differentiable dynamical systems on V? (Similar question for smooth Γ-manifolds.)

Example 1.5. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbf{D}_6$-equivariant. Can f have an attractor with $\mathbf{D}_3$ symmetry? Answer: NO - even for continuous maps - by results of Dellnitz, Golubitsky & Melbourne [14]. ♡

Unlike Dellnitz et al., our emphasis will be on smooth invertible dynamical systems (diffeomorphisms and flows). For continuous maps, see [14, 5, 36].

(2) If Γ is finite, what can be said about the *structure* of attractors associated to Γ-equivariant dynamical systems?

In general, this is a difficult, and not at all well understood, classification problem. We confine ourselves to describing a broad class of examples where there is either a uniform hyperbolic structure or controlled non-uniformly hyperbolic structure. These examples very much develop from the standard non-equivariant theory of differentiable dynamical systems. However, we shall shortly give one exotic example that depends crucially on the underlying symmetry ...

(3) If Γ is compact, but not finite, what can be said about the statistical properties of attractors (invariant sets) with full Γ-symmetry? Do there even exist interesting examples of attractors will full Γ-symmetry?

An exotic equivariant flow on $\mathbb{R}^4$

We describe some recent work with X Peng. More details and computations can be found in [20, Appendix].

Let the symmetric group S_5 act on $\mathbb{R}^5$ by permutation of coordinates. Let H be the zero locus of $x_1 + \ldots + x_5$. Obviously, H is S_5-invariant. Furthermore S_5 acts (absolutely) irreducibly on H (that is, there are no proper S_5-invariant subspaces of H). Identify H with $\mathbb{R}^4$.

Let $\hat{\Gamma} \subset S_5$ be the group generated by

$$s = (12345), \quad t = (2453).$$

It is straightforward to verify that $|\hat{\Gamma}| = 20$ and that

$$st = ts^2.$$

It is easy to show that $\hat{\Gamma}$ is isomorphic to $\mathrm{Aff}_1(\mathbb{F}_5)$ – the invertible affine linear transformations of the field with five elements. The representation of S_5 on $\mathbb{R}^4$ yields a representation of $\hat{\Gamma}$ on $\mathbb{R}^4$ and it follows from the double transitivity of $\hat{\Gamma}$ that the representation is absolutely irreducible. Let $\Gamma = \langle \hat{\Gamma}, -I \rangle$. It follows from the above that $|\Gamma| = 40$ and Γ acts absolutely irreducibly on $\mathbb{R}^4$.

1.4. A basis for the action of Γ on $\mathbb{R}^4 = \mathbb{C}^2$. Let V be the four dimensional real subspace of $\mathbb{C}^4$ defined by

$$V = \{(z_1, z_2, \bar{z}_2, \bar{z}_1) \mid z_1, z_2 \in \mathbb{C}\}.$$

Let $\omega = \exp(\frac{2\pi\imath}{5})$. It may be shown (see [20]) that the action of Γ on $\mathbb{R}^4$ is equivalent to the action of Γ on V defined by

$$\begin{aligned} s(z_1, z_2) &= (\omega z_1, \omega^2 z_2), \\ t(z_1, z_2) &= (\bar{z}_2, z_1), \\ -I(z_1, z_2) &= (-z_1, -z_2). \end{aligned}$$

(Here we have identified (z_1, z_2) with $(z_1, z_2, \bar{z}_2, \bar{z}_1) \in V$.)

1.5. Geometry of the representation (V, Γ). We take (complex) coordinates (z_1, z_2) on V and corresponding real coordinates (x_1, y_1, x_2, y_2), where $z_i = x_i + \imath y_i$, $i = 1, 2$.

Lemma 1.6. *Representative proper fixed point spaces of (V, Γ) are given by*

(A) $\mathbf{A} = \mathbb{R}(1, 0, 1, 0) = V^{(t)}$ *(axis of symmetry).*
(B) $\mathbf{B} = \mathbb{R}(1, 0, -1, 0) = V^{(-t)}$ *(axis of symmetry).*
(S) $\mathbf{S} = \{(x_1, 0, x_2, 0) \mid x_1, x_2 \in \mathbb{R}\} = V^{(t^2)}$ *(submaximal stratum).*
(P) $\mathbf{P} = \{(0, y_1, 0, y_2) \mid y_1, y_2 \in \mathbb{R}\} = V^{(-t^2)}$ *(maximal orbit stratum).*

There are exactly five axes of type (A), five axes of type (B), five planes of type (S) and five planes of type (P). There are no three dimensional fixed point subspaces.

1.6. Cubic equivariant vector fields on V.

Lemma 1.7. *A basis for the homogeneous cubic equivariant polynomial maps from V to V is given by*

$$R(z_1, z_2) = (|z_1|^2 + |z_2|^2)(z_1, z_2), \quad p(z_1, z_2) = (\bar{z}_1^2 \bar{z}_2, z_1 \bar{z}_2^2), \quad q(z_1, z_2) = (z_2^3, \bar{z}_1^3)$$

We consider the following Γ-equivariant family of vector fields on V.

$$z_1' = \lambda z_1 - (|z_1|^2 + |z_2|^2) z_1 + \beta \bar{z}_1^2 \bar{z}_2 + \gamma z_2^3, \tag{1.1}$$

$$z_2' = \lambda z_2 - (|z_1|^2 + |z_2|^2) z_2 + \beta z_1 \bar{z}_2^2 + \gamma \bar{z}_1^3. \tag{1.2}$$

It turns out that if $|\beta + \gamma| < 2$ then there is a supercritical bifurcation at $\lambda = 0$ and that for $\lambda > 0$, all trajectories are forward asymptotic to an attracting Γ-invariant sphere. Henceforth, we assume $|\beta + \gamma| < 2$.

1.7. Equilibria along axes of symmetry. Rewriting the system (1.1,1.2) in real coordinates, we find that

$$\begin{aligned} x_1' &= \lambda x_1 - \|x\|^2 x_1 + \beta(x_1^2 x_2 - y_1^2 x_2 - 2x_1 y_1 y_2) + \gamma(x_2^3 - 3x_2 y_2^2), \\ y_1' &= \lambda y_1 - \|x\|^2 y_1 + \beta(-x_1^2 y_2 + y_1^2 y_2 - 2x_1 y_1 x_2) + \gamma(-y_2^3 + 3x_2^2 y_2), \\ x_2' &= \lambda x_2 - \|x\|^2 x_2 + \beta(x_1 x_2^2 - x_1 y_2^2 + 2y_1 x_2 y_2) + \gamma(x_1^3 - 3x_1 y_1^2), \\ y_2' &= \lambda y_2 - \|x\|^2 y_2 + \beta(y_1 x_2^2 - y_1 y_2^2 - 2x_1 x_2 y_2) + \gamma(y_1^3 - 3x_1^2 y_1). \end{aligned}$$

Equilibria on the axis **A**. If $\lambda \geq 0$, then there is a pair of equilibria $\pm a(\lambda)$ on the axis **A**. Computing, we find that

$$a(\lambda) = (\sqrt{\frac{-\lambda}{\beta+\gamma-2}}, 0, \sqrt{\frac{-\lambda}{\beta+\gamma-2}}, 0).$$

The eigenvalues of the linearization of the cubic system at $\pm a(\lambda)$ are

$$[-2\lambda : \frac{4\lambda\gamma}{\beta+\gamma-2} : \frac{\lambda}{\beta+\gamma-2}[(\gamma+3\beta) \pm \imath(3\gamma-\beta)].$$

Note that -2λ is the eigenvalue associated to the radial direction and that the eigenspace of the eigenvalue $\frac{4\lambda\gamma}{\beta+\gamma-2}$ lies in the x_1, x_2-plane $V^{(t^2)}$. The invariant space associated to the remaining pair of eigenvalues is the y_1, y_2-plane $V^{(-t^2)}$.

Equilibria on the axis **B**. For $\lambda \geq 0$, there is a pair of equilibria $\pm b(\lambda)$ on the axis **B** given by

$$b(\lambda) = (\sqrt{\frac{\lambda}{\beta+\gamma+2}}, 0, -\sqrt{\frac{\lambda}{\beta+\gamma+2}}, 0).$$

The eigenvalues of the linearization of the cubic system at $\pm b(\lambda)$ are

$$[-2\lambda : \frac{4\lambda\gamma}{\beta+\gamma+2} : \frac{\lambda}{\beta+\gamma+2}[(\gamma+3\beta) \pm \imath(3\gamma-\beta)].$$

The eigenspace of the eigenvalue $\frac{4\lambda\gamma}{\beta+\gamma+2}$ lies in the x_1, x_2-plane $V^{(t^2)}$. The invariant space associated to the remaining pair of eigenvalues is the y_1, y_2-plane $V^{(-t^2)}$.

1.8. Dynamics on and near the x_1, x_2-plane S. Dynamics on **S** are governed by the system

$$x_1' = \lambda x_1 - (x_1^2 + x_2^2)x_1 + \beta x_1^2 x_2 + \gamma x_2^3, \tag{1.3}$$

$$x_2' = \lambda x_2 - (x_1^2 + x_2^2)x_2 + \beta x_1 x_2^2 + \gamma x_1^3. \tag{1.4}$$

Since we are assuming that $|\beta+\gamma| < 2$, (1.3,1.4) has a globally attracting invariant circle $C(\lambda)$, for $\lambda > 0$. Necessarily, $\pm a(\lambda), \pm b(\lambda) \in C(\lambda)$.

Lemma 1.8. *Let $\lambda > 0$. If $\gamma \neq 0$ and $|\beta+\gamma| < 2$, then $\pm a(\lambda)$, $\pm b(\lambda)$ are the only nonzero equilibrium points of (1.3,1.4).*

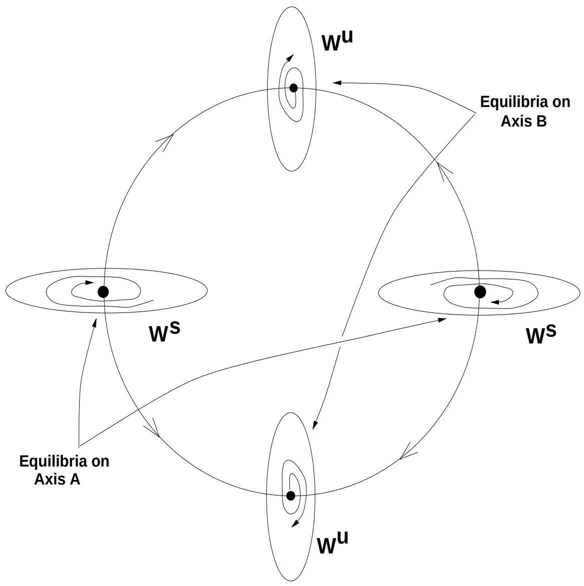

FIGURE 5. Dynamics near x_1, x_2 plane, case III, III*

Lemma 1.9. *Let $\lambda > 0$, $\gamma \neq 0$, and $|\beta + \gamma| < 2$. Stabilities of $a(\lambda)$, $b(\lambda)$ in the x_1, x_2-plane are as follows.*

(a) *If $\gamma > 0$, $\pm a(\lambda)$ are sinks, $\pm b(\lambda)$ are saddles and*

$$C(\lambda) = W^u(b(\lambda)) \cup W^u(-b(\lambda)) \cup \{a(\lambda), -a(\lambda)\}.$$

(b) *If $\gamma < 0$, $\pm b(\lambda)$ are sinks, $\pm a(\lambda)$ are saddles and*

$$C(\lambda) = W^u(a(\lambda)) \cup W^u(-a(\lambda)) \cup \{b(\lambda), -b(\lambda)\}.$$

Lemma 1.10. *Suppose that $\lambda > 0$ and and $|\beta + \gamma| < 2$.*

(1) *If $\gamma > 0$ and $\gamma + 3\beta < 0$, then $dim(W^u(a(\lambda))) = 2$, $dim(W^s(b(\lambda))) = 3$, $W^u(a(\lambda))$ and $W^s(b(\lambda))$ are transverse to $\mathbf{S}$ and there is a one-dimensional connection from $b(\lambda)$ to $a(\lambda)$ in $\mathbf{S}$.*
(2) *If $\gamma < 0$ and $\gamma + 3\beta > 0$, then $dim(W^u(b(\lambda))) = 2$, $dim(W^s(a(\lambda))) = 3$, $W^u(b(\lambda))$ and $W^s(a(\lambda))$ are transverse to $\mathbf{S}$ and there is a one-dimensional connection from $a(\lambda)$ to $b(\lambda)$ in $\mathbf{S}$.*
(3) *If the conditions of (1) and (2) are not satisfied and $\gamma(\gamma + 3\beta) \neq 0$, then exactly one of $a(\lambda)$, $b(\lambda)$ is a sink.*

1.9. Dynamics on the y_1, y_2-plane P. We refer the reader to [20, Appendix] for detailed results. Suffice it to say that either there are 8 equilibria on an invariant circle or there is a limit cycle. In all cases, equilibria and cycles are saddles.

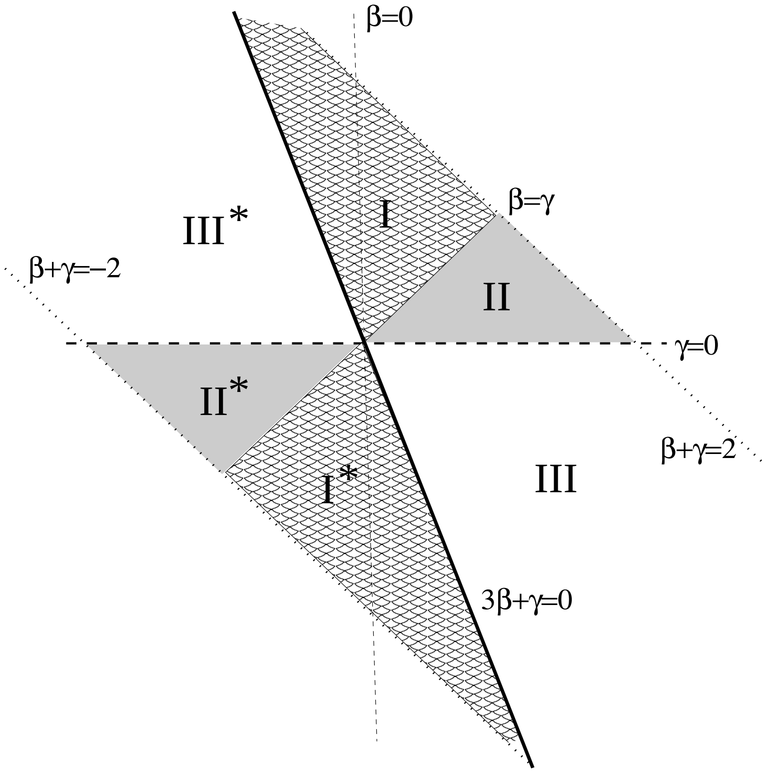

FIGURE 6. Stability of equilibria

1.10. Equilibria.

Lemma 1.11. *Let $\lambda > 0$. Provided that $\gamma(3\beta + \gamma) \neq 0$, the equilibria of (1.1,1.2) are hyperbolic and lie on the Γ-orbits of the axes* **A**, **B** *and the plane* **P**.

In Figure 6, we have divided the β, γ-plane into six stability regions. In regions I and II, (1.1,1.2) has ten hyperbolic sinks, all lying on the Γ-orbit of the axis **A**. In regions III there are no sinks (or sources). In regions II and III there are five limit cycles lying in the Γ orbit of **P**. We have an analogous situation in the regions I*, II* and III*, with A and B interchanged.

1.11. Dynamics. Using the dynamical systems package *dstool* [6], we have investigated the dynamics of (1.1,1.2). Granted our assumption $|\beta + \gamma| < 2$ and the homogeneity of higher order terms, we may fix $\lambda = 1$.

1.12. The regions I,I*, and II, II*. In these regions, there are always ten hyperbolic sinks lying on the Γ-orbits of one or other of the axes **A**, **B**. Typically, dynamics are asymptotic to one of these sinks.

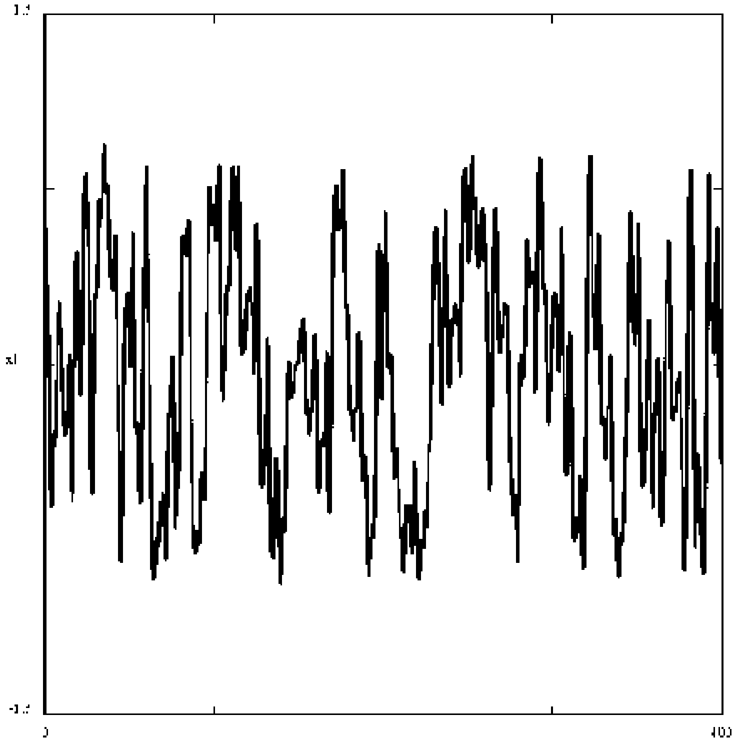

FIGURE 7. Time series for x_1 when $\beta = -0.35$, $\gamma = 1$

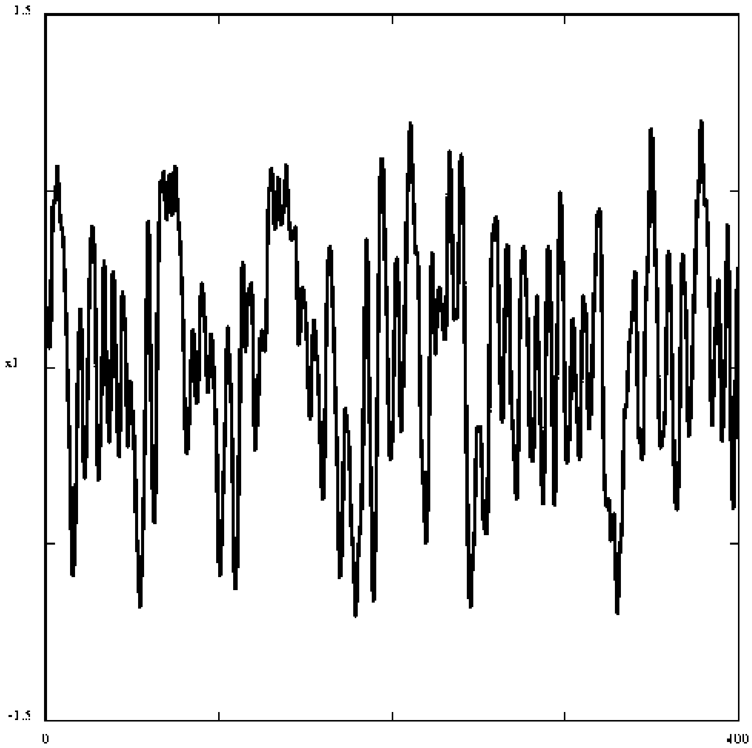

FIGURE 8. Time series for x_1 when $\beta = 1$, $\gamma = -0.6$

1.13. The regions III, III*. For $\beta, \gamma \in \mathrm{III} \cup \mathrm{III}^\star$, there are no equilibria which are sinks (or sources). Moreover, the limit cycle in the plane **P** is neither attracting nor repelling.

Restricting to the attracting flow-invariant three sphere, we see that there is a possibility that there are heteroclinic cycles connecting equilibria on axes of type **A** and **B**. Indeed, there is always a one-dimensional connection between $a(1)$ and $b(1)$ in the plane **S**. If, for example, $a(1)$ is a sink for the dynamics restricted to **S**, then $\dim(W^u(a(1))) = 2$, $\dim(W^s(b(1))) = 2$ (restricted to the invariant sphere) and so there is the possibility of a persistent (transverse) intersection between $W^u(a(1))$ and (the Γ-orbit) of $W^s(b(1))$. This situation, similar to what occurs in the Shilnikov bifurcation [28, 6.5], can be expected to lead to very complex dynamics. In our case, the symmetry is likely to result in the formation of a 'Shilnikov network' of heteroclinic cycles. Since the limit cycle in the plane **P** is a hyperbolic saddle, there are the additional possibilities of horseshoe dynamics, due to transverse intersections between the stable and unstable manifolds of the limit cycles, as well as intersections between the invariant manifolds of equilibria and limit cycles.

Numerical simulations confirm the complexity of the dynamics that arise in this system. In particular, it is likely that there are heteroclinic cycles connecting equilibria on axes of type **A** and **B**. In Figures 7, 8, we show time series obtained using *dstool*. In Figure 7, we show the plot of the variable x_1 against time when β, γ is close to the boundary of the region $\mathrm{III}^\star$. In the second plot, we show the time series when β, γ is in region III and well away from the boundary. In both cases, we have used a time step of $\Delta t = 0.002$ and plotted the result of 200,000 steps of a 4th order Runge-Kutta algorithm. Finally, we note that for least some values of $\beta, \gamma \in \mathrm{III} \cup \mathrm{III}^\star$, (1.1,1.2) has an attracting asymmetric limit cycle.

Finally, in Figure 4, we show the projection into the x_1, y_1-plane of a characteristic flow (parameters $\beta = 1$, $\gamma = -0.6$)

An attractor in $\mathbb{R}^3$

For the remainder of this lecture, we describe Williams' construction of expanding hyperbolic attractors [45] (see also the lecture notes by Newhouse [37]). Rather than giving the general theory, we star by looking at one example in $\mathbb{R}^3$ (no symmetry) that contains most of the features of the general case.

Abstract set up. Let $f : S^1 \to S^1$ be the map $f(z) = z^2$. Obviously f is an *expanding map*: $\|Tf(X)\| = 2\|X\|$, all $X \in TS^1$. We define the classical *solenoid* $\mathcal{S} \subset \Pi^\infty S^1$ as the inverse limit

$$\mathcal{S} = \{(z_i)_{i \geq 0} \mid f(z_i) = z_{i-1}, i \geq 1\}.$$

The map f induces a *shift homeomorphism* $\sigma : \mathcal{S} \to \mathcal{S}$ by

$$\sigma(z)_j = z_{j+1}, \quad (z \in \mathcal{S}, j \geq 0).$$

It is straightforward to verify that periodic points are dense in $\mathcal{S}$ and that $\sigma : \mathcal{S} \to \mathcal{S}$ is topologically transitive (this is already true for $f : S^1 \to S^1$).

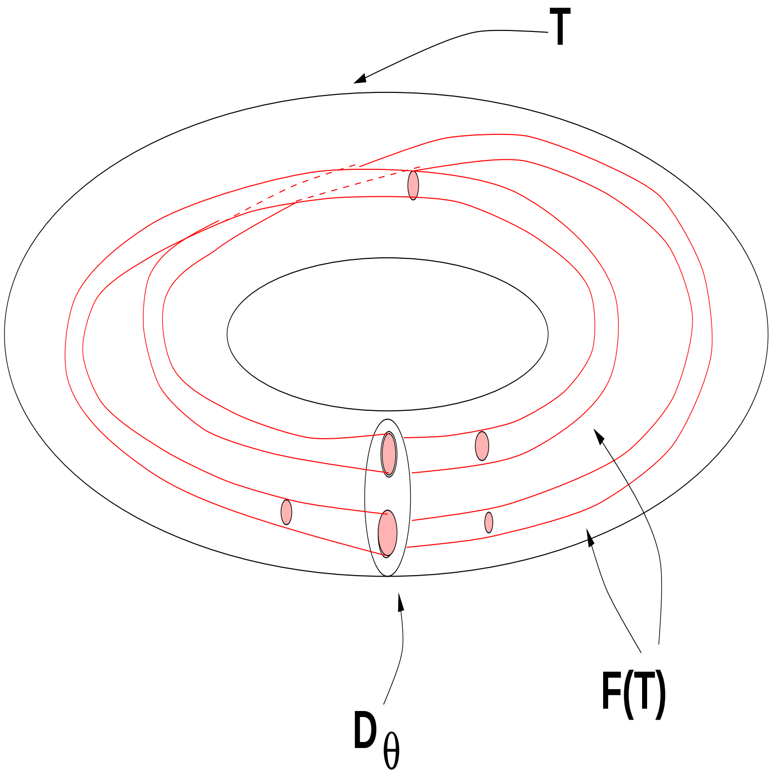

FIGURE 9. The image $F(\mathbf{T})$

Geometric realization. We want to realize $\sigma : \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{S}}$ as a hyperbolic attractor of a smooth diffeomorphism of $\mathbb{R}^3$.

Let $\mathbf{D} \subset \mathbb{C}$ denote the closed disk of radius one half. Consider the solid torus $\mathbf{T} = \mathbf{D} \times S^1$ Regard $\mathbf{T}$ as embedded in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ so that the core $\{0\} \times S^1$ is the unit circle in the (x, y)-plane.

Denoting coordinates on $\mathbf{T}$ by (z, θ), let $f^\star : \mathbf{T} \to \mathbf{T}$ be defined by $f^\star(z, \theta) = (0, 2\theta)$. Clearly, $f^\star$ is a smooth extension of $f : S^1 \to S^1$ to $\mathbf{T}$. Obviously, $f^\star$ is not injective and our next step will be to deform $f^\star$ to an embedding $F : \mathbf{T} \to \mathbf{T}$. Specifically, we define

$$F(z, \theta) = (\frac{z}{8} + \frac{1}{4}\exp(\imath\theta), 2\theta).$$

It is easy to verify that F is a smooth embedding. Indeed, we can make an arbitrarily C^∞-small perturbation of $f^\star$ to an embedding of $\mathbf{T}$ in $\mathbf{T}$.

For $\theta \in S^1$, let $\mathbf{D}_\theta \subset \mathbf{T}$ denote the transverse disk $\mathbf{D} \times \{\theta\}$. It follows from our construction of F that F preserves the transverse disks. That is, for each $\theta \in S^1$, we have

$$F(\mathbf{D}_\theta) \subset \mathbf{D}_{2\theta}.$$

Indeed, $F(\mathbf{T}) \cap \mathbf{D}_{2\theta} = F(\mathbf{D}_\theta) \cup F(\mathbf{D}_{\theta+\pi})$.

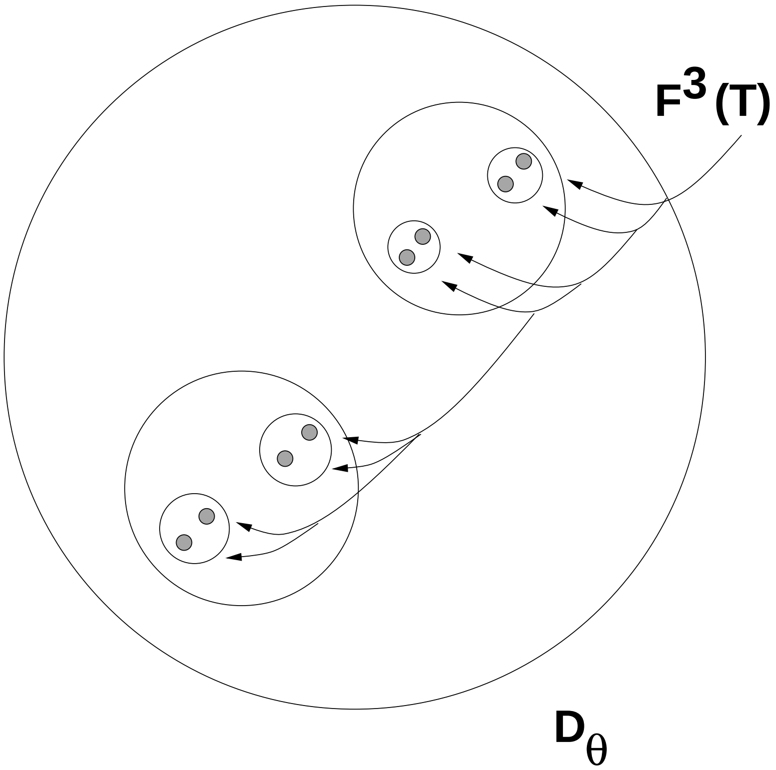

FIGURE 10. $F^3(\mathbf{T}) \cap \mathbf{D}_\theta$

Define $\Lambda = \cap_{n\geq 0} F^n(\mathbf{T})$. Certainly Λ is an F-invariant compact subset of $\mathbf{T}$. It follows easily from the construction that Λ is homeomorphic to the solenoid $\mathcal{S}$ we defined above and that $F|\Lambda$ is conjugate to the shift map (points in the $\mathcal{S}$ or Λ have unique history).

Furthermore, Λ has a *hyperbolic* structure. The contracting subspaces are given by the transverse disks $\mathbf{D}_\theta$. More precisely, given $\lambda = (z, \theta) \in \Lambda$, let $\mathbb{E}^s_\lambda$ be the 2-dimensional linear subspace of $\mathbb{R}^3$ tangent to $\mathbf{D}_\theta$ and $\mathbb{E}^u_\lambda$ be the 1-dimensional subspace tangent to Λ at λ. In this way, we define a 2-dimensional TF-invariant contracting bundle $\mathbb{E}^u$ and 1-dimensional TF-invariant expanding bundle $\mathbb{E}^u$ over Λ such that

$$T_\Lambda \mathbb{R}^3 = \mathbb{E}^u \oplus \mathbb{E}^s.$$

We also have the hyperbolic estimates

$$\begin{aligned} \|TF(X)\| &= \frac{1}{8}\|X\|, \ \ (X \in \mathbb{E}^s), \\ \|TF(X)\| &= 2\|X\|, \ \ (X \in \mathbb{E}^u). \end{aligned}$$

The problem remains of extending F to a diffeomorphism of $\mathbb{R}^3$. For this we need the isotopy theorem of differential topology. In our case, we are given a smooth embedding $F : \mathbf{T} \rightarrow \mathbb{R}^3$. It follows from the isotopy theorem that F extends to a diffeomorphism of $\mathbb{R}^3$

if F is smoothly isotopic[1] to the identity map of $\mathbf{T}$. This is easy to verify for our example ($F(\mathbf{T})$ is unknotted) and so F extends to a smooth diffeomorphism of $\mathbb{R}^3$.

To summarize the method: Start with a smooth expanding map of a 1-dimensional space. Embed in Euclidean space. Thicken to a manifold with boundary, foliated by transverse disks (tubular neighborhood). Perturb to an embedding. Make sure that the perturbed map is isotopic to the identity. Extend to ambient space. The contracting directions correspond to transverse disks, the expanding directions are approximately those of the core 1-dimensional manifold.

Notes on Lecture 1

Useful references on compact (non-finite) Lie groups and their representations include the books by Adams [1] and Bröcker & tom Dieck [10].

Haar measure. Every compact Lie group Γ carries a unique left and right translation invariant probability measure $d\mu$ – Haar measure. Haar measure provides a vital tool in constructing symmetric objects. For example, suppose X is a smooth vector field on a Γ-manifold M. If we define

$$\tilde{X}(x) = \int_\Gamma \gamma^{-1} X(\gamma x)\, d\mu, \quad (x \in M),$$

then it is easy to see that $\tilde{X}$ is smooth and Γ-equivariant. Using Haar measure, we can average any Riemannian metric on M to obtain a Γ-invariant Riemannian metric on M. In case $M = \mathbb{R}^n$ and $\Gamma \subset \mathrm{GL}(n,\mathbb{R})$, it follows that we can always choose a Γ-invariant inner product on $\mathbb{R}^n$ relative to which $\Gamma \subset \mathrm{O}(n)$.

Lie groups. Let $\mathrm{O}(n)$ denote the group of $n \times n$ orthogonal matrices and $\mathrm{SO}(n)$ denote the determinant one subgroup of $\mathrm{O}(n)$ – the special orthogonal group. Both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact *submanifolds* of $\mathrm{GL}(n,\mathbb{R})$. In particular, group composition and inversion define smooth (in fact algebraic) operations on $\mathrm{O}(n)$ and $\mathrm{SO}(n)$.

It may be shown that every closed subgroup of a Lie group is a Lie group. In particular, every closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ is a Lie group. Furthermore, if Γ is a *compact* Lie group then Γ may be represented as a subgroup of an orthogonal group. This follows from the fact that every compact Lie group has a *faithful* representation.

Example 1.12. The group $\mathbb{K}$ is Abelian. For $m \geq 1$, let $\mathbb{K}^m$ denote the m-fold product of $\mathbb{K}$. We refer to $\mathbb{K}^m$ as the m-torus. Every compact connected m-dimensional Abelian Lie group is (smoothly) isomorphic to $\mathbb{K}^m$. ♡

Let Γ be a connected Lie group with center $Z(\Gamma)$. We recall that

- Γ is *simple* if Γ has no nontrivial normal subgroups.
- Γ is *almost simple* if every normal subgroup of Γ is finite.

[1] That is, there exists a smooth map $H : \mathbf{T} \times [0,1] \to \mathbb{R}^3$ such that $H_t : \mathbf{T} \to \mathbb{R}^3$ is an embedding, $t \in [0,1]$, and $H_0 = F$, $H_1 = I_{\mathbf{T}}$.

- Γ is semisimple if $Z(\Gamma)$ is finite.

Remark 1.13. Let $\mathfrak{g}$ denote the Lie algebra of Γ. Since Γ is connected, every *finite* normal subgroup of Γ is a subgroup of $Z(\Gamma)$. Consequently, Γ is simple as a group if and only if $\mathfrak{g}$ is a simple Lie algebra and $Z(\Gamma)$ is trivial. If $\mathfrak{g}$ is simple, then Γ is almost simple. $\diamondsuit$

Examples 1.14. (1) For $n > 2$, the groups $\mathrm{SO}(n)$ are not Abelian. In fact, $\mathrm{SO}(2n)$ is almost simple, $n > 1$, and $\mathrm{SO}(2n+1)$ is simple $n > 0$.
(2) For $n \geq 2$, the special unitary groups $\mathrm{SU}(n)$ are almost simple.
(3) For $n \geq 2$, the symplectic groups $\mathrm{Sp}(n)$ are compact, connected and almost simple ($\mathrm{Sp}(1) = \mathrm{SU}(2)$).
(4) There are five 'exceptional' almost simple Lie groups: $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$.
(5) If we let $G^\star$ denote the universal cover of G, then it is known that every compact connected Lie group Γ is a quotient by a finite group of a product $\mathbb{K}^m \times G_1^\star \times \dots G_N^\star$, where $G_1, \dots, G_N$ are groups listed in (1–4). $\heartsuit$

Representations and actions. Suppose that Γ is a compact Lie group. A representation of Γ on $\mathbb{R}^n$ is a homomorphism $\rho : \Gamma \to \mathrm{O}(n)$. Associated to ρ, we have an action of Γ on $\mathbb{R}^n$ defined by

$$\gamma(X) = \rho(\gamma)(X), \ (\gamma \in \Gamma, X \in \mathbb{R}^n).$$

In the sequel, we usually write γX rather than $\gamma(X)$.

More generally, if M is a smooth manifold with diffeomorphism group $\mathrm{Diff}(M)$, and $\rho : \Gamma \to \mathrm{Diff}(M)$ is a homomorphism then we may define an action of Γ on M by

$$\gamma(m) = \rho(\gamma)(m), \ (\gamma \in \Gamma, m \in M).$$

Example 1.15. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. If we are given a representation of Γ on $\mathbb{R}^{n+1}$ then this action restricts to give a smooth action of Γ on S^n. $\heartsuit$

We say that a map $f : M \to M$ is Γ-*equivariant* (or just equivariant) if

$$f(\gamma m) = \gamma f(m), \quad (m \in M, \gamma \in \Gamma).$$

If TM denotes the tangent bundle of M, we have a natural action of Γ on TM defined by $\gamma(X) = T\gamma(X)$, $X \in TM$, $\gamma \in \Gamma$. We say that a vector field X on M is equivariant if $X : M \to TM$ is Γ-equivariant with respect to the actions of Γ on M and TM.

Orbits and isotropy groups. Suppose that the compact Lie Γ acts smoothly on M. Given $m \in M$, define

$$\Gamma m = \{\gamma m \mid \gamma \in \Gamma\}.$$

We refer to Γm as the Γ-*orbit* through m or just the Γ-orbit of m. Γm is invariant by Γ and so we have an induced action of Γ on Γm.

Given $m \in M$, we define the *isotropy subgroup* of Γ at m by

$$\Gamma_m = \{\gamma \in \Gamma \mid \gamma m = m\}.$$

Obviously, Γ_m is a closed (therefore Lie) subgroup of Γ.

Using the easily proven fact that the homogeneous space $\Gamma \to \Gamma/\Gamma_m$ admits local smooth sections, it is not hard to show that Γm is a smooth submanifold of M, all $m \in M$. For this, and other details about compact Lie group actions, see [7], especially Chapter 6.

We have the following elementary properties relating group orbits and isotropy groups.

- For all $\gamma \in \Gamma$, $m \in M$, $\Gamma_{\gamma m} = \gamma \Gamma_m \gamma^{-1}$.
- Γm is Γ-equivariantly diffeomorphic to Γ/Γ_m, where we let Γ act on Γ/Γ_m by left translation. Conversely, Γ_m is Γ-equivariantly diffeomorphic to Γ_n if and only if Γ_m and Γ_n are *conjugate* subgroups of Γ: $\exists \gamma \in \Gamma$ such that $\Gamma_n = \gamma \Gamma_m \gamma^{-1}$.

If H is an isotropy group for the action of Γ on M, let (H) denote the conjugacy class of H in Γ. It follows easily from the differentiable slice theorem (see below) that if M is compact or a Γ-representation, then there are only finitely many isotropy types.

Mappings and isotropy groups. Suppose M and N are Γ-manifolds and that $f : M \to N$ is Γ-equivariant. It follows from the equivariance of f that

$$\Gamma_x \subset \Gamma_{f(x)}, \quad (x \in M). \tag{1.5}$$

Furthermore, if f is 1:1 then $\Gamma_x = \Gamma_{f(x)}$, all $x \in M$.

If H is a subgroup of Γ, let M^H denote the fixed point space of the action of H on M. That is,

$$M^H = \{x \in M \mid hx = x, \text{ for all } h \in H\}.$$

It follows from (1.5) that if $f : M \to N$ is Γ-equivariant, then $f(M^H) \subset N^H$ for all subgroups H of Γ. This observation has been made the basis of a small industry in case f is a vector field on M and M^H is 1-dimensional [26].

Slice theorem. From the technical point of view, the most important property of the smooth action of a compact Lie group is the existence of *slices*. Roughly speaking, the differentiable slice theorem says that compact Lie group actions are locally linearizable transverse to the group action.

Theorem 1.16. *Let M be a Γ-manifold. Given $m \in M$, there exists a Γ_m-representation (V, Γ_m) and a smooth Γ_m-equivariant embedding $j : V \to M$ satisfying the following properties:*

1. *$j(0) = m$.*
2. *$T_m M = T_m \Gamma m \oplus T_m j(V)$ (j is transverse to Γm at m).*
3. *$\gamma j(V) \cap j(V) \neq \emptyset$ if and only if $\gamma \in \Gamma_m$.*
4. *$\cup_{\gamma \in \Gamma} \gamma j(V)$ is an open neighborhood of Γ_m.*

The set $j(V)$ is called a differentiable slice for the action at m.

It is useful to reformulate the slice theorem. To this end, observe that if (V, Γ_m) is a Γ_m-representation, then we can define a free action of Γ on the product $\Gamma \times V$ by

$$\gamma(g, v) = (g\gamma^{-1}, gv), \quad ((g, v) \in \Gamma \times V, \gamma \in \Gamma).$$

Let $\Gamma \times_{\Gamma_m} V$ denote the orbit space $(\Gamma \times V)/\Gamma_m$. Since the action of Γ on $\Gamma \times V$ is free, $\Gamma \times_{\Gamma_m} V$ inherits the structure of a smooth Γ-manifold[2]. The space $\Gamma \times_{\Gamma_m} V$ is usually called the *twisted product* of Γ and V.

Fix a Γ-invariant Riemannian metric on M. Let $V = T_m\Gamma m^{\perp}$. Since $T_m\Gamma m$ is obviously Γ_m-invariant, it follows that V is a Γ_m-representation. The normal bundle of Γm is easily seen to be isomorphic to the twisted product $\Gamma \times_{\Gamma_m} V$. Hence the slice theorem follows from the equivariant version of the tubular neighborhood theorem. Of course, the existence of slices follows rather easily if M is a Γ-representation.

Let M be a smooth Γ-manifold. We say that a point $m \in M$ has *principal isotropy type* if given any $x \in M$ there exists $\gamma \in \Gamma$ such that $\Gamma_m \subset \gamma\Gamma_x\gamma^{-1}$. That is, the isotropy of m is as small as possible.

An important, and easy, consequence of the differentiable slice theorem is the following result.

Theorem 1.17. *Let M be a connected smooth Γ-manifold. The set of points in M with principal isotropy type form an open and dense subset of M.*

Remark 1.18. If Γ is finite and M is connected, it is no loss of generality to assume that points with principal isotropy type have trivial isotropy. Indeed, suppose $m \in M$ has principal isotropy. Let $U \subset M$ be the subset consisting of points with isotropy group Γ_m. It follows from the slice theorem that U is an open subset of M. If $x \in \partial U$ and S is a slice at x, then Γ_m fixes all points in the open subset $U \cap S$ and so must fix all points in S. It follows that Γ_m fixes all points in M. Since $\Gamma_{\gamma m} = \gamma\Gamma_m\gamma^{-1}$, it follows that Γ_m is a normal subgroup of Γ. Now replace Γ by Γ/Γ_m.

Although this result does not generalize to general compact Lie groups, it does hold for 'most' representations. $\diamondsuit$

A second important application of the differentiable slice theorem is in the construction of equivariant vector fields and maps. For example, suppose that $j : V \to M$ is a slice at m. Let X be a Γ_m-equivariant vector field defined on $j(V)$ (that is, a smooth section of $T_{j(V)}M$. The vector field X extends by Γ-equivariance to all of $\Gamma(j(V))$. Indeed, for $v \in j(V)$, $\gamma \in \Gamma$, define $\tilde{X}(\gamma v) = \gamma X(v)$. If X has compact support, the same is true for $\tilde{X}$ and so $\tilde{X}$ extends by zero to all of M.

Isotopy theorem. Let M be a smooth Γ-manifold and suppose that N is a closed Γ-invariant subset of M. We shall shortly encounter situations where we have defined a smooth equivariant embedding $i : N \to M$ and we want to extend i to a smooth Γ-equivariant diffeomorphism of M. The equivariant version of the isotopy theorem tells us when we can do this (for the detailed proof, see Bredon [7, Chapter VI]).

Theorem 1.19. *Let N be a compact Γ-invariant subset of the Γ-manifold M. Suppose that $i : N \to M$ is a smooth equivariant embedding and that i is smoothly equivariantly isotopic to the identity map of N. Then i extends to a smooth equivariant diffeomorphism of M.*

[2]In fact, $\Gamma \times_{\Gamma_m} V$ has the structure of a real non-singular algebraic variety.

Proof. We follow the proof of the standard isotopy theorem in differential topology. The main part of that proof is to construct a smooth vector field on $M \times [0, 1]$ which extends the vector field defined by the isotopy. Average this vector field over Γ and use the fact that the flow of an equivariant vector field is equivariant. $\square$