# Approximation of Convex Bodies by Rhombi and by other Axially Symmetric Bodies

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**Abstract.** Let $C$ be an arbitrary planar convex body. We prove that $C$ contains an axially symmetric convex body of area at least $\frac{2}{3}|C|$. Also approximation by some specific axially symmetric bodies is considered. In particular, we can inscribe a rhombus of area at least $\frac{1}{2}|C|$ in $C$, and we can circumscribe a homothetic rhombus of area at most $2|C|$ about $C$. The homothety ratio is at most 2. Those coefficients $\frac{1}{2}$ and 2 and also the homothety ratio 2 cannot be improved.

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**Key words:** convex body, axially symmetric body, rhombus, isosceles triangle, approximation, area, homothety.

Our aim is to approximate convex bodies of Euclidean plane $E^2$ by arbitrary axially symmetric bodies. We are interested in three kinds of such approximation. For a given convex body $C \subset E^2$ we are looking for an axially symmetric convex body $A$ of possibly large area contained in $C$, for an axially symmetric convex body $B$ of possibly small area containing $C$, and for an axially symmetric convex body $D$ such that $D \subset C \subset h(D)$, where $h$ is a homothety with possibly small positive homothety ratio. We are also looking for some specific axially symmetric bodies like rhombi, isosceles triangles and some axially symmetric hexagons which well approximate $C$.

The area of a convex body $C \subset E^2$ is denoted by $|C|$ and the length of the segment $ab$ by $|ab|$. Nohl [19] proves that every centrally symmetric convex body $C \subset E^2$ contains an axially symmetric convex body of area at least $2(\sqrt{2}-1)|C|$ and that the coefficient $2(\sqrt{2}-1)$ cannot be enlarged. Krakowski [16] shows that every convex body $C \subset E^2$ contains an axially symmetric body of area at least $\frac{5}{8}|C|$. We improve this coefficient up to $\frac{2}{3}$. We also show that $C$ is contained in an axially symmetric convex body of area at most $\frac{31}{16}|C|$.

In the survey article about measures of symmetry of convex bodies, Grünbaum suggests to consider also measures of axiality of convex bodies (see [13] p. 263). A few such

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measures of axiality are proposed by deValcourt [6], who lists a number of related estimates without proofs. In next two papers [7] and [8] he presents detailed considerations. In particular, deValcourt considers approximation of any convex body $C \subset E^2$ by isosceles triangles, by axially symmetric octagons, and by some axially symmetric quadrangles called kits.

He also recalls the result of Radziszewski [20] that every convex body $C \subset E^2$ contains a rectangle of area at least $\frac{1}{2}|C|$ and observes that $C$ is contained in a rectangle of area at most $2|C|$. Of course, the classic theorem of John [15] about ellipses containing and contained in $C$ is of a similar nature. DeValcourt [7] pays attention that nothing is known about a rhombus of possibly large area contained in $C$ and about a rhombus of possibly small area containing $C$. Our aim is to give the answers: every convex body $C$ contains a rhombus of area at least $\frac{1}{2}|C|$ and it is contained in a rhombus of area at most $2|C|$. Both the coefficients cannot be improved which follows from the example of a triangle in the part of $C$.

A *regular convex body* in the plane is a convex body such that only one supporting line can be provided through every its boundary point, and such that every supporting line meets this body only at one point (see [9], p. 31).

## 1. Approximation by rhombi

There exists a rectangle $R$ inscribed in $C$ such that its homothetic copy of positive ratio at most 2 is circumscribed about $C$ (see [21], [18] and [22]). This fact about approximation by a pair of homothetic rectangles is analogical to the planar version of the classic theorem of John [15] on a pair of homothetic ellipses with positive ratio at most 2 approximating $C$. Our aim is to give one more theorem of this kind.

**THEOREM.** *Let $C \subset E^2$ be a convex body. There exists a rhombus $R$ inscribed in $C$ such that a homothetic copy $R'$ of $R$ is circumscribed about $C$. The ratio of homothety is at most 2.*

*Proof.* First we consider a regular convex body $C$. For every direction $\theta$ we can inscribe exactly one rhombus $R(\theta)$ in $C$ with a diagonal of direction $\theta$. This follows from [5], [10] and [23].

Denote by $a(\theta)$, $b(\theta)$, $c(\theta)$ and $d(\theta)$ the consecutive vertices of $R(\theta)$; assume that the vector $ab$ is parallel to $\theta$ and oriented as $\theta$.

As we rotate $\theta$, the vertices of $R(\theta)$ change continuously. This easily follows from the Bolzano-Weierstrass theorem (comp. [5], where the continuity of the position of the angles of the rhombus $R(\theta)$ is shown).

For every $R(\theta)$ we construct the circumscribed parallelogram $R'(\theta)$ of $C$ with sides parallel to the sides of $R(\theta)$. It is clear that $R'(\theta)$ exists and is unique for every $\theta$.

Denote by $a'(\theta)$, $b'(\theta)$, $c'(\theta)$, $d'(\theta)$ the corresponding vertices of $R'(\theta)$ (we mean that the vectors $a(\theta)b(\theta)$ are $a'(\theta)b'(\theta)$ are parallel and have a common orientation). Consider four strips, each determined by pair of parallel segments contained in the bounding lines of the strip: the strip $S(\theta)$ determined by the segments $a(\theta)b(\theta)$ and $c(\theta)d(\theta)$, the strip $S'(\theta)$ determined by the segments $a'(\theta)b'(\theta)$ and $c'(\theta)d'(\theta)$, the strip $T(\theta)$ determined by the segments $b(\theta)c(\theta)$ and $d(\theta)a(\theta)$, the strip $T'(\theta)$ determined by the segments $b'(\theta)c'(\theta)$ and $d'(\theta)a'(\theta)$. Denote by $f(\theta)$ the ratio of the width of $S'(\theta)$ to the width of $S(\theta)$. Let $g(\theta)$ mean the ratio of the width of $T'(\theta)$ to the width of $T(\theta)$. From the continuity of the position of the vertices of $R(\theta)$ we see that $f(\theta)$ and $g(\theta)$ are continuous functions. Since $f(\theta_1) = g(\theta_2)$ for perpendicular directions $\theta_1$ and $\theta_2$, we conclude that there exists a direction $\theta_0$ such that $f(\theta_0) = g(\theta_0)$. Thus $R'(\theta_0)$ is a homothetic copy of $R(\theta_0)$. Hence our Theorem is shown in the case of a regular convex body $C$.

Arbitrary convex body $C \subset E^2$ can be presented as a limit of a sequence of regular convex bodies $C_1, C_2, \dots$ (see [9]). As it is shown above, for every $C_i$ there is a pair of homothetic inscribed and circumscribed homothetic rhombi $R_i$ and $R'_i$. By four consecutive choices (each for one vertex), from the sequence $R_1, R_2, \dots$ we can select a subsequence which is convergent to a rhombus *abcd* inscribed in $C$. This rhombus and the circumscribed parallelogram $a'b'c'd'$ with parallel sides are homothetic because $a'b'c'd'$ is just the limit of a subsequence of the sequence of rhombi $R'_1, R'_2, \dots$. Thus we have obtained the promised pair $R, R'$ of homothetic rhombi.

From Lemma in [18], where $s_1 = s_2$, it follows that the positive ratio of homothety is at most 2, which ends the proof. ∎

The ratio 2 in our Theorem cannot be lessened because of the example of an arbitrary triangle as $C$. It cannot be also lessened for centrally symmetric convex bodies. It follows from the example of any rectangle with the ratio of lengths of the perpendicular sides at least 2.

Analogically as in Part 6 of the proof of Theorem in [18], we show that the inscribed rhombus $R$ in our Theorem has area at least $\frac{1}{2}|C|$, and that the circumscribed rhombus $R'$ has area at most $2|C|$. Thus we obtain the following Corollary.

**COROLLARY 1.** *Let $C$ be a planar convex body. We can inscribe a rhombus of area at least $\frac{1}{2}|C|$ in $C$, and we can circumscribe a rhombus of area at most $2|C|$ about $C$.*

## 2. Approximation by isosceles triangles

**COROLLARY 2.** *Let $C \subset E^2$ be a convex body. There exists an isosceles triangle $T \subset C$ such that its homothetic copy $T'$ of ratio at most $\frac{7}{2}$ contains $C$. We have $|T'| \le \frac{49}{16}|C|$.*

*Proof.* Consider the inscribed rhombus $R = abcd$ and the homothetic circumscribed rhombus $R' = a'b'c'd'$ as in Theorem. Denote by $\rho$ the positive ratio of this homothety. When we prolong the sides of the rhombus $abcd$, they intersect the sides of the rhombus $a'b'c'd'$. We obtain four parallelograms at the vertices of $a'b'c'd'$ (see Figure 1).

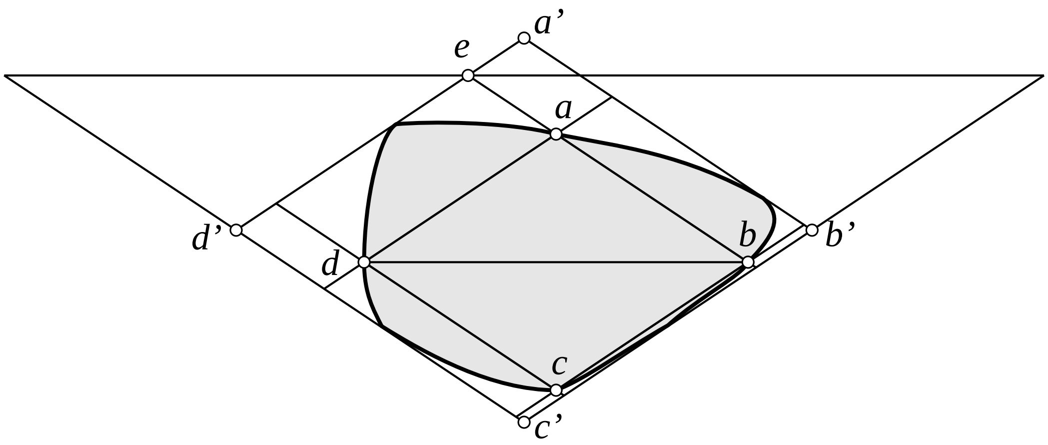


*Figure 1.*

One of the parallelograms contains translates of the three other parallelograms. We do not loose the generality of the considerations assuming that this "largest" parallelogram $K$ is at vertex $a'$. Of course, the triangle $T = bcd$ is isosceles. Now we describe the homothetic copy $T'$ promised in the formulation of Corollary 2. Two sides of $T'$ contain the segments $b'c'$ and $c'd'$. The third side of $T'$ is parallel to the segment $bd$ and passes through a vertex $e$ of $K$ in such a way that $T'$ contains all vertices of $K$ different from $a'$. This side is disjoint with the interior of $C$. This follows from our construction and from the convexity of $C$. So we have $T \subset C \subset T'$. Of course, $T'$ is a homothetic copy of $T$. Let us estimate the ratio $\delta$ of this homothety and the area of $|T'|$. Since the ratio of lengths of two segments (and also the ratio of the areas of two bodies) does not change under affine transformations, we do not make our considerations narrower assuming that $R$ is a square of side 1. Put $\kappa = |ea'|$. Our assumption about $K$ implies that $\kappa \geq \frac{1}{2}(\rho - 1)$. We have $\delta = 2\rho - \kappa \leq 2\rho - \frac{1}{2}(\rho - 1) = \frac{3}{2}\rho + \frac{1}{2} \leq \frac{7}{2}$. The last inequality follows from $1 \leq \rho \leq 2$. Since $R$ is inscribed in $C$ and since $C$ touches the sides of $|R'|$, we get $|C| \geq 1 + 2 \cdot \frac{1}{2}(\rho - 1) = \rho$. Moreover $|T'| = \frac{1}{2}(2\rho - \kappa)^2 \leq \frac{1}{2}\big((2\rho - \frac{1}{2}(\rho - 1)\big)^2 = \frac{1}{8}(9\rho^2 + 6\rho + 1)$. Thus $|T'|/|C| \leq \frac{1}{8}(9\rho + 6 + \rho^{-1}) \leq \frac{49}{16}$, which results from $1 \leq \rho \leq 2$.

■

The question is how much the ratio from Corollary 2 can be lessened. Surely not below $1 + \frac{1}{2}\sqrt{5}$ because of the example of the regular pentagon as $C$; see [11], [17] and [1], where analogical approximation by a pair of triangles (not necessarily isosceles) is considered. What is more, in the following example the best ratio is over $1 + \frac{1}{2}\sqrt{5}$. In the approximation of the regular pentagon by a pair of homothetic triangles, the best possible ratio $1 + \frac{1}{2}\sqrt{5}$ is attained exactly for five symmetric positions of a triangle inscribed in the regular pentagon. Each of the triangles is regular and has a vertex at a vertex of the pentagon (see Figure 2 in [17]). When we apply an affine transformation which stretches the regular pentagon along one of its axes of symmetry, then only one of those five triangles remains isosceles. It is the only isosceles triangle in $C$ for which the ratio $1 + \frac{1}{2}\sqrt{5}$ is attained. Thus if we lessen the stretched pentagon by moving a little the vertex which is on the axis of symmetry along the axis of symmetry, the best homothety ratio for a pair of approximating homothetic isosceles triangles is over $1 + \frac{1}{2}\sqrt{5}$. It is matter of a tedious calculation to find the optimum estimate from below here.

DeValcourt [8] presents two "theorems" which say that every convex body $C \subset E^2$ contains an isosceles triangle of area at least $\frac{3}{8}|C|$ and that it is contained in an isosceles triangle of area at most $2|C|$. He presents a proof of the first "theorem" and explains that the second proof is analogical. The given proof is incorrect and the second one cannot be provided by the proposed approach. The author considers a triangle $T_\varphi$ with a side of a given direction $\varphi$ which is contained in (a strictly convex) $C$. He changes $\varphi$ in order to apply continuity arguments which should lead to the conclusion that there is a specific $\varphi_0$ for which the triangle $T_{\varphi_0}$ is isosceles. The proof is based on the claim about the continuity of the function $f(\varphi) = a'/a''$, where $a'$ and $a''$ arthe distances of the two vertices of $T_\varphi$ lying in e a line of direction $\varphi$ from the orthogonal projection of the remaining vertex of $T_\varphi$ on this line. But this is not true because the position of $T_\varphi$ may vary non-continously. For instance, let $C$ be the convex hull of the union of the unit circle centered at $(0, 0)$, of the point $(0.01, 1.01)$ and of a small piece of the concentric circle of radius 1.01 with an endpoint at $(0, -1.01)$ which is in the fourth quarter of our coordinate system (if we wish to have a strictly convex body, we can exchange the four segments in the boundary by some pieces of circles which differ so little that the horizontal lines through the points $(0.01, 1.01)$ and $(0, -1.01)$ still support $C$). For the direction $\varphi = 0$ of the axis $Ox$ we have two triangles $T_\varphi$ with the maximum possible area; one has a vertex at $(0.01, 1.01)$ and the other at $(0, -1.01)$. The ratios $a'/a''$ in those two triangles are different. Even if we choose one of them, the trouble remains because the ratio $a'/a''$ does not change continously at $\varphi = 0$. Just the one-sided limits of this ratio at $\varphi = 0$ are different.

The proof is incorrect also in the case if by $T_\phi$ the author means the triangle $abc$ inscribed in $C$ of the maximum possible area such that the vector $ab$ is of the direction $\phi$ and such that $a, b, c$ are in the counterclockwise order on the boundary of $C$. When such a triangle $T_\phi$ becomes isosceles for a direction $\phi$, the quoted proof of Hodges [14] does not guarantee that its area is at least $\frac{3}{8}|C|$. This proof only claims that $|abc| \geq \frac{3}{8}$ or that $|a'b'c'| \geq \frac{3}{8}$, where $a'b'c'$ is an analogical triangle with vertices in the clockwise order. Of course, the proof of deValcourt remains valid if $C$ is centrally symmetric.

Also the "theorem" of deValcourt about an isosceles triangle of the area at most $2|C|$ containing $C$ can be proved for centrally symmetric $C$ by his approach. It is likely that every convex body $C \subset E^2$ is contained in an isosceles triangle of area at most $2|C|$ (this is true when we do not require that the triangle is isosceles, as proved by Gross [12]). For arbitrary convex body $C \subset E^2$ we have up to now only the estimate $\frac{49}{16}|C|$ from Corollary 1.

DeValcourt [8] conjectures that every convex body $C \subset E^2$ contains an isosceles triangle of area at least $\frac{3\sqrt{3}}{4\pi}|C|$ (this is true when we do not require that the triangle is isosceles, as proved by Gross [12]). The following example disproves this conjecture. Let $E$ be the ellipse $\frac{1}{4}x^2 + y^2 = 1$. It is an affine image of the unit disk $U$. The corresponding images of the regular triangles inscribed in $U$ are the only triangles of the area $\frac{3\sqrt{3}}{4\pi}|E|$ in $E$. Exactly four of them are isosceles; they are the images of the regular triangles inscribed in $U$ with a vertical or a horizontal side. Thus if we take a convex body $F$ which is obtained from $E$ by a sufficiently slight "enlargement" at the point $(\sqrt{2}, \frac{1}{2}\sqrt{2})$, those four isosceles triangles are still the only isosceles triangles of maximum area in $F$, which disproves the conjecture.

It is likely that every convex body $C \subset E^2$ is contained in an isosceles triangle of area at most $2|C|$. Of course, this is true when we do not require that the triangle is isosceles, as proved by Gross [12]. It is likely that every convex body $C \subset E^2$ is contained in an isosceles triangle of area at most $2|C|$. Of course, this is true when we do not require that the triangle is isosceles, as proved by Gross [12].

**REMARK.** Similarly as in Corollary 2, from the result about the approximation of a convex body by a pair of rectangles ([21], [18], [22]), we obtain the following corollary. *For every convex body $C \subset E^2$ there exists a rectangular triangle $T \subset C$ such that its homothetic copy $T'$ of ratio $\frac{7}{2}$ contains $C$.* How much can be the ratio $\frac{7}{2}$ lessened? Surely not below $\sqrt{2}+1$ as the example of a disk as $C$ shows. But even this is not the infimum as it follows from an example analogical the to the example of the set $F$ presented just before this Remark. Also the questions appear about a rectangular triangle of possibly large area contained in $C$ and about a rectangular triangle of possibly small area containing $C$ (a rectangular triangle of area at most $\frac{49}{16}|C|$ containing $C$ can be constructed similarly like in Corollary 1).

## 3. Approximation by arbitrary axially symmetric bodies

Remember that by *affine regular hexagon* we mean any non-degenerated hexagon which is an affine image of the regular hexagon. In other words, a non-degenerated hexagon $abcdef$ is affine regular if it is centrally symmetric, and if the segment $ab$ is parallel to the chord $fc$ and is exactly two times shorter. We call a non-degenerated hexagon $abcdef$ *axially regular* if the straight line through $f$ and $c$ is an axis of symmetry of this hexagon and if the segment $ab$ parallel to the chord $fc$ and exactly two times shorter. Here is a lemma analogical to the well known result of Besicovitch [2] that an affine regular hexagon can be inscribed in an arbitrary convex body $C \subset E^2$.

**LEMMA.** *In every convex body $C \subset E^2$ we can inscribe an axially regular hexagon.*

*Proof.* Assume that $C$ is a regular convex body; at the end of the proof we will consider an arbitrary convex body $C \subset E^2$.

For any given direction $\theta$ there are three chords $ab$, $fc$, $ed$ in $C$ of this direction so that $|ab| = |ed|$, $fc$ is equidistant from the straight lines containing $ab$ and $ed$, and $|fc| = 2|ab|$ (see [2]). Since $C$ is a regular body, those three chords are unique. The chords $ad$, $be$, $fc$ meet in a point $o$. The chords $ad$ and $be$ are bisected at $o$. For arbitrary $\theta$ consider the angles $\alpha = \angle foa$, $\beta = \angle aob$ and $\gamma = boc$. Let $H(\theta)$ denote the hexagon $abcdef$ determined by $\theta$. Of course, for any direction $\theta$ there is a unique $H(\theta)$.

Besicovitch [2] rotates the direction $\theta$ and applies continuity arguments in order to show that there exists a direction $\theta_1$ for which the position $o_1$ of $o$ is the center of the corresponding chord $f_1c_1$. Thus $H(\theta_1) = a_1b_1c_1d_1e_1f_1$ is an affine-regular hexagon. Denote by $\theta_2$ the direction of the chord $a_1d_1$ and by $\theta_3$ the direction of the chord $b_1e_1$.

Let $H(\theta_i) = a_ib_ic_id_ie_if_i$ and $\alpha_i = \angle f_ioa_i$, $\beta_i = \angle a_iob_i$ and $\gamma_i = b_ioc_i$ for $i \in \{1,2,3\}$. From the uniqueness of $H(\theta)$ and since $H(\theta_1)$ is an affine-regular hexagon, we see that $H(\theta_1) = H(\theta_2) = H(\theta_3)$. Thus there is an index $i \in \{1,2,3\}$ such that $\alpha_i$ is not greater than $\beta_i$ and not greater than $\gamma_i$. Let for instance $\alpha_1 \le \min\{\beta_1, \gamma_1\}$. The further consideration is similar in the remaining cases.

When rotating the direction $\theta$ from $\theta_1$ to $\theta_2$, all the six vertices of $H(\theta)$ and the point $o$ change the positions continuously and also the angles $\alpha$, $\beta$, $\gamma$ change continuously: $\alpha$ from $\alpha_1$ to $\alpha_2 = \beta_1$, $\beta$ from $\beta_1$ to $\beta_2 = \gamma_1$, and $\gamma$ from $\gamma_1$ to $\gamma_2 = \alpha_1$. Thus there is an angle $\theta^*$ between $\theta_1$ and $\theta_2$ for which the corresponding angles $\alpha^*$ and $\gamma^*$ are equal. Of course, $H(\theta^*)$ is an axially regular hexagon inscribed in $C$.

The above solution for a regular convex body leads to a solution for an arbitrary convex body $C \subset E^2$. Just $C$ is a limit of a sequence of regular convex bodies (see [9]). From the corresponding sequence of axially regular hexagons inscribed in those regular bodies we can select a subsequence convergent to an axially regular hexagon inscribed in $C$.

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Below we improve the result of Krakowski [16] that every convex body $C \subset E^2$ contains an axially symmetric body of area at least $\frac{5}{8}|C|$.

**PROPOSITION 1.** *Every convex body $C \subset E^2$ contains an axially symmetric convex body of area at least $\frac{2}{3}|C|$.*

*Proof.* According to Lemma, we can inscribe an axially regular hexagon $H = abcdef$ in $C$ . Without loss of generality we may assume that $|fo| \geq |oc|$.

Consider the triangle $pqr$ such that $fa \subset pq$, $bc \subset qr$ and $de \subset rp$, and also the triangle $p'q'r'$, where $p', q', r'$ are points symmetric to the points $p, q, r$ with respect to the straight line containing $fc$. Since $H$ is inscribed in $C$, from the convexity of $C$ it follows that $C$ is contained in the starshaped set $S = pqr \cup p'q'r'$ (see Figure 2).

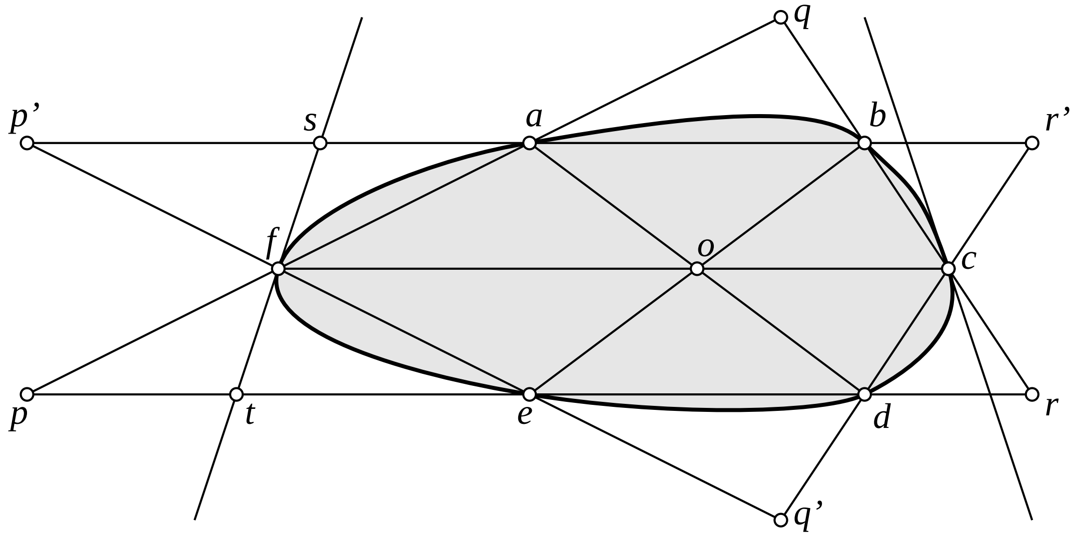


*Figure 2.*

We provide supporting lines of $C$ at points $a, c, e, f$ and we consider the half-planes bounded by them and containing $C$. Let $Q$ denote the intersection of those four half-planes and of $S$. The supporting lines at $f$ and at $c$ cut off from $S$ four triangles of total area $\frac{1}{3}|H|$. Denote by $W$ the starhaped set which remains from $S$ after cutting off those four triangles. Consider the two triangles $fsa$ and $fet$, where $s$ and $t$ are the intersections of the supporting line through $f$ with the segments $p'a$ and $pe$, respectively. We do not make our considerations narrower assuming that $\angle ofs \leq 90^o$ and that $\angle tfo \geq 90^o$.

The supporting line of $C$ at $a$ cuts off two triangles from $W$. Since $|fa| = |aq|$ and $|sa| \leq |ab|$, the total area of those two triangles is minimized when the supporting line contains $fa$ (this is when we cut off the triangle $fsa$). Analogically, the supporting line of $C$ through $e$ cuts off two triangles from $W$. Since $|fe| = |eq'|$, the total area of those two triangles is minimized when we cut off the triangle $fet$ or if we cut off the triangle $edq'$. The first case is when $|te| \leq |ed|$, and the second case is when $|te| \geq |ed|$. In each of the two worst cases, the sum of the areas of all the pieces cut off from $S$ by the four considered supporting lines is at least $\frac{1}{2}|H|$; the reason is that the area of the pieces which are cut off is at least $|pfe| + |p'af| + \frac{1}{2}(|cdr| + |bcr'|) \geq \frac{1}{2}|H|$ in the first case, and at least $\frac{1}{2}(|pfe| + |p'af| + |cdr| + |bcr'|) + |edq'| = \frac{1}{3}|H| + \frac{1}{6}|H| = \frac{1}{2}|H|$ in the second case. From $|S| = 2|H|$ we see that $|Q| \le \frac{3}{2}|H|$. By $C \subset Q$, we have $|C| \le \frac{3}{2}|H|$. Consequently, $|H| \ge \frac{2}{3}|C|$. Since $H$ is axially symmetric, we obtain the thesis of Proposition 1.

∎

It does not seem that the estimate $\frac{2}{3}|C|$ obtained in Proposition 1 is the best possible. A reasonable way to deal with this question would be in finding an axially symmetric hexagon (not obligatory axially regular) of a large area in $C$. The examples of some parallelograms (see [19]) and triangles (see [3]) show that a value over $(2\sqrt{2} - 2)|C|$ is not possible here.

Proposition 1 implies the existence of an axially symmetric set (usually non-convex) of the area at most $\frac{4}{3}|C|$ containing $C$. This set is the union of $C$ and of its image in the straight line containing the segment $fc$ (see the proof of Proposition 1). What is more, the following three conditions are equivalent: (1) *$C$ contains an axially symmetric convex body of area $\lambda|C|$*, (2) *$C$ is contained in an axially symmetric set (not necessarily convex) of area $(2 - \lambda)|C|$*, (3) *$C$ can be dissected into two parts by a straight line $L$ such that after folding one of the two obtained parts of $C$ along $L$, the area of the union of the folded part and of the other part of $C$ is $(1 - \frac{1}{2}\lambda)|C|$*. We leave the short proof as an exercise for the reader.

This last approach leads to the question how large piece of $C$ may be folded along a straight line into $C$. An estimate is given in the following Proposition.

**PROPOSITION 2.** *For every convex body $C \subset E^2$ there is a straight line $L$ which cuts off from $C$ such a subset of area at least $\frac{1}{8}|C|$ which remains in $C$ after folding it along $L$.*

*Proof.* We will use the notation introduced in the proof of Proposition 1. We provide the supporting line $M$ of $C$ parallel to the segment $ae$ and passing through the triangle $pef$. Denote by $u$ a point of support and by $w$ the intersection of $M$ with the segment $pe$. Let $\sigma = |we|/|ed|$. Since $|fo| \ge |fc|$, we have $\frac{1}{2} \le \sigma \le 2$. The required line $L$ is parallel to $M$ and its position depends on $\sigma$. Below we consider two cases in which we determine the position of $L$. By $C_1$ we denote the part of $C$ which is the intersection of $C$ with the halfplane bounded by $L$ and containing the point $u$. Let $C_1'$ denote the image of $C_1$ after folding it along $L$.

Case 1, when $1 \le \sigma \le 2$. We provide $L$ such that it bisects the segment $wd$ (see Figure 3). Since $C$ is convex and since the axially regular hexagon $abcdef$ is inscribed in $C$, we conclude that $C_1' \subset C$.

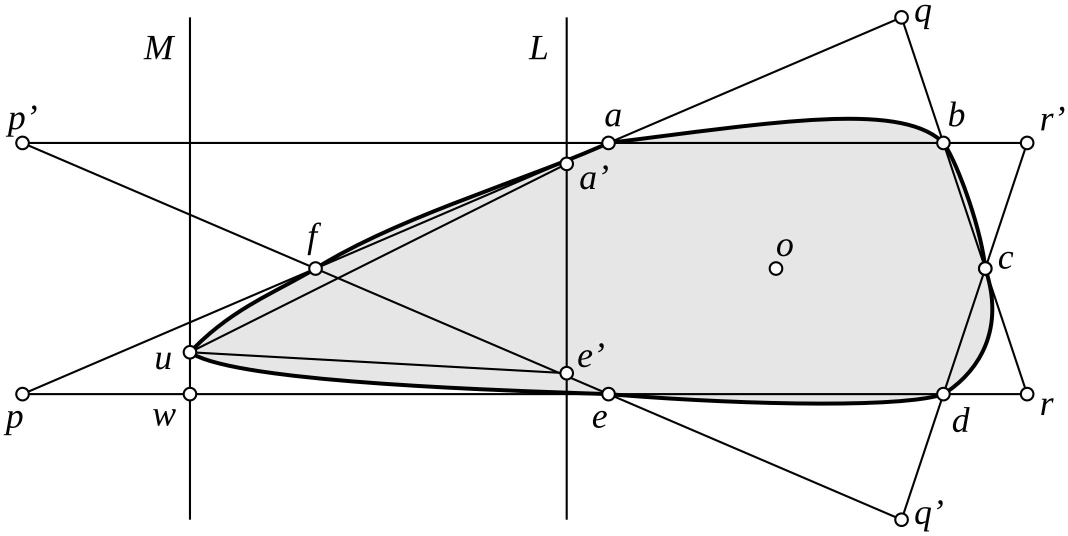


*Figure 3.*

Of course, $C_1$ contains the triangle $a'e'u$, where $a'$ and $e'$ are the intersections of $L$ with the segments $af$ and $ef$. A short calculation shows that we have $|a'e'u| = \frac{1}{12}(\sigma+1)^2\sigma^{-1}|H|$. Since $1 \le \sigma \le 2$, this is at least $\frac{1}{3}|H|$. From the inequality $|H| \ge \frac{2}{3}|C|$ established in Proposition 1 we obtain that $|C_1| \ge \frac{2}{9}|C| > \frac{1}{8}|C|$.

Case 2, when $\frac{1}{2} \le \sigma \le 1$. We provide the line $L$ through points $a$ and $e$. From the convexity of $C$ and since the axially regular hexagon $abcdef$ is inscribed in $C$, we obtain that $C_1' \subset C$. We provide supporting lines of $C$ through the points $b$ and $d$. They cut off four triangles from $S$. Since $|qb| = |bc|$ and since $|ab| \ge |br'|$, we see that the area of the two triangles cut off from $S$ by the supporting line through $b$ is minimized when the line contains the segment $bc$. Analogically, the area of the two triangles cut of by the second line is minimized when this line contains $cd$. Thus $|C| \le |C_1| + |aqcq'e|$. Consequently, $|C| \le |C_1| + \frac{7}{6}|H|$. Thus $|C_1|/|C| \ge |C_1|/(|C_1| + \frac{7}{6}|H|)$. Since $|C_1| \ge \frac{1}{6}|H|$, we get $|C_1|/|C| \ge \frac{1}{8}$. Hence $|C_1| \ge \frac{1}{8}|C|$.

$\blacksquare$

Proposition 2 leads to the problem formulated below. Let $C$ be a convex body in Euclidean $d$-space $E^d$. Denote by $\mu(C)$ the greatest number with the property that there exists a hyperplane $L$ which cuts off from $C$ a subset of volume at least $\mu|C|$ whose mirror image obtained by reflection about $L$ belongs to $C$.

**PROBLEM.** *Find the infimum $\mu_d$ of the numbers $\mu(C)$ over all convex bodies $C \subset E^d$.*

By the way, observe that the number $2\mu(C)$ is a very natural measure of symmetry with respect to a hyperplane (i.e. a measure of axiality for $d = 2$), comp. [13], [4] and [6].

From Proposition 2 we immediately obtain the following Corollary.

**COROLLARY 3.** *Every convex body $C \subset E^2$ contains an axially symmetric convex body of area at least $\frac{1}{4}|C|$ whose symmetric half of the boundary is a subset of the boundary of $C$.*

There is an open problem about possibly small, with respect to the area, axially symmetric convex body containing a convex body $C \subset E^2$. Of course, always such an axially symmetric body of area at most $2|C|$ exists. We can take here the rectangle of area at most $2|C|$ containing $C$ (see [20]). We can also use the circumscribed rhombus $R'$ from Corollary 1. We can get a slight improvement of the ratio 2 by lessening $R'$ up to the axially symmetric pentagon $P = R' \cap T'$ (here and below we use the notation of the proof of Corollary 2). Remember that $|C| \geq \rho|R|$. From the definition of $P$ we obtain that $|P| = (\frac{7}{8}\rho^2 + \frac{1}{4}\rho - \frac{1}{8})|R|$. Hence $|P|/|C| \leq \frac{7}{8}\rho + \frac{1}{4} - \frac{1}{8}\rho^{-1}$. Since $\rho \in [1,2]$, this value is always at most $\frac{31}{16}$. Thus we obtain the following corollary.

**COROLLARY 4.** *Every convex body $C \subset E^2$ is contained in an axially symmetric convex body of area at most $\frac{31}{16}|C|$.*

For every convex body $C \subset E^2$ there is the smallest positive $\tau(C)$ such that $C$ contains an axially symmetric convex body $D$ and that $D \subset C \subset h(D)$ for a homothety $h$ of ratio $\tau(C)$. The number $1/\tau(C)$ is one more reasonable measure of axiality. A natural question is about the infimum $\tau$ of $\tau(C)$ over all convex bodies $C \subset E^2$. Of course, $\tau \leq 2$ which follows from the John's approximation of $C$ by a pair of homothetic ellipses with ratio at most 2 (this follows also from the analogical theorems on pairs of rectangles and on pairs of rhombi). A similar question is when we additionally require that the homothetic bodies $D$ and $h(D)$ have a common axis of symmetry (like the pair of ellipses in John's theorem).

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