FINITE THREE DIMENSIONAL PARTIAL ORDERS WHICH ARE NOT SPHERE ORDERS

STEFAN FELSNER, PETER C. FISHBURN, AND WILLIAM T. TROTTER

ABSTRACT. Given a partially ordered set $\mathbf{P} = (X, P)$, a function F which assigns to each $x \in X$ a set $F(x)$ so that $x \leq y$ in P if and only if $F(x) \subseteq F(y)$ is called an inclusion representation. Every poset has such a representation, so it is natural to consider restrictions on the nature of the images of the function F. In this paper, we consider inclusion representations assigning to each $x \in X$ a sphere in $\mathbb{R}^d$, d-dimensional Euclidean space. Posets which have such representations are called sphere orders. When $d = 1$, a sphere is just an interval from $\mathbb{R}$, and the class of finite posets which have an inclusion representation using intervals from $\mathbb{R}$ consists of those posets which have dimension at most two. But when $d \geq 2$, some posets of arbitrarily large dimension have inclusion representations using spheres in $\mathbb{R}^d$. However, using a theorem of Alon and Scheinerman, we know that not all posets of dimension $d + 2$ have inclusion representations using spheres in $\mathbb{R}^d$. In 1984, Fishburn and Trotter asked whether every finite 3-dimensional poset had an inclusion representation using spheres (circles) in $\mathbb{R}^2$. In 1989, Brightwell and Winkler asked whether every finite poset is a sphere order and suggested that the answer was negative. In this paper, we settle both questions by showing that there exists a finite 3-dimensional poset which is not a sphere order. The argument requires a new generalization of the Product Ramsey Theorem which we hope will be of independent interest.

1. INTRODUCTION

Given a partially ordered set (poset) $\mathbf{P} = (X, P)$, a function F which assigns to each $x \in X$ a set $F(x)$ is called an *inclusion representation* of $\mathbf{P}$ if $x \leq y$ in P if and only if $F(x) \subseteq F(y)$. Every poset has such a representation. For example, just take $F(x) = \{y \in X : y \leq x \text{ in } P\}$. In recent years, there has been considerable interest in inclusion representations where the images of the function F are required to be geometric objects of a particular type, with attention focused on circles and spheres. We refer the reader to [7] for a summary of results in this area and an extensive bibliography.

As is well known, the finite posets of dimension at most two are just those which have inclusion representations using closed intervals of the real line $\mathbb{R}$. Because a closed interval of $\mathbb{R}$ can also be considered as a sphere in $\mathbb{R}^1$, it is natural to ask which posets have inclusion representations using disks (circles) in $\mathbb{R}^2$. For historical reasons, these posets are called *circle orders*. Fishburn [4] showed that all interval orders are circle orders. Also, the so called *standard examples* of n-dimensional

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posets, the 1-element and $(n-1)$-element subsets of $\{1, 2, \ldots, n\}$, ordered by inclusion, are circle orders. So among the circle orders are *some* posets of arbitrarily large dimension.

Call a poset $\mathbf{P}$ a *sphere order* if there is some $d \geq 1$ for which it has an inclusion representation using spheres in $\mathbb{R}^d$. Using the "degrees of freedom" theorem of Alon and Scheinerman [1], it follows that not all posets of dimension $d+2$ have inclusion representations using spheres in $\mathbb{R}^d$. In particular, when $d = 2$, we conclude that there are 4-dimensional posets which are not circle orders. In this case, an explicit example can be given, as Sidney et al. [22] have shown that the 4-dimensional poset consisting of the 14 proper nonempty subsets of $\{1, 2, 3, 4\}$ ordered by inclusion is not a circle order.

In [21], Scheinerman and Wierman used a very nice Ramsey theoretic argument to show that the countably infinite 3-dimensional poset $\mathbb{Z}^3$ is not a circle order. They also noted that $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} \times \mathbb{N}$ is not a circle order when n is sufficiently large. Additional contributions along this line appear in Hurlbert [11], Lin [13] and Fon-Der-Flaass [9]. The last of these proves that $\{1, 2\} \times \{1, 2, 3\} \times \mathbb{N}$ is not a circle order.

These results leave open the following question:

Question 1. *Is every finite 3-dimensional poset a circle order?* □

This question was raised by Fishburn and Trotter at the Banff meeting on ordered sets in 1984 but has also been posed by other researchers. Although the results in the preceding paragraph suggest that the answer is negative, some evidence supports a positive answer. As shown in [24], for every finite 3-dimensional poset $\mathbf{P}$ and every integer $n \geq 3$, $\mathbf{P}$ has an inclusion representation using regular n-gons in the plane. So it is natural to surmise that as $n \to \infty$, we may be able to pass to a limit and obtain the desired inclusion representation using circles.

Some of the motivation for questions involving inclusion representations for posets comes from the parallel concept of intersection graphs. For example, Maehara [14] showed that for every finite graph $\mathbf{G} = (V, E)$, there is some $d \geq 1$ so that $\mathbf{G}$ is the intersection graph of a family of spheres in $\mathbb{R}^d$. The corresponding question for posets was posed independently by Brightwell and Winkler [3] and by Meyer [15]. Brightwell and Winkler also conjectured that the answer is negative.

Question 2. *Is every finite poset a sphere order?* □

This paper settles Question 1 and Question 2 with the following result.

Theorem 1.1. *There exists a finite 3-dimensional poset which is not a sphere order.*

Inclusion representations that use circles and spheres have other applications and have been studied for a variety of reasons. For example, Scheinerman [18] proved that a graph $\mathbf{G} = (V, E)$ is planar if and only if the poset formed by its vertices and edges, ordered by inclusion, is a circle order. Knight [12] has studied representation problems using non-standard analysis, while Meyer [15], [16], [17] and Brightwell and Gregory [2] have investigated the modeling of time and space with spheres, an approach of interest to physicists.

Additional information on circle and sphere orders appears in Scheinerman [19], [20], while more general geometric objects are considered in Fishburn and Trotter [6], Sidney et al. [22], Tanenbaum [23], Urrutia [27] and other papers cited in Fishburn and Trotter [7].

The remainder of the paper is organized as follows. Section 2 provides basic notation and terminology. Section 3 outlines the proof. Section 4 gathers important Ramsey theoretic tools essential to our argument, tools which we feel will have applications beyond this paper. In Sections 5–10, we present the proof of Theorem 1.1. Section 11 discusses related problems and research directions.

2. Notation and Terminology

For positive integers n and t, let $\mathbf{n}$ denote the chain $0 < 1 < \cdots < n-1$, and let $\mathbf{n}^t$ denote the cartesian product of t copies of $\mathbf{n}$, so that $(i_1, i_2, \ldots, i_t) \leq (j_1, j_2, \ldots, j_t)$ in $\mathbf{n}^t$ if $i_k \leq j_k$ in $\mathbb{R}$ for $k = 1, 2, \ldots, t$. Also let $\mathbb{R}_0$ denote the set of all positive real numbers.

Given a poset $\mathbf{P} = (X, P)$, recall that the the minimum cardinality of a family of linear extensions of P whose intersection is P is called the *dimension* of $\mathbf{P}$ and is denoted by $\dim(\mathbf{P})$. We refer the reader to [24] for additional background material on the subject of dimension for partially ordered sets and to [25] and [26] for more discussion of connections between graphs and posets. Here we will need only the well known fact that a finite poset has dimension at most t if and only if there is an integer n for which it is isomorphic to a subposet of $\mathbf{n}^t$. Hence, to prove Theorem 1.1, it then suffices to establish the following result.

Theorem 2.1. *There exists an integer n_0 so that if $n \geq n_0$, the finite 3-dimensional poset $\mathbf{n}^3$ is not a sphere order.*

For positive integers n, d and t, we consider inclusion representations of the poset $\mathbf{n}^t$ using spheres from $\mathbb{R}^d$. We use the letters u, v, w, x, y, z, B and T to denote elements of $\mathbf{n}^t$. For example, the coordinates of x for $t = 3$ would be $\big(x(1), x(2), x(3)\big)$. Also, we write, for example, $x = (5, 4, 7)$ to indicate the element in $\mathbf{n}^3$ with $x(1) = 5$, $x(2) = 4$ and $x(3) = 7$.

Given an inclusion representation F of $\mathbf{n}^3$, using spheres in $\mathbb{R}^d$, the center of the sphere $F(x)$ will be denoted by $c(x)$. We never refer explicitly to the coordinates of $c(x)$, as we wish to emphasize that our argument is independent of the value of d.

We will also use the symbol s (with various subscripts) to denote points in $\mathbb{R}^d$ which may or may not be centers of spheres in our representation. We denote the Euclidean distance between points s_1 and s_2 from $\mathbb{R}^d$ by $\rho(s_1, s_2)$. When x and y are points in $\mathbf{n}^3$, we abbreviate $\rho\big(c(x), c(y)\big)$ by $\rho(x, y)$. Accordingly, the inclusion rule may be stated as follows:

$$x \leq y \text{ in } \mathbf{n}^3 \text{ if and only if } r(y) - r(x) \geq \rho(x, y). \tag{1}$$

In other words, one sphere is contained in another when the difference in their radii is at least as large as the distance between the centers. Technically speaking, we should write $\rho_F(x, y)$ because the distance between $c(x)$ and $c(y)$ depends on F. However, in our proof, once an inclusion representation F is determined, we make at most two modifications to the representation, and both leave the distance between centers invariant.

Given two points s_1 and s_2 in $\mathbb{R}^d$, let $L(s_1, s_2)$ denote the line they determine. The line $L(c(x), c(y))$ will be abbreviated by $L(x, y)$.

Given three non-collinear points s_1, s_2 and s_3, let $\phi(s_1, s_2, s_3)$ denote the angle at s_1 determined by $L(s_1, s_2)$ and $L(s_1, s_3)$. Also let $\gamma(s_1, s_2, s_3)$ denote the angle formed at s_3 by $L(s_1, s_3)$ and $L(s_2, s_3)$. Then let $p(s_1, s_2, s_3)$ denote the unique

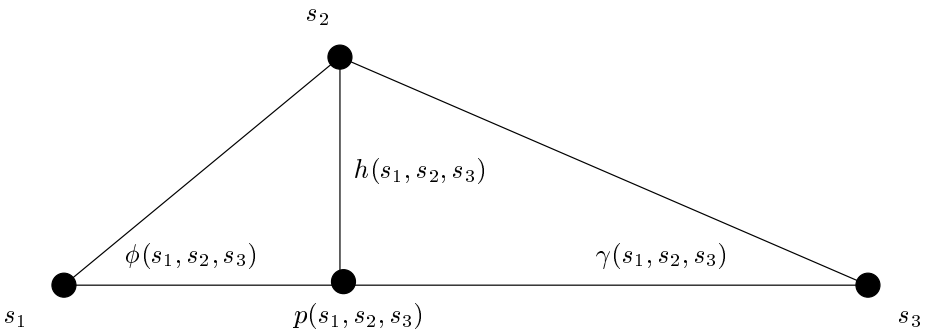

FIGURE 1

point on $L(s_1, s_3)$ which is closest to s_2, and let $h(s_1, s_2, s_3) = \rho\big(s_2, p(s_1, s_2, s_3)\big)$ (see Figure 1). As usual, when discussing centers, we will just write $\phi(x, y, z)$, $\gamma(x, y, z)$, $p(x, y, z)$ and $h(x, y, z)$.

The proof of our main theorem uses a large constant N which we somewhat arbitrarily take as $N = 10^{100}$. More modest values would work but would undermine the Ramsey theoretic perspective we have adopted. More importantly, in a certain sense, the perspective we have taken is forced. Given *any* collection of spheres, elementary Lorentz transformations may be applied to relocate the centers so that they are *very* close to being collinear. So this paper can be viewed as an effort to work with small errors—a task that sometimes requires large constants.

The following notation is used throughout. When e_1 and e_2 are positive quantities, we write

$$e_1 << e_2 \text{ when } Ne_1 < e_2.$$

Also, we write

$$e_1 \lesssim e_2 \text{ when } e_1 < e_2(1 + 1/N).$$

We use $e_2 >> e_1$ as an alternative for $e_1 << e_2$, and $e_2 \gtrsim e_1$ for $e_1 \lesssim e_2$. When $e_1 \lesssim e_2 \lesssim e_1$, we will write $e_1 \approx e_2$. Furthermore, our inequalities will be strong enough to allow the natural notion that if $e_1 \approx e_2$ and $e_2 \approx e_3$, then $e_1 \approx e_3$.

When arguing to a contradiction using quantities compared with this notation, we must be careful to avoid such traps as believing that

$$e_1 < e_2 \lesssim e_3 < e_4 < e_5 < e_1$$

results in a contradiction, because it only leads to the conclusion that the five quantities are approximately the same. So to obtain a contradiction, we will always show (at least) something like

$$e_1 \lesssim e_2 \text{ and } 2e_2 \lesssim e_1.$$

Also, our argument will make extensive use of a principle which we call *differentiation* and develop in Section 4. To illustrate this principle, consider an injective function $f : \mathbf{n}^3 \to \mathbb{R}$, and let x and y be distinct elements of $\mathbf{n}^3$. Because f is injective, $f(x) \neq f(x)$. In arguments that follow, we will control the behavior of f so that one of the following three situations always obtains:

$$f(x) << f(y).$$

$$f(x) >> f(y).$$

$$f(x) \approx f(y).$$

In other words, we want to exclude the middle ground where, for example,

$$f(x)(1 + 1/N) \leq f(y) \leq Nf(x).$$

When f maps distinct points x and y so that $f(x) \approx f(y)$, we will need to examine how differences behave. In this case, when $f(x) < f(y) < f(z)$, we want to have either

$$f(y) - f(x) << f(z) - f(y),$$

or

$$f(y) - f(x) >> f(z) - f(y),$$

and never

$$f(y) - f(x) \approx f(z) - f(y).$$

Let e_1, e_2 and e_3 be positive positive real numbers. We write

$$e_1 - e_2 = \text{zero}(e_3)$$

when $e_3 << e_1$, $e_2 \lesssim e_1 + e_3$, and $e_1 \lesssim e_2 + e_3$. The basic idea here is that we will have three quantities satisfying a weak version of the triangle inequality

$$e_i + e_j \gtrsim e_k$$

for all distinct $i, j, k \in \{1, 2, 3\}$. We will then discover that e_1 is much larger than e_3, leading to the conclusion that e_1 and e_2 are almost exactly the same size.

3. Outline of the Proof

We will assume that we have an inclusion representation of $\mathbf{n}^3$ using spheres in $\mathbb{R}^d$ and argue to a contradiction—provided n is sufficiently large.

The basic idea of the proof is straightforward. We envision the centers of the spheres as being nearly collinear along some line in $\mathbb{R}^d$. Each sphere will have as its radius a value which is almost exactly the same as the distance from its center to the center $c(B)$ of the bottom point $B = (0, 0, 0)$. Given any two other points x and y in $\mathbf{n}^3$, the center of one will be much closer to $c(B)$, say by a mulitiplicative factor of $N = 10^{100}$ or more.

For distinct points x and y from $\mathbf{n}^3$, we define

$$\text{gap}(x, y) = r(y) - r(x) - \rho(x, y).$$

When $x < y$, $\text{gap}(x, y) > 0$, and when x is incomparable to y, $\text{gap}(x, y) < 0$. However, in all cases, $\rho(x, y)$ and $|r(y) - r(x)|$ will be approximately equal, so we will need to pay careful attention to the magnitude of the error terms.

For three distinct points x, y and z, let

$$\Delta(x, y, z) = \rho(x, y) + \rho(y, z) - \rho(x, z).$$

Clearly, $\Delta(x, y, z) \geq 0$, and $\Delta(x, y, z) > 0$ when the centers are not collinear.

The proof of our main theorem focuses on a 2-element chain $x < z$ and the quantity $\mathrm{gap}(x, z)$. We will obtain upper bounds on $\mathrm{gap}(x, z)$ by considering a point incomparable to both x and z. For example, suppose v is such a point. Then

$$r(z) - r(x) = \big(r(v) - r(x)\big) + \big(r(z) - r(v)\big) < \rho(x, v) + \rho(v, z),$$

so that

$$\mathrm{gap}(x, z) < \Delta(x, v, z).$$

Since this bound holds for any point incomparable to both x and z, we may consider several candidate points and take the best bound they produce.

To obtain a lower bound, we consider an integer k and a chain C of $2k+1$ points having x as its bottom element and z as its top element. Let $C = \{x = u_1 < u_2 < \cdots < u_{2k+1} = z\}$ be such a chain. Then

$$\begin{aligned}
r(z) - r(x) &= r(u_{2k+1}) - r(u_1) \\
&= \sum_{i=1}^{2k} \big[r(u_{i+1}) - r(u_i)\big] \\
&> \sum_{i=1}^{2k} \rho(u_{i+1}, u_i) \\
&= \sum_{i=1}^{k} \big[\rho(u_{2i+1}, u_{2i-1}) + \Delta(u_{2i-1}, u_{2i}, u_{2i+1})\big] \\
&\geq \rho(u_1, u_{2k+1}) + \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}). \\
&= \rho(x, z) + \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}).
\end{aligned}$$

Setting

$$\Delta(x, C, z) = \sum_{i=1}^{k} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}),$$

we conclude that

$$\mathrm{gap}(x, z) > \Delta(x, C, z).$$

In all cases, we will obtain a contradiction by carefully choosing a point v, with v incomparable to both x and z, and a chain C having x and z as its bottom and top elements so that

$$\Delta(x, v, z) < \Delta(x, C, z).$$

The chain C will often consist of x, z and one intermediate point, but there are cases that need several intermediate points.

The argument depends heavily on Ramsey theory to assure that our representation is suitably regular. However, we must avoid any dependence on the dimension of the space from which the spheres in the representation are taken.

Finally, we encourage the reader to observe the key role played by dominating coordinates, a concept which is introduced in the next section.

4. Extensions of The Product Ramsey Theorem

Given a finite set S and an integer k with $0 \leq k \leq |S|$, we denote the set of all k-element subsets of S by $\binom{S}{k}$. Given integers t and k and finite sets $S_1, S_2, \ldots, S_t$, we call an element of $\binom{S_1}{k} \times \binom{S_2}{k} \times \cdots \times \binom{S_t}{k}$ a *grid* (also, a $\mathbf{k}^t$ grid). The sets $S_1, S_2, \ldots, S_t$ are called *factor sets* of the grid. Using the natural order, a set of n integers is just an n-element chain, so considered as a poset, $S_1 \times S_2 \times \cdots \times S_t$ is isomorphic to $\mathbf{n}_1 \times \mathbf{n}_2 \times \cdots \times \mathbf{n}_t$, where $n_i = |S_i|$ for $i = 1, 2, \ldots, t$.

The following Product Ramsey Theorem, stated here in poset form, will be used extensively in making certain uniformizing assumptions about the inclusion representation. We refer the reader to [10] for the proof and additional material on Ramsey theory.

Theorem 4.1. *Given positive integers m, k, r and t, there exists an integer n_0 so that if $n \geq n_0$ and f is any map which assigns to each $\mathbf{k}^t$ grid of $\mathbf{n}^t$ a color from $\{1, 2, \ldots, r\}$, then there exists a subposet $\mathbf{P}$ isomorphic to $\boldsymbol{m}^t$ and a color $\alpha \in \{1, 2, \ldots, r\}$ so that $f(g) = \alpha$ for every $\mathbf{k}^t$ grid g from $\mathbf{P}$.* □

We will refer to the least n_0 for which the conclusion of the preceding theorem holds as the Product Ramsey number $\mathrm{PR}(m, k, r, t)$.

Recall that $x \leq y$ in $\mathbf{n}^t$ if and only if $x(i) \leq y(i)$ for $i = 1, 2, \ldots, t$. So it does not follow that $x(i) < y(i)$ for $i = 1, 2, \ldots, t$ when $x < y$ in $\mathbf{n}^t$. Nevertheless, the following elementary proposition allows us to assume that if $x \neq y$, then $x(i) \neq y(i)$ for $i = 1, 2, \ldots, t$. We view this proposition as a "spacing" tool in that it allows us to assume that distinct points have all coordinates distinct and separated by some reasonable amount.

Proposition 4.2. *Let m, n and G be positive integers with $n \geq Gm^t$. Then the function $I : \boldsymbol{m}^t \to \mathbf{n}^t$ defined (cyclically) by*

$$I(x)(i) = G \sum_{j=1}^{t} x(i + j - 1)(m - 1)^{t-j-1}$$

is an embedding. Furthermore,

1. *If $x, y \in \boldsymbol{m}^t$, $i \in \{1, 2, \ldots, t\}$ and $x(i) < y(i)$, then $I(x)(i) < I(y)(i)$.*
2. *If $x, y \in \boldsymbol{m}^t$ and $x \neq y$, then $|I(x)(i) - I(y)(i)| \geq G$ for $i = 1, 2, \ldots, t$.*

□

In what follows, we refer to the integer G in the preceding theorem as the *gap size* of the embedding I.

Let $\mathbf{P}$ be a poset and let f map $\mathbf{P}$ into $\mathbb{R}$. We say f is *monotonic* if it is either order-preserving or order-reversing. Now consider an order-preserving function f which maps $\mathbf{n}^t$ (or a subposet of $\mathbf{n}^t$) to $\mathbb{R}$. We say that f is *dominated* by coordinate α if for all x and y from its domain, $f(x) < f(y)$ whenever $x(\alpha) < y(\alpha)$. Dually, given an order-reversing function f, we say that f is *dominated* by α if for all x and y from its domain, $f(x) > f(y)$ whenever $x(\alpha) < y(\alpha)$.

In [8], Fishburn and Graham used the Product Ramsey Theorem to obtain the following result.

Theorem 4.3. *Given integers m and t, there exists an integer n_0 so that if $n \geq n_0$ and f is any injective function from $\mathbf{n}^t$ to $\mathbb{R}$, then there exist a coordinate $\alpha \in \{1, 2, \ldots, t\}$ and a subposet $\mathbf{P}$ isomorphic to $\boldsymbol{m}^t$ so that the restriction of f to $\mathbf{P}$ is monotonic and dominated by coordinate α.* □

We stated the preceding theorem (and all to follow) in terms of injective functions, because all the functions we consider may be assumed to be injective. If this assumption is dropped, then a modestly more complicated concept of domination is needed, and the conclusions of the theorems have additional cases. However, the basic principles we discuss here apply to arbitrary functions.

Here is one elementary consequence of coordinate domination.

Proposition 4.4. *Let f, g and h be monotonic injective functions from $\mathbf{n}^t$ to $\mathbb{R}_0$, each dominated by a coordinate. If $h(x) = f(x)g(x)$ for all x in $\mathbf{n}^t$, then two of the three functions are dominated by the same coordinate.*

Proof. We provide the proof when f is order-preserving and g is order-reversing, all other cases being similar.

Suppose the conclusion fails and f, g and h are dominated by distinct coordinates, say f by coordinate 1, g by coordinate 2 and h by coordinate 3. Then consider the points $x_1 = (1, 3, 2, 0, 0, \ldots, 0)$, $x_2 = (2, 2, 3, 0, 0, \ldots, 0)$, and $x_3 = (3, 1, 1, 0, 0, \ldots, 0)$. Observe that $h(x_1) < h(x_2) < h(x_3)$, $x_1(3) = 2$, $x_2(3) = 3$ and $x_3(3) = 1$. Thus h cannot be dominated by coordinate 3, regardless of whether it is order-preserving or order-reversing. □

Note that if f is a monotonic function from $\mathbf{n}^t$ to $\mathbb{R}_0$ and f is dominated by coordinate α, then the reciprocal of f is also dominated by coordinate α, as is the square of f.

One central concept in our proof is the notion of how fast a function changes. Now a sequence, even a strictly increasing sequence, doesn't have to change very much at all, but in this case, differences can change dramatically.

To provide further motivation, we present the following elementary proposition.

Proposition 4.5. *For positive integers m and N with $N > 1$, there exists an integer n_0 so that if $n \geq n_0$ and $a_1 < a_2 < \cdots < a_n$ is any strictly increasing sequence of real numbers, then there exists a subsequence $a_{p_1} < a_{p_2} < \cdots < a_{p_m}$ so that for all i, j, k, l with $1 \leq i < j < k < l \leq m$, either*

$$a_{p_j} - a_{p_i} >> a_{p_l} - a_{p_k},$$

or

$$a_{p_j} - a_{p_i} << a_{p_l} - a_{p_k}.$$

□

We will be studying functions defined on $\mathbf{n}^t$ in what follows. Setting $u_i = (i, i, \ldots, i)$, the values of $f(u_i)$ form a long sequence, and we will want (at least) to control the behavior of f on a long subchain in a manner indicated by the conclusions of Proposition 4.5.

With these comments in mind, we present the basic definitions which will describe how a function changes. We say an order-preserving function $f : X \to \mathbb{R}_0$ *advances conservatively in magnitude* if $f(y) >> f(x)$ whenever $f(y) > f(x)$. Similarly, we say that an order-reversing function f *retreats aggressively in magnitude* if $f(y) << f(x)$ whenever $f(y) < f(x)$. We abbreviate these two definitions with the symbols **ACM** and **RAM**, respectively. The basic idea is that an **ACM** function advances in a manner that postpones large changes as long as possible. Dually, a **RAM** function retreats rapidly, making large changes as soon as possible. Both properties are defined in terms of a parameter N, which we fix in this paper by

setting $N = 10^{100}$. Nevertheless, our definitions make complete sense with any value of N which exceeds 1.

We say a function $f : X \to \mathbb{R}_0$ is *nearly constant* if $f(x) \approx f(y)$ for all $x, y \in X$. We abbreviate this property with the notation **NC**. Evidently, the three properties **ACM**, **RAM** and **NC** are mutually exclusive. However, a function can be **NC** without being monotonic.

As discussed in Section 3, when a function is nearly constant, we still need to describe how its differences behave. Accordingly, when f is an **NC** order-preserving function, we say that f *advances conservatively* if $f(y) - f(x) << f(z) - f(y)$ whenever $f(x) < f(y) < f(z)$. Similarly, we say that an order-preserving **NC** function f *advances aggressively* if $f(y) - f(x) >> f(z) - f(y)$, whenever $f(x) < f(y) < f(z)$.

Dually, if f is an **NC** order-reversing function, we say that f *retreats conservatively* if $f(x) - f(y) << f(y) - f(z)$ whenever $f(x) > f(y) > f(z)$, and we say that f *retreats aggressively* if $f(x) - f(y) >> f(y) - f(z)$ whenever $f(x) > f(y) > f(z)$.

We use the abbreviations **AC**, **AA**, **RC**, and **RA** for the four properties defined in the preceding two paragraphs, so for example, the statement f is **RC** means that f is an **NC** order-reversing function which retreats conservatively. Note that we have defined these last four properties only for **NC** functions.

Let $\mathcal{L} = \{\mathbf{AC}, \mathbf{AA}, \mathbf{RC}, \mathbf{RA}, \mathbf{ACM}, \mathbf{RAM}\}$. We call the elements of $\mathcal{L}$ *change labels*. Now for any function f, at most one of these change labels applies—and for many functions, none of them is appropriate. The $6t$ elements of $\mathcal{L} \times \{1, 2, \ldots, t\}$ are called *change patterns*. A function $f : \mathbf{n}^t \to \mathbb{R}_0$ is said to be *uniform* if there exists a change pattern $(\mathbf{L}, \alpha)$ so that f is $\mathbf{L}$ and is dominated by coordinate α. In this case, we say that f *satisfies the change pattern* $(\mathbf{L}, \alpha)$.

With this background material in mind, we state a theorem which is only a gentle extension of Theorem 4.3. However, we will need an even stronger result, one for which the following theorem is an immediate corollary.

Theorem 4.6. *Given positive integers m, t and N with $N > 1$, there exists an integer n_0 so that if $n \geq n_0$ and $f : \mathbf{n}^t \to \mathbb{R}_0$ is any injective function, then there exist a subposet $\mathbf{Q}$ isomorphic to $\boldsymbol{m}^t$ and a change pattern $(\boldsymbol{L}, \alpha)$, so that the restriction of f to $\mathbf{Q}$ is a uniform function satisfying $(\boldsymbol{L}, \alpha)$.* □

To prove our main theorem, we need to uniformize a large number of functions, a number which goes to infinity with n. The preceding result would allow us to handle only a bounded number of functions. Fortunately, the functions we need to uniformize have additional structure.

Let k and s be positive integers with $1 \leq s \leq k$, and let A be a function which maps the $\mathbf{k}^t$ grids of $\mathbf{n}^t$ to $\mathbb{R}_0$. Then for each $(\mathbf{k} - \mathbf{1})^t$ grid g, we can define a function $A_{g,s}$ on certain points in $\mathbf{n}^t$, namely on those points x (the set of such points may be vacuous) in $\mathbf{n}^t$ so that for each $i = 1, 2, \ldots, t$, the coordinate $x(i)$ is larger than the smallest $s - 1$ integers in the i^{th} factor set of g and less than the largest $k - s$. Of course, when the i^{th} coordinate of x is added to the i^{th} factor set of g for $i = 1, 2, \ldots, t$, we obtain a $\mathbf{k}^t$ grid g'. So we can define $A_{g,s}(x) = A(g')$. Note that the function $A_{g,s}$ has as its domain a poset which is a product of t chains—although in general the lengths of these chains is not constant. We call $A_{g,s}$ a (k, s)-*induced* function.

To make this more concrete, suppose we have an inclusion representation of $\mathbf{n}^3$ using spheres from $\mathbb{R}^d$. Then we can define a function A which maps the

$\mathbf{3}^3$ grids from $\mathbf{n}^3$ to $\mathbb{R}_0$ as follows. With each $\mathbf{3}^3$ grid g', we associate a chain $x < y < z$, and then define $A(g') = \phi(x, y, z)$, the angle at x formed by $L(x, y)$ and $L(x, z)$. Now consider, for example, the value $s = 2$. Then consider the $\mathbf{2}^t$ grid $g = \{10, 23\} \times \{47, 90\} \times \{18, 45\}$. It follows that the $(3, 2)$-induced function $A_{g,2}$ is defined on a subposet isomorphic to $\mathbf{12} \times \mathbf{42} \times \mathbf{26}$. Of course, the size of the subposet on which the function $A_{g,s}$ is defined depends both on g and s. However, if the set of points on which $A_{g,s}$ is defined is non-empty, we can discuss the issue of whether $A_{g,s}$ is uniform.

We are ready to present the main uniformizing theorem needed to prove Theorem 2.1. In the proof, we sketch those details which are included in the proof of Theorem 4.3 and concentrate on those which are new to this paper.

Theorem 4.7. *Given positive integers m, t, k and N with $N > 1$, there exists an integer n_0 so that if $n \geq n_0$ and A is any injective function which maps the $\mathbf{k}^t$ grids of $\mathbf{n}^t$ to $\mathbb{R}_0$, then there exist k change patterns $(\mathbf{L}_1, \alpha_1), (\mathbf{L}_2, \alpha_2), \ldots, (\mathbf{L}_k, \alpha_k)$ and a subposet $\mathbf{P}$ isomorphic to $\boldsymbol{m}^t$ so that for every $s = 1, 2, \ldots, k$ and every $(\mathbf{k} - \mathbf{1})^t$ grid g in $\mathbf{P}$, the (k, s)-induced function $A_{g,s}$ is uniform and satisfies change pattern $(\mathbf{L}_s, \alpha_s)$.*

Proof. Before beginning the proof, we comment that it is essential that the change pattern of an induced function $A_{g,s}$ depends only on s, not on g. There are only k choices for s, but the number of choices for g can be much larger than n. To help the reader keep track of sizes, we will always use g, g' and g'' (with subscripts) to denote grids of size $(\mathbf{k} - \mathbf{1})^t$, $\mathbf{k}^t$ and $(\mathbf{k} + \mathbf{1})^t$, respectively.

Set $q = 100m^t N \log N + 4k$ and $l = k(2^{4t} + 3 \cdot 2^{2t})$. Then set $r = 2^l$. The value of q is taken to insure that q is comfortably larger than m^t, $N \log N$ and k. We now show that the value $n_0 = \mathrm{PR}(q, k + 1, r, t)$ satisfies the conclusion of our theorem. To accomplish this, we start with a poset $\mathbf{P} = \mathbf{P}_0$ isomorphic to $\mathbf{n}_0^t$. We will then determine subposets $\mathbf{P}_1$, $\mathbf{P}_2$ and $\mathbf{P}_3$ with $\mathbf{P}_{i+1}$ a subposet of $\mathbf{P}_i$ for $i = 0, 1, 2$. For each $i = 0, 1, 2, 3$, $\mathbf{P}_i$ will be isomorphic to $\mathbf{n}_i^t$. The values of the other parameters are $n_1 = q$, $n_2 = n_1 - 4k$ and $n_3 = m$.

To show that the specified value of n_0 works, we first describe a coloring of the $(\mathbf{k} + \mathbf{1})^t$ grids in $\mathbf{n}^t$.

Let A be any injective function which maps the $\mathbf{k}^t$ grids of $\mathbf{n}_0^t$ to $\mathbb{R}_0$. We use A to define a coloring of the $(\mathbf{k} + \mathbf{1})^t$ grids of $\mathbf{n}_0^t$ using r colors.

Given a $(\mathbf{k} + \mathbf{1})^t$ grid g'', we temporarily relabel the factor sets so that each is just $\{1, 2, \ldots, k + 1\}$. For each $s = 1, 2, \ldots, k$, we consider the set G_s of all $\mathbf{k}^t$ grids having factor sets of the form $\{1, 2, \ldots, s - 1, e, s + 2, s + 3, \ldots, k + 1\}$, where $e \in \{s, s + 1\}$. For each s, there is a natural correspondence between grids in G_s and subsets of $\{1, 2, \ldots, t\}$. So we can label these grids as $g'(S, s)$ where $S \subseteq \{1, 2, \ldots, t\}$. With this convention, $g'(\emptyset, s)$ corresponds to the subgrid in which every factor set is $\{1, 2, \ldots, s, s + 2, s + 3, \ldots, k + 1\}$. When the value of s is fixed, we may just refer to a grid as a subset of $\{1, 2, \ldots, t\}$.

Now fix a value of s. Then consider all the images of the grids in G_s under the map A, using the abbreviation $A(S)$ for $A(g'(S, s))$. As a consequence, some of the following statements will be true (T) and some will be false (F).

1. $A(S_1) < A(S_2)$.
2. $A(S_1) << A(S_2)$.
3. $A(S_1) \lesssim A(S_2)$.
4. $A(S_1) - A(S_2) << A(S_3) - A(S_4)$.

To emphasize that these statements actually depend on both g'' and s, we refer to them collectively as $\Sigma(g'', s)$.

In each of the first three forms, there are 2^{2t} ordered pairs of variables for which the statement can be meaningfully expressed. In the last form, there are 2^{4t} ordered 4-tuples for which the statement makes sense. So summing over all s, there are $l = k(2^{4t} + 3 \cdot 2^{2t})$ statements altogether. It follows that we may associate with g'' a boolean string of T's and F's of length l. There are $r = 2^l$ such strings.

Since $n_0 = \mathrm{PR}(q, k+1, r, t)$, there is a subposet $\mathbf{P}_1$ isomorphic to $\mathbf{q}^t$ so that all $(\mathbf{k}+\mathbf{1})^t$ grids in $\mathbf{P}_1$ receive the same color. This uniform color is then an assignment of truth values so that the issue of whether statements in $\Sigma(g'', s)$ are true or false depends only on s and not on g''. Accordingly, for the subposet $\mathbf{P}_1$ in which all grids receive the same color, we can refer to statements in the family $\Sigma(s)$, deleting g'' from our earlier notation.

Now let $\mathbf{P}_2$ denote all those $x \in \mathbf{P}_1$ so that $2k \leq x(i) \leq n_1 - 2k$ for $i = 1, 2, \ldots, t$. Then $\mathbf{P}_2$ is isomorphic to $\mathbf{n}_2$ with $n_2 = n_1 - 4k$ as promised. (This technical step is just to save some space at the top and bottom of $\mathbf{P}_2$.)

Since $n_2 > 3m^t$, we may use the spacing proposition to choose a subposet $\mathbf{P}_3$ of $\mathbf{P}_2$, with $\mathbf{P}_3$ isomorphic to $\mathbf{m}^t$, so that $\mathbf{P}_3$ is embedded by I in $\mathbf{P}_2$ with gap size 3.

In the remainder of the proof, we concentrate on points from $\mathbf{P}_3$, but we discuss their coordinates in $\mathbf{P}_2$— via the embedding I.

Now fix a value of s. We show that there exists a change pattern $(\mathbf{L}, \alpha)$ so that if g is any $(\mathbf{k}-\mathbf{1})^t$ grid in $\mathbf{P}_3$, the induced (k, s) function $A_{g,s}$ is uniform and satisfies the change pattern $(\mathbf{L}, \alpha)$.

Let g be any $(\mathbf{k}-\mathbf{1})^t$ grid in $\mathbf{P}_3$. We may assume without loss of generality that the subposet $\mathbf{Q}$ of points in $\mathbf{P}_3$ on which $A_{g,s}$ is defined is non-trivial, else there is nothing to prove.

If x and y are distinct points from $\mathbf{Q}$, then the coordinates of x and y together with the grid g form a $(\mathbf{k}+\mathbf{1})^t$ grid g''. Although g'' depends on g, x and y, all $(\mathbf{k}+\mathbf{1})^t$ grids receive the same color, so we can track the behavior of their images in some canonical grid, say the one in which all factor sets are just $\{1, 2, \ldots, k+1\}$.

As before, we associate x and y with subsets of $\{1, 2, \ldots, t\}$. If $x < y$, then $x = \emptyset$ and $y = \{1, 2, \ldots, t\}$, so $A_{g,s}$ is order-preserving if and only if the statement

$$A(\emptyset) < A(\{1, 2, \ldots, t\})$$

from $\Sigma(s)$ is true.

Now suppose $A_{g,s}$ is order-preserving. We explain why $A_{g,s}$ is dominated by a coordinate α which depends only on s.

Consider the grids corresponding to the singleton sets $\{1\}, \{2\}, \ldots, \{t\}$, and the order of their images under A. Suppose that the largest of these is $A(\{\alpha\})$. We now show that $A_{g,s}$ is dominated by coordinate α, and assume that $\alpha = 1$ without loss of generality.

Now consider the points $v_1, v_2, \ldots, v_t$ in $\mathbf{P}_2$ where

1. $v_i(1) = i - 1$;
2. $v_i(j) = n_1 - 1 \quad$ for $j = 2, 3, \ldots, t - i + 1$; and
3. $v_i(j) = 0 \quad$ for $j = t - i + 2, \ldots, t$.

We claim that $A_{g,s}(v_i) < A_{g,s}(v_{i+1})$ for $i = 1, 2, \ldots, t-1$. To see this, note that for each i, we can add a $(\mathbf{k}-\mathbf{1})^t$ grid to the coordinates of v_i and v_{i+1} to form a $(\mathbf{k}+\mathbf{1})^t$ grid for which $v_i = \{1\}$ and $v_{i+1} = \{t - i + 2\}$.

It follows that $A_{g,s}(0, t-1, t-1, \ldots, t-1) < A_{g,s}(t-1, 0, 0, 0, \ldots, 0)$.

Now let x and y be distinct points from $\mathbf{Q}$ with $x(1) < y(1)$. We show that $A_{g,s}(x) < A_{g,s}(y)$. This is certainly true if $x < y$, so we assume that x and y are incomparable. Since the coloring of grids is uniform, and $A_{g,s}$ is order-preserving, we conclude that $A_{g,s}(x) \leq A_{g,s}(x(1), n_1 - 1, n_1 - 1, \ldots, n_1 - 1) < A_{g,s}(y(1), 0, 0, 0, \ldots, 0) \leq A_{g,s}(y)$.

The situation when $A_{g,s}$ is order-reversing is dual.

We now show that the restriction of $A_{g,s}$ to $\mathbf{P}_3$ is uniform and has a change pattern which depends only on s. Suppose first that $A_{g,s}$ is order-preserving.

Consider the following statement from $\Sigma(s)$:

$$A(\{1, 2, \ldots, t\}) \lesssim A(\emptyset).$$

Suppose first that this statement is false. Then we know that $A_{g,s}(y) \geq (1 + 1/N)A_{g,s}(x)$ for every 2-element chain from $\mathbf{Q}$.

Recall that $q > 10N \log N$. Consider the chain $u_0 < u_1 < \cdots < u_q - 1$ in $\mathbf{P}_2$, where $u_i = (i, i, \ldots, i)$ (coordinate values in $\mathbf{P}_2$). Then consider two other auxiliary chains $v_1 < v_2 < \cdots < v_k$ and $w_1 < w_2 < \cdots < w_k$ from $\mathbf{P}_1 - \mathbf{P}_2$, where $v_i(j) = i$ for $i = 1, 2, \ldots, k$ and all $j = 1, 2, \ldots, t$. Also, $w_i(j) = n_1 - k - 1 + i$ for $i = 1, 2, \ldots, k$ and all $j = 1, 2, \ldots, t$. These coordinate values are defined in $\mathbf{P}_1$, so $v_k < u_0$ and $u_{q-1} < w_1$.

Then for each $i = 0, 1, \ldots, q - 1$, we may consider the grid g'_i determined by u_i and appropriate portions of the two auxiliary chains, i.e, $s - 1$ of the points from the bottom and $k - s$ from the top. Together, they form a $\mathbf{k}^t$ grid g'_i. For simplicity, we write $A(u_i)$ rather than $A(g'_i)$.

For each $i, j = 0, 1, \ldots, q - 1$, with $i < j$, we may consider the points u_i and u_j together with portions of the auxiliary chains as forming a $(\mathbf{k} + \mathbf{1})^t$ grid in which u_i and u_j occur as levels s and $s + 1$. In such a grid, $u_i = \emptyset$ and $u_j = \{1, 2, \ldots, t\}$.

With the choice $j = i + 1$, we conclude that $A(u_{i+1}) \geq (1 + 1/N)A(u_i)$ for $i = 1, 2, \ldots, q - 1$. Since $q > 10N \log N$, it follows that $A(u_1)) << A(u_{q-1})$. Therefore the statement

$$A(\emptyset) << A(\{1, 2, \ldots, t\})$$

from $\Sigma(s)$ is true, and $A_{g,s}(x) << A_{g,s}(y)$ for every 2-element chain $x < y$.

Now suppose that x and y are any two points from $\mathbf{Q}$ and that $A_{g,s}(x) < A_{g,s}(y)$. Since $A_{g,s}$ is dominated by coordinate α, we know that $x(\alpha) < y(\alpha)$. Since the gap size is at least 3, we may choose an integer β so that $x(\alpha) < \beta < \beta + 1 < y(\alpha)$. Now let u and v be any two points in $\mathbf{Q}$ so that $u < v$, $u(\alpha) = \beta$ and $v(\alpha) = \beta + 1$. Then $A_{g,s}(x) < A_{g,s}(u)$, $A_{g,s}(u) << A_{g,s}(v)$ and $A_{g,s}(v) < A_{g,s}(y)$. It follows that $A_{g,s}(x) << A_{g,s}(y)$, so that $A_{g,s}$ is **ACM**.

Now suppose that the statement

$$A(\{1, 2, \ldots, t\}) \lesssim A(\emptyset)$$

from $\Sigma(s)$ is true. Then $A_{g,s}(y) \lesssim A_{g,s}(x)$ for every 2-element chain $x < y$ from $\mathbf{Q}$. Let B be the bottom element of $\mathbf{Q}$ and let T be the top element. Then $A_{g,s}(B)) < A_{g,s}(x) < A_{g,s}(T)$ for every other point x from $\mathbf{P}_1$. This shows that $A_{g,s}$ is **NC**.

We now show that $A_{g,s}$ is either **AC** or **AA**. Suppose first that the statement

$$A(\{1\}) - A(\emptyset) << A(\{1, 2, \ldots, t\}) - A(\{1\})$$

from $\Sigma(s)$ is true. Then it follows that for every 3-element chain $x < y < z$ in $\mathbf{Q}$, $A_{g,s}(y) - A_{g,s}(x) << A_{g,s}(z) - A_{g,s}(y)$. Now let x, y and z be any three

points from $\mathbf{P}$ with $A_{g,s}(x) < A_{g,s}(y) < A_{g,s}(z)$. Then, since the gap size in $\mathbf{P}_2$ is 3 and $A_{g,s}$ is dominated by coordinate α, we may find a 3-element chain $w_1 < w_2 < w_3$ so that $w_1(\alpha) < x(\alpha) < y(\alpha) < w_2(\alpha) < w_3(\alpha) < z(\alpha)$. Since $A_{g,s}(y) - A_{g,s}(x) < A_{g,s}(w_2) - A_{g,s}(w_1)$ and $A_{g,s}(w_3) - A_{g,s}(w_2) < A_{g,s}(z) - A_{g,s}(y)$, it follows that $A_{g,s}(y) - A_{g,s}(x) << A_{g,s}(z) - A_{g,s}(y)$. We conclude that $A_{g,s}$ is **AC**.

Dually, if the statement

$$A(\{1\}) - A(\emptyset) >> A(\{1, 2, \ldots, t\}) - A(\{1\})$$

from $\Sigma(s)$ is true, then $A_{g,s}$ is **AA**.

Now suppose that both statements from $\Sigma(s)$ are false. Then, referring to the chain $u_0 < u_1 < \ldots, u_{q-1}$ discussed earlier in the proof, we note that if $0 \leq i < j < k < l \leq q-1$, we have:

$$\big(A(u_j) - A(u_i)\big)/N \leq A(u_l) - A(u_k) \leq N\big(A_{g,s}(u_j) - A_{g,s}(u_i)\big).$$

It follows that the interval $[A(u_0), A(u_{q-1})]$ is divided up into $q-1$ disjoint subintervals. Choose an integer j with $1 \leq j \leq q-2$ so that the length of the interval $[A(u_j), A(u_{j+1})]$ is as small as possible. Then set $i = 0$, $k = j+1$ and $l = q-1$ to conclude that the length of $[A(u_0), A(u_i)]$ is at most N times the length of $[A(u_j), A(u_k)]$. Similarly, the length of $[A(u_l), A(u_{q-1})]$ is at most N times the length of $[A(u_j), A(u_k)]$. Being generous, we can conclude that $j \leq N$ and $q - j \leq N$, so that $q \leq 2N$. This contradicts the fact that $q > 10N \log N$.

A dual argument shows that when $A_{g,s}$ is order-reversing, it is either **RAM** or **NC**. When it is **NC**, it is either **RC** or **RA**.

□

Note that Theorem 4.6 is just the special case of Theorem 4.7 obtained when $k = 1$. Although we stated Theorem 4.7 in terms of a single function A, it is clear that we can apply it to a bounded number of functions. In fact, this result—and for that matter, all the Ramsey theoretic material discussed here—can be treated in much greater generality.

Before leaving this section, we point out two important implications of the preceding Theorem 4.7. Let $f : \mathbf{n}^3 \to \mathbb{R}_0$ be a uniform function. Now let $x, y \in \mathbf{n}^3$ and suppose that we know that $f(y) \geq (1 + 1/N)f(x)$. Even with no information as to which change pattern f satisfies, not even knowing whether it is order preserving or order reversing, we may still conclude that $f(y) > Nf(x)$. We call this phenomenon the *prinicple of differentiation*. It results from using the Ramsey theoretic tools developed in this section to eliminate the case in which $(1+1/N)f(x) \leq f(y) \leq Nf(x)$.

Second, the theorem allows us to recover from errors. For example, in arguments to follow, we will say that $e_1 \approx e_2$ and $e_3 \approx e_4$ imply $e_1 + e_3 \approx e_2 + e_4$. Similarly, if we know that $e_1 < 1/N$ and $e_2 < 1/N$, we will conclude that $e_1 + e_2 < 1/N$. Technically speaking, this may not quite be true. But by restricting to a subposet, we can strengthen the bounds so that such conclusions can be made (at least a bounded number of times) with impunity.

5. Part 1: Uniformizing the Representation

This section begins the proof of Theorem 1.1. As discussed in Section 2, we prove Theorem 1.1 by showing that if n is sufficiently large, the finite 3-dimensional poset

$\mathbf{n}^3$ is not a sphere order. We start with the assumption that we have an inclusion representation F using spheres for $\mathbf{n}^3$ and then argue to a contradiction—provided n is sufficiently large. The issue of how large n must be is decided in six steps. We begin by setting $n = n_0$ and $\mathbf{P} = \mathbf{P}_0 = \mathbf{n}_0^3$. Then, for each $i = 1, 2, \ldots, 6$, we will choose an appropriate subposet $\mathbf{P}_i$ of $\mathbf{P}_{i-1}$, with $\mathbf{P}_i$ isomorphic to $\mathbf{n}_i^3$. At each step, we increase the uniformity of the inclusion representation for the remaining points. The final poset $\mathbf{P}_6$ is isomorphic to $\mathbf{11}^3$, which is certainly of modest size in comparison to other quantities we have discussed. But to obtain this final poset, we must start with a very large poset. The relative sizes between $n_0, n_1, \ldots, n_6$ will be clear from the material to follow.

To begin, we assume that the spheres used in our representation are in "general position," i.e.:

1. No two spheres are tangent.
2. All centers are distinct.
3. No three centers are collinear.
4. No four centers are coplanar.
5. All radii are distinct and positive.
6. The angles determined by any three centers are distinct.
7. The distances from any center to the line passing through two other centers are all distinct.

This assumption is allowed by the fact that we may add (in an order preserving manner) a small quantity to each radius without disturbing the inclusion relation. We may then make small perturbations in the center locations.

Assuming that n_0 is sufficiently large in terms of n_1, we may apply Theorem 4.6 to find a subposet $\mathbf{P}_1$ isomorphic to $\mathbf{n}_1^3$ on which the radius function r is uniform.

When $x < y$, we know that $r(x) < r(y)$, so the function r must be order-preserving on $\mathbf{P}_1$. Without loss of generality, we assume that it is dominated by coordinate 1. So r satisfies one of the following three change patterns: $(\mathbf{ACM}, 1)$, $(\mathbf{AC}, 1)$, or $(\mathbf{AA}, 1)$. However, we want to assume that r is $\mathbf{ACM}$.

Should r be $\mathbf{AA}$, we choose a large positive number R_0, with $r(x) < R_0$ for every $x \in \mathbf{P}_1$. We then take a new representation by setting $\hat{r}(x) = R_0 - r(x)$. Note that we are merely taking advantage of the well known fact that the dual of a finite sphere order is again a sphere order—together with the trivial observation that $\mathbf{n}^3$ is self dual. Now that the change has been made, we drop the hat and use $r(x)$ to denote the new radius function. Obviously, the new function is again uniform.

So now we have a representation of $\mathbf{P}_1$ on which the radius function is either $\mathbf{AC}$ or $\mathbf{ACM}$. If it is $\mathbf{AC}$, we let $B_1 = (0, 0, 0)$ and $r_0 = r(B_1)$. We then define a new radius function $\hat{r}(x) = r(x) - r_0$. Since $\hat{r}(y) - \hat{r}(x) = r(y) - r(x)$ for every x and y in $\mathbf{P}_1$, we could equally well use $\hat{r}$ as our radius function.

Now let $x < y$ be any two elements of $\mathbf{P}_1$ with $\hat{r}(x) < \hat{r}(y)$ and $x > B_1$. Then $r(B_1) < r(x) < r(y)$. It follows that $N\hat{r}(x) = N(r(x) - r_0) = N(r(x) - r(B_1)) < r(y) - r(x) < r(y) - r(B_1) = \hat{r}(y)$. It follows that $\hat{r}$ is a uniform $\mathbf{ACM}$ function. Again, we drop the hats and use r to denote the new radius function. However, we now have a representation where the least element has a circle of radius zero. Since the criteria for uniformity are expressed in terms of strict inequalities, we add a small quantity to the radius of the bottom element.

We next describe three functions A, B and C to which we will apply Theorem 4.7. In each case, we take the value $k = 3$. With each $\mathbf{3}^3$ grid g in $\mathbf{P}_1$, we associate a

3-element chain $x < y < z$ and then set $A(g) = \phi(x,y,z)$, $B(g) = h(x,y,z)$ and $C(g) = h(x,y,z)\phi(x,y,z)/2$.

After applying Theorem 4.7 three times, once for each of these functions, we may assume that we have a subposet $\mathbf{P}_2$ isomorphic to $\mathbf{n}_2^3$ so that we have nine change patterns, one for each ordered pair from $\{A,B,C\} \times \{1,2,3\}$, so that the nine classes of $(3,s)$-induced functions they produce are uniform and have a change pattern depending only on the class.

We are only concerned with five of these nine classes:

1. The $(3,2)$ and $(3,3)$ functions induced by A.
2. The $(3,1)$ and $(3,2)$ functions induced by B.
3. The $(3,2)$ function induced by C.

We find it convenient to use the symbols Φ, Θ, K, H, and G to denote these functions, so that:

1. For each 2-element chain $x < z$, the $(3,2)$-induced function $\Phi(x,y,z)$ is defined on those y with $x < y < z$ by setting $\Phi(x,y,z) = \phi(x,y,z)$.
2. For each 2-element chain $x < y$, the $(3,3)$-induced function $\Theta(x,y,z)$ is defined on those z with $x < y < z$ by setting $\Theta(x,y,z) = \phi(x,y,z)$.
3. For each 2-element chain $y < z$, the $(3,1)$-induced function $K(x,y,z)$ is defined on those x with $x < y < z$ by setting $K(x,y,z) = h(x,y,z)$.
4. For each 2-element chain $x < z$, the $(3,2)$-induced function $H(x,y,z)$ is defined on those y with $x < y < z$ by setting $H(x,y,z) = h(x,y,z)$.
5. For each 2-element chain $x < z$, the $(3,2)$-induced function $G(x,y,z)$ is defined on those y with $x < y < z$ by setting $G(x,y,z) = h(x,y,z)\phi(x,y,z)/2$.

We will return to the discussion of these induced functions after we develop some geometric aspects of our construction.

6. Part 2: Geometric Implications

Now let $u_1 = (0, n_2 - 1, n_2 - 1)$, and let $\epsilon_0 = r(u_1)$. Setting $n_3 = n_2 - 2$, and letting $\mathbf{P}_3$ consist of all $x \in \mathbf{P}_2$ whose coordinates satisfy $0 < x(i) < n_2 - 1$ for $i = 1,2,3$, it follows that $\mathbf{P}_3$ is isomorphic to $\mathbf{n}_3^3$. Also, note that $N\epsilon_0 < r(x)$ for every x from $\mathbf{P}_3$.

Claim 1. *For all $x, y \in \mathbf{P}_3$ with $x(1) < y(1)$, $r(y) \approx \rho(x,y)$.*

Proof. Let $u \in \mathbf{P}_3$. Then $B_1 < u$, so that $F(B_1) \subset F(u)$, and $r(u) - r(B_1) > \rho(u, B_1)$, so $r(u) > \rho(u,B_1) + r(B_1) > \rho(u,B_1)$. On the other hand, u is incomparable to u_1, so $r(u) - r(u_1) < \rho(u,u_1) < \rho(u,B_1) + \rho(B_1,u_1) < \rho(u,B_1) + r(u_1) = \rho(u,B_1) + \epsilon_0$. It follows that

$$\rho(x,B_1) < r(u) < \rho(x,B_1) + 2\epsilon_0$$

for every $u \in \mathbf{P}_3$. Thus $r(u) \approx \rho(u,B_1)$, for every u in $\mathbf{P}_3$.

Noting that $\rho(y,B_1) < \rho(x,B_1) + \rho(x,y)$ and $\rho(x,y) < \rho(y,B_1) + \rho(x,B_1)$, we see that $r(y) - \rho(x,y) = \text{zero}\big(r(x)\big)$. Therefore, $r(y) \approx \rho(x,y)$. □

When x, y and z are distinct points in $\mathbf{P}_3$, we know that $\Delta(x,y,z) = \rho(x,y) + \rho(y,z) - \rho(x,z) > 0$. However, we can actually write the following elementary identity:

$$\Delta(x,y,z) \quad = \quad \rho(x,y)(1 - \cos\phi(x,y,z)) + \rho(y,z)(1 - \cos\gamma(x,y,z)). \tag{2}$$

Note that

$$\rho(x,y)\sin\phi(x,y,z) = h(x,y,z) = \rho(y,z)\sin\gamma(x,y,z).$$

Also, $\rho(x,y) \approx r(y)$, $\rho(y,z) \approx r(z)$ and $r(y) << r(z)$ imply that $\gamma(x,y,z) << \phi(x,y,z)$. We conclude that

$$\Delta(x,y,z) \approx r(y)(1-\cos\phi(x,y,z)).$$

When $x(1) < y(1) < z(1)$, the preceding discussion shows that the angle $\gamma(x,y,z)$ is very small. But at this point, we cannot make such a claim for $\phi(x,y,z)$. However, we now show that we may assume that all $\phi(x,u,z)$ are very small, provided $x < u < z$.

To accomplish this, we use the Product Ramsey Theorem. With each $\mathbf{3}^3$ grid g, we associate a chain $x < u < z$. Color the grid red if $\phi(x,u,z) < 1/N$; otherwise, color it blue. Setting $n_3 = \mathrm{PR}(n_4,3,2,3)$, we may find a subposet $\mathbf{P}_4$ isomorphic to $\mathbf{n}_4$ so that all $\mathbf{3}^3$ grids in $\mathbf{P}_4$ receive the same color. Now set $n_4 = n_5^3$ and $n_5 = 17$. We may then choose a subposet $\mathbf{P}_5$ isomorphic to $\mathbf{n}_5^3$ via the embedding I defined in the spacing proposition.

Claim 2. *For every 3-element chain $x < u < z$ in $\mathbf{P}_5$, $\phi(x,u,z) < 1/N$.*

Proof. Suppose to the contrary that $\phi(x,u,z) \geq 1/N$ for some 3-element chain in $\mathbf{P}_5$. Considering coordinates in $\mathbf{P}_4$, we see that $\mathbf{P}_4$ contains a blue $\mathbf{3}^t$ grid. Thus all $\mathbf{3}^3$ grids in $\mathbf{P}_4$ are blue.

Then consider the following points in $\mathbf{P}_4$: $x = u_1 = (1,1,1)$, $v = (2,0,7)$, $u = u_5 = (5,5,5)$ and $z = u_6 = (6,6,6)$. Because r is dominated by coordinate 1, we know that $N^3 r(v) < r(u)$.

Since

$$\Delta(x,u,z) \gtrsim r(u)\big(1-\cos\phi(x,u,z)\big)$$

and $\phi(x,u,z) \geq 1/N$, we conclude that $2N^2\Delta(x,u,z) \gtrsim r(u)$. On the other hand, since $\big(1-\cos\phi(x,v,z)\big) \leq 1$, we know that $\Delta(x,v,z) \lesssim r(v)$. Thus, $r(u) \lesssim 2N^2 r(v)$. But, since $r(u) > N^3 r(v)$, this last inequality fails badly. The contradiction completes the proof. □

For the remainder of the proof, we will use the symbols $B = (0,0,0)$ and $T = (16,16,16)$ to denote the top and bottom elements of $\mathbf{P}_5$. Also, we let $B' = (1,1,1)$, $B'' = (2,2,2)$, $T'' = (14,14,14)$, $T' = (15,15,15)$. We then let $\mathbf{P}_6$ consist of those points x in $\mathbf{P}_5$ with $2 < x(i) < 14$ for $i = 1,2,3$. Then $B < B' < x < T' < T$ for every x in $\mathbf{P}_6$. As anticipated, $n_6 = n_5 - 6 = 11$.

Also, in $\mathbf{P}_6$, we will let $C = \{u_1 < u_2 < \cdots < u_9\}$ denote the nine element chain with $u_i = (i,i,i)$, for $i = 1,2,\ldots,9$. Of course, we intend that the coordinates of the points in C are given in $\mathbf{P}_6$ rather than in $\mathbf{P}_5$.

For emphasis, we point out that the triangle inequality holds for angles in $\mathbb{R}^d$.

Proposition 6.1. *Let x_1, x_2, x_3 and x_4 be distinct points from $\mathbf{n}^3$. Then*

$$\phi(x_1,x_2,x_3) + \phi(x_1,x_3,x_4) > \phi(x_1,x_2,x_4), \text{ and} \tag{3}$$

$$\gamma(x_1,x_2,x_4) + \gamma(x_2,x_3,x_4) > \gamma(x_1,x_3,x_4). \tag{4}$$

If s_1, s_2 and s_3 are distinct points from $\mathbb{R}^d$, then $\phi(s_1,s_2,s_3) = \phi(s_1,s_3,s_2)$ and $\gamma(s_1,s_2,s_3) = \gamma(s_2,s_1,s_3)$. So the triangle inequalities in Proposition 6.1 can be written in several different forms.

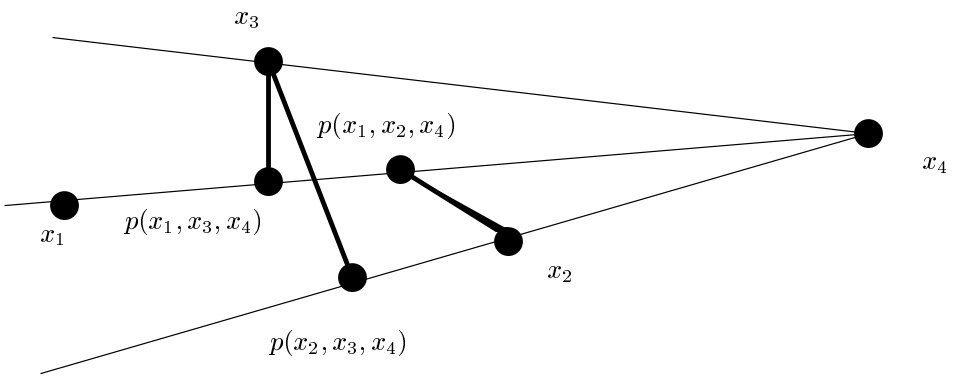

FIGURE 2

On the other hand, note that $h(s_1, s_2, s_3) \neq h(s_2, s_1, s_3)$ in general. In fact, the two quantities can be far apart. However, due to the uniform behavior of the radius function, we do have approximate symmetry in the first two coordinates for centers.

Proposition 6.2. *Let x_1, x_2 and x_3 be distinct points from $\mathbf{n}^3$ with $x_1(1) < x_3(1)$ and $x_2(1) < x_3(1)$. Then*

$$h(x_1, x_2, x_3) \approx h(x_2, x_1, x_3) \approx r(x_3)\gamma(x_1, x_2, x_3).$$

Proof. Observe that

$$h(x_1, x_2, x_3) = \rho(x_2, x_3) \sin \gamma(x_1, x_2, x_3)$$

and

$$h(x_2, x_1, x_3) = \rho(x_1, x_3) \sin \gamma(x_2, x_1, x_3).$$

Because $\gamma(x_1, x_2, x_3) = \gamma(x_2, x_1, x_3)$ and $\rho(x_2, x_3) \approx r(x_3) \approx \rho(x_1, x_3)$, it follows that

$$h(x_1, x_2, x_3) \approx h(x_2, x_1, x_3) \approx r(x_3)\gamma(x_1, x_2, x_3).$$

□

We consider the next corollary as providing a "weak" version of the triangle inequality for the height function (see Figure 2). From an intuitive standpoint, we consider this the "view back from infinity."

Corollary 6.3. *Let x_1, x_2, x_3 and x_4 be points from $\mathbf{n}^3$ with $x_i(1) < x_4(1)$ for $i = 1, 2, 3$. Then*

$$h(x_1, x_2, x_4) + h(x_2, x_3, x_4) \gtrsim h(x_1, x_3, x_4). \tag{5}$$

Proof. We know that $\gamma(x_1, x_2, x_4) + \gamma(x_2, x_3, x_4) > \gamma(x_1, x_3, x_4)$. From Proposition 6.2, we note that

1. $h(x_1, x_2, x_4) \approx r(x_4)\gamma(x_1, x_2, x_4)$,
2. $h(x_2, x_3, x_4) \approx r(x_4)\gamma(x_2, x_3, x_4)$,
3. $h(x_1, x_3, x_4) \approx r(x_4)\gamma(x_1, x_3, x_4)$.

Clearly, these statements imply the conclusion of the corollary. □

Because the expression $h(x_1, x_2, x_3)$ is (approximately) symmetric in the first two coordinates, we can write the (weak) triangle inequalities for height in several different forms, just as was the case for angles. For example, we could have written

$$h(x_1, x_2, x_4) + h(x_1, x_3, x_4) \gtrsim h(x_2, x_3, x_4).$$

Claim 3. *For all x, y and z in $\mathbf{P}_6$ with $x(1) < y(1) < z(1)$, $\phi(x, y, z) < 1/N$.*

Proof. Using the weak triangle inequality, we see that $h(x, y, z) \lesssim h(B, x, z) + h(B, y, z)$, so that

$$\sin\phi(x, y, z) \approx h(x, y, z)/r(y) \lesssim r(x)\sin\phi(B, x, z)/r(y) + \sin\phi(B, y, z).$$

Now $\phi(B, x, z) < \phi(B, x, T) + \phi(B, z, T) < 1/N$. Similarly, $\phi(B, y, z) < 1/N$. Thus $\sin\phi(x, y, z) < 1/N$. □

We may now use the following estimates for any three points x, y and z with $x(1) < y(1) < z(1)$:

$$\Delta(x, y, z) \approx r(y)\phi^2(x, y, z)/2 \quad \text{and} \quad h(x, y, z) \approx r(y)\phi(x, y, z).$$

7. Part 3: Applications of Uniformity

This section develops properties of the various functions involving angles and distances. Already, we know that the radius function r is **ACM** and dominated by coordinate 1.

Without loss of generality, we may assume :

1. There is a coordinate α_1 and a change label $\mathbf{L}_1 \in \mathcal{L}$ so that for every 2-element chain $x < z$ in $\mathbf{P}_5$, the map $\Phi(x, y, z)$, defined on those y with $x < y < z$ is uniform and satisfies change pattern $(\mathbf{L}_1, \alpha_1)$.
2. There is a coordinate α_2 and a change label $\mathbf{L}_2 \in \mathcal{L}$ so that for every 2-element chain $x < y$ in $\mathbf{P}_5$, the map $\Theta(x, y, z)$, defined on those z with $x < y < z$ is uniform and satisfies change pattern $(\mathbf{L}_2, \alpha_2)$.
3. There is a coordinate α_3 and a change label $\mathbf{L}_3 \in \mathcal{L}$ so that for every 2-element chain $y < z$ in $\mathbf{P}_5$, the map $K(x, y, z)$, defined on those x with $x < y < z$ is uniform and satisfies change pattern $(\mathbf{L}_3, \alpha_3)$.
4. There is a coordinate α_4 and a change label $\mathbf{L}_4 \in \mathcal{L}$ so that for every 2-element chain $x < z$ in $\mathbf{P}_5$, the map $H(x, y, z)$, defined on those y with $x < y < z$ is uniform and satisfies change pattern $(\mathbf{L}_4, \alpha_4)$.
5. There is a coordinate α_5 and a change label $\mathbf{L}_5 \in \mathcal{L}$ so that for every 2-element chain $x < z$ in $\mathbf{P}_5$, the map $G(x, y, z)$, defined on those y with $x < y < z$ is uniform and satisfies change pattern $(\mathbf{L}_5, \alpha_5)$.

When $x \in \mathbf{P}_6$, we use the shorthand notations: $\Phi(x) = \Phi(B, x, T)$, $\Theta(x) = \Theta(B, B', x)$, $K(x) = K(x, T', T)$, $H(x) = H(B, x, T)$ and $G(x) = G(B, x, T)$. Also, for example, when we say that Φ is dominated by coordinate α_1, we mean that $\Phi(x) = \Phi(B, x, T)$ is dominated by α_1. It is important to remember that, for example, for all $x < z$, the function $\Phi(x, y, z)$, defined on y with $x < y < z$, satisfies the same change pattern as $\Phi(x)$.

We now begin to gather some information about other patterns present in $\mathbf{P}_5$. For reasons which will become clear, we concentrate on the $(3, 2)$-induced functions Φ and H, and we make extensive use of the principle of differentiation.

Claim 4. *The function Φ cannot be **ACM**.*

Proof. Suppose to the contrary that Φ is **ACM**. Then for every 2-element chain $x < z$, the map $\Phi(x, y, z)$ defined on those y with $x < y < z$ is **ACM**. It follows that

1. $\phi(u_1, u_3, u_4) - \phi(u_1, u_2, u_3) = \text{zero}\big(\phi(u_1, u_2, u_4)\big)$,
2. $\phi(u_1, u_3, u_5) - \phi(u_1, u_2, u_3) = \text{zero}\big(\phi(u_1, u_2, u_5)\big)$,
3. $\phi(u_1, u_4, u_5) - \phi(u_1, u_3, u_4) = \text{zero}\big(\phi(u_1, u_3, u_5)\big)$.

Therefore, $\phi(u_1, u_4, u_5) \approx \phi(u_1, u_3, u_4) \approx \phi(u_1, u_2, u_3) \approx \phi(u_1, u_3, u_5)$. It follows that $\phi(u_1, u_4, u_5) \approx \phi(u_1, u_3, u_5)$. However, the fact that Φ is **ACM** requires that $\phi(u_1, u_4, u_5) >> \phi(u_1, u_3, u_5)$. □

The next claim is dual to the preceding one—except for the fact that it uses the weak version of the triangle inequality.

Claim 5. *The function H cannot be **RAM**.*

Proof. Suppose to the contrary that H is **RAM**. Then for every 2-element chain $x < z$, the map $H(x, y, z)$ defined on those y with $x < y < z$ is **RAM**. It follows that

1. $h(u_2, u_3, u_5) - h(u_3, u_4, u_5) = \text{zero}\big(h(u_2, u_4, u_5)\big)$,
2. $h(u_1, u_3, u_5) - h(u_3, u_4, u_5) = \text{zero}\big(h(u_1, u_4, u_5)\big)$,
3. $h(u_1, u_2, u_5) - h(u_2, u_3, u_5) = \text{zero}\big(h(u_1, u_3, u_5)\big)$.

Therefore, $h(u_1, u_2, u_5) \approx h(u_2, u_3, u_5) \approx h(u_3, u_4, u_5) \approx h(u_1, u_3, u_5)$. It follows that $h(u_1, u_2, u_5) \approx h(u_1, u_3, u_5)$. However, the fact that H is **RAM** requires $h(u_1, u_2, u_5) << h(u_1, u_3, u_5)$. □

Next we begin to consider the issue of coordinate domination. The next two claims are again dual.

Claim 6. *If Φ is **NC**, then H is **ACM** and dominated by coordinate 1.*

Proof. Let $s_1 = (1, 2, 2)$ and $s_2 = (2, 1, 1)$. Since $\Phi(s_1) \approx \Phi(s_2)$, and $r(s_1) << r(s_2)$, we know that $H(s_1) << H(s_2)$. From the preceding claim, we know that H cannot be **RAM**. Evidently, it is not **NC**, so it must be **ACM**. Furthermore, it must be dominated by coordinate 1, since $s_1(i) > s_2(i)$ for $i = 2, 3$.

□

Claim 7. *If H is **NC**, then Φ is **RAM** and dominated by coordinate 1.*

Proof. Again, let $s_1 = (1, 2, 2)$ and $s_2 = (2, 1, 1)$. Since $H(s_1) \approx H(s_2)$, and $r(s_1) << r(s_2)$, we know that $\Phi(s_1) << \Phi(s_2)$. We know that Φ cannot be **ACM**. Evidently, it is not **NC**, so it must be **RAM**. Furthermore, it must be dominated by coordinate 1, since $s_1(i) > s_2(i)$ for $i = 2, 3$. □

Here is another useful property.

Claim 8. *If Φ is dominated by coordinate 2, then H is **ACM** and dominated by coordinate 1.*

Proof. Once again, consider $s_1 = (1, 2, 2)$ and $s_2 = (2, 1, 1)$. The inequalities $r(s_1) << r(s_2)$ and $\phi(s_2) > \phi(s_1)$ imply $H(s_1) << H(s_2)$, so H is **ACM** and dominated by coordinate 1. □

The remainder of the argument is by cases which depend on the change patterns of Φ and H. Originally, this would have resulted in $324 = 18^2$ cases, which would have been unbearable even for the most patient of readers. But in view of the results of the claims in this section, we only have 3 cases:

Case 1. Φ is **RAM**; H is **ACM**.
Case 2. Φ is **NC**; H is **ACM**.
Case 3. H is **NC**; Φ is **RAM**.

Moreover, in Case 2, we know that H is dominated by coordinate 1, while in Case 3, we know that Φ is dominated by coordinate 1. Also, following the pattern evidenced in this section, Cases 2 and 3 will be dual.

Since we are arguing by contradiction, we will show that each of the cases is impossible. When this is accomplished, our proof will be complete.

8. Part 4: Case 1 of 3

In this section, we assume Φ is **RAM** and H is **ACM**. We assume without loss of generality that α_1, the coordinate which dominates Φ is either 1 or 2.

Let y be a point with $x(1) < y(1) < z(1)$ and $x < z$. We obtain some estimates on $\phi(x, y, z)$ and $\Delta(x, y, z)$.

First, note that

$$\phi(B, x, T) - \phi(B, x, z) = \text{zero}\big(\phi(B, z, T)\big),$$

so that $\Phi(x) = \phi(B, x, T) \approx \phi(B, x, z)$.

Furthermore, exactly one of the following statements is true:

1. $\phi(B, y, T) - \phi(B, y, z) = \text{zero}\big(\phi(B, z, T)\big)$.
2. $\phi(B, z, T) - \phi(B, y, z) = \text{zero}\big(\phi(B, y, T)\big)$.

The issue as to which of the two statements is true is decided by the order of $y(\alpha_1)$ and $z(\alpha_1)$. Noting that $\phi(B, y, T) = \Phi(y)$ and $\phi(B, z, T) = \Phi(z)$, we can then say that $\phi(B, y, z) \approx \max\{\Phi(y), \Phi(z)\}$.

Now suppose that $x < u < z$ is a chain. We know that $\Phi(z) << \Phi(u)$, so that $\Phi(B, u, z) \approx \Phi(u)$. Since H is uniform and **ACM**, we know that $h(B, x, Z) < h(B, u, z)$. It follows that:

$$h(B, u, z) - h(x, u, z) = \text{zero}\big(h(B, x, z)\big).$$

Thus $h(B, u, z) \approx h(x, u, z)$. Therefore, $\Phi(u) \approx h(B, u, z)/r(u) \approx h(x, u, z)/r(u) \approx \phi(x, u, z)$, i.e., $\Phi(u) \approx \phi(x, u, z)$.

Recall that $G(x) = H(x)\Phi(x)/2$. It follows that $\Delta(x, u, z) \approx G(u)$. The important fact here is that this estimate is *independent* of both x and z.

For the remainder of this case, we will fix notation for the following points in $\mathbf{P}_6$: $x = (1, 1, 1)$, $z = (9, 9, 9)$, $v = (5, 0, 10)$ and $w = (5, 10, 0)$. Note that x and z are just the bottom and top elements of the chain $C = \{u_1 < u_2 < \cdots < u_9\}$.

As outlined in Section 3, we have the following lower bound on $\text{gap}(x, z)$.

$$\text{gap}(x, z) > \Delta(x, C, z) = \sum_{i=1}^{4} \Delta(u_{2i-1}, u_{2i}, u_{2i+1}).$$

Since $\Delta(u_{2i-1}, u_{2i}, u_{2i+1}) \gtrsim G(u_{2i})$, we can write

$$\Delta(x, C, z) \gtrsim G(u_2) + G(u_4) + G(u_6) + G(u_8).$$

We now turn our attention to the problem of finding relatively tight upper bounds on $\mathrm{gap}(x, z)$.

To do this, we consider the points v and w, but we need to consider subcases depending on the coordinate that dominates Φ.

Subcase 1a. Φ is dominated by coordinate 1.

In this subcase, it is straightforward to verify that:

1. $\phi(B, v, z) \approx \Phi(v)$.
2. $\phi(B, w, z) \approx \Phi(w)$.
3. $h(B, v, z) \approx H(v)$.
4. $h(B, w, z) \approx H(w)$.

Using the property that H is **ACM**, we know that exactly one of the following statements is true:

1. $H(x) >> H(v)$.
2. $H(v) >> H(x)$.

Recall that $\phi(B, x, z) \approx \Phi(x)$, so that $H(x) \approx r(x)\Phi(x) \approx r(x)\phi(B, x, z) \approx h(B, x, z)$. Also, $h(B, v, z) \approx r(v)\phi(x, v, z) \approx r(v)\Phi(v) \approx H(v)$. If the first statement listed above is true, then

$$h(B, x, z) - h(x, v, z) = \mathrm{zero}\big(h(B, v, z)\big),$$

and thus $h(B, x, z) \approx h(x, v, z)$. In this case, we see that $\phi(x, v, z) \approx r(x)\Phi(x)/r(v)$.

Now if the second statement holds, then

$$h(B, v, z) - h(x, v, z) = \mathrm{zero}\big(h(B, x, z)\big),$$

and $h(B, v, z) \approx h(x, v, z)$. In this case, we conclude that $\phi(x, v, z) \approx \Phi(v)$. So we may then write:

$$\phi(x, v, z) \approx \max\{r(x)\Phi(x)/r(v), \Phi(v)\}.$$

Applying the same argument to w, we can write:

$$\phi(x, w, z) \approx \max\{r(x)\Phi(x)/r(w), \Phi(w)\}.$$

Therefore,

$$\Delta(x, v, z) \approx \max\{r(x)G(x)/r(v), G(v)\},$$

and

$$\Delta(x, w, z) \approx \max\{r(x)G(x)/r(w), G(w)\}.$$

Now we consider the implications of the (weak) inequality

$$\Delta(x, C, z) \lesssim \min\{\Delta(x, v, z), \Delta(x, w, z)\}.$$

At this point, the argument depends on the coordinate dominating G. Suppose first that G is dominated by coordinate 1. If G is order-preserving, then $\Delta(x, C, z) \geq 2G(v)$, but $G(v) \approx \max\{r(x)G(x)/r(v), G(v)\}$, which is a contradiction.

Now suppose G is order-reversing. Then $\Delta(x, C, z) \gtrsim 2G(w)$ and $\Delta(x, C, z) \gtrsim 2G(v)$, which implies that $r(x)G(x)/r(v) >> G(v)$ and $r(x)G(x)/r(w) >> G(w)$. Thus $H(x) >> H(v)$ and $H(x) >> H(w)$. However, there is no coordinate $i \in \{1, 2, 3\}$ for which $x(i) > v(i)$ and $x(i) > w(i)$. We conclude that G is not dominated by coordinate 1.

Because the definitions of v and w are symmetric between coordinates 2 and 3, we can assume without loss of generality that G is dominated by coordinate 2. If G is order-preserving, then $\max\{r(x)G(x)/r(v), G(v)\} \leq G(x)$, but $\Delta(x, C, z) \gtrsim 2G(x)$.

So G must be order-reversing. Now $\Delta(x, C, z) \gtrsim 2G(w)$, so $r(x)G(x)/r(w) > G(w)$. This implies that $H(x) > H(w)$, so that H must be dominated by coordinate 3. This is impossible, because Φ is dominated by coordinate 1, G by coordinate 2 and $G \approx H\phi/2$. The contradiction completes the proof of this subcase.

Subcase 1b. Φ is dominated by coordinate 2.

In this subcase, we know from Claim 8 that H is dominated by coordinate 1. It follows without loss of generality that we may assume G is dominated by coordinate 1 or 2.

Now it is straightforward to verify that:

1. $\phi(B, v, z) \approx \Phi(v)$.
2. $\phi(B, w, z) \approx \Phi(z)$.
3. $h(B, v, z) \approx H(v)$.
4. $h(B, w, z) \approx r(w)\Phi(z)$.

Since H is **ACM** and dominated by coordinate 1, we know that $H(v) >> H(x)$ and $H(w) >> H(x)$. Therefore, $h(B, v, z) - h(x, v, z) = \text{zero}\big(h(B, x, z)\big)$, so that $h(B, v, z) \approx h(x, v, z)$, $\Phi(v) \approx \phi(x, v, z)$ and $\Delta(x, v, z) \approx G(v)$.

Now $r(w)\Phi(z) >> r(w)\Phi(w) \approx H(w) >> H(x)$, so $h(B, w, z) - h(x, w, z) = \text{zero}\big(h(B, x, z)\big)$. Therefore $\phi(x, w, z) \approx \Phi(z)$ and $\Delta(x, w, z) \approx r(w)G(z)/r(z)$.

We now consider the implications of $\Delta(x, C, z) \lesssim \min\{G(v), r(w)G(z)/r(z)\}$. Regardless of whether G is order-preserving or order-reversing, since G is dominated by coordinate 1 or 2, we see that $\Delta(x, C, z) \gtrsim 2G(v)$. The contradiction completes both the proof of the subcase as well as Case 1.

9. Part 5: Case 2 of 3

In this case, we assume Φ is **NC**. By Claim 6, H is **ACM** and dominated by coordinate 1. Without loss of generality, we may assume that α_2, the coordinate which dominates Θ, is either 1 or 2.

Claim 9. *The function Θ is **ACM**.*

Proof. Suppose to the contrary that Θ is not **ACM**. Let $x < y < z < w$ be a 4-element chain in $\mathbf{P}_5$. Since Φ is **NC**, we know $\phi(x, y, w) \approx \phi(x, z, w)$. Since Θ is not **ACM**, we know $\phi(x, y, w) \lesssim \phi(x, y, z)$, and thus $\phi(x, z, w) \lesssim \phi(x, y, z)$.

Since H is **ACM**, we know that $h(x, z, w) >> h(x, y, w)$, so that $h(x, z, w) - h(y, z, w) = \text{zero}\big(h(x, y, w)\big)$. Thus $h(x, z, w) \approx h(y, z, w)$ so that $\phi(x, z, w) \approx \phi(y, z, w)$. It follows that

$$\phi(x, y, z) \gtrsim \phi(x, z, w) \approx \phi(x, y, w) \approx \phi(y, z, w).$$

In particular, $\phi(x, y, z) \gtrsim \phi(y, z, w)$.

On the other hand, $\phi(x, y, z) < \phi(x, y, w) + \phi(x, z, w) \lesssim 2\phi(y, z, w)$. It follows that $\phi(x, y, z) \approx \phi(y, z, w)$.

Now let $w_1 < w_2 < \cdots < w_6$ be a chain in $\mathbf{P}_5$. It follows that

$$\phi(w_1, w_2, w_3) \approx \phi(w_2, w_3, w_4) \approx \phi(w_3, w_4, w_5) \approx \phi(w_4, w_5, w_6),$$

and therefore

$$\phi(x_1, y_1, z_1) \approx \phi(x_2, y_2, z_2)$$

for any two 3-element chains $x_1 < y_1 < z_1$ and $x_2 < y_2 < z_2$ from $\mathbf{P}_5$.

Now consider the following points in $\mathbf{P}_6$: $x = (1,1,1)$, $v = (2,0,4)$, $u = (3,3,3)$ and $z = (4,4,4)$. Since $x < u < z$ is a 3-element chain, we know that $\phi(x,u,z) \approx \phi(B, B', B'')$ so that $\Delta(x,u,z) \approx r(u)\phi^2(B,B',B'')/2$.

On the other hand, $\phi(x,v,z) < \phi(x,z,T) + \phi(x,v,T)$. Also, $h(B,v,T) - h(x,v,T) = \text{zero}\big(h(B,x,T)\big)$ implies $h(x,v,T) \approx h(B,v,T)$ so that $\phi(x,v,T) \approx \phi(B,v,T)$. Thus $\phi(x,y,z) \lesssim 2\phi(B,B',B'')$.

In turn, this implies that $\Delta(x,v,z) \lesssim 2r(v)\phi^2(B,B',B'')$, and thus $r(u) \leq 2r(v)$. However, $r(u) >> r(v)$. The contradiction completes the proof. □

Claim 10. *For all 4-element chains $x < y < z < w$, $\phi(x,y,w) \approx \phi(y,z,w) \approx \Theta(w)$.*

Proof. Since Φ is **NC**, we know that $\phi(x,y,w) \approx \phi(x,z,w)$. Thus $h(x,y,w) << h(x,z,w)$. This implies that

$$h(x,z,w) - h(y,z,w) = \text{zero}\big(h(x,y,w)\big).$$

Thus $h(x,z,w) \approx h(y,z,w)$ and $\phi(x,z,w) \approx \phi(y,z,w)$. It follows that $\phi(x,y,w) \approx \phi(y,z,w)$.

Observing that this pattern holds for any 4-element chain, we may also conclude that

$$\Theta(w) = \phi(B,B',w) \approx \phi(B',x,w) \approx \phi(x,y,z).$$

□

So for chains, the behavior of Θ depends only on the last coordinate. The next claim extends this to certain triples which are not chains.

Claim 11. *If $x(1) < y(1) < z(1)$, $x(2) < z(2)$ and $y(2) < z(2)$, then $\phi(x,y,z) \approx \Theta(z)$.* □

Proof. Since Θ is **ACM** and dominated by coordinate 1 or 2, we know that $\phi(B,B',z) = \Theta(z) >> \Theta(y) = \phi(B,B',y)$. Thus

$$\phi(B,B',z) - \phi(B,y,z) = \text{zero}\big(\phi(B,B',y)\big),$$

and $\Theta(z) \approx \phi(B,y,z)$.

Similarly, we know that $\Theta(z) \approx \phi(B,x,z)$. Now H is dominated by coordinate 1, so $h(B,y,z) >> h(B,x,z)$. Thus

$$h(B,y,z) - h(x,y,z) = \text{zero}\big(h(B,x,z)\big).$$

It follows that $h(B,y,z) \approx h(x,y,z)$ and thus $\Theta(z) \approx \phi(B,y,z) \approx \phi(x,y,z)$. □

Now we consider the following points in $\mathbf{P}_6$: $x = (1,1,1)$, $v = (2,0,5)$, $u = (3,3,3)$ and $z = (4,4,4)$.

From Claim 11, it follows that $\phi(x,u,z) \approx \Theta(z) \approx \phi(x,v,z)$. Thus $\Delta(x,u,z) \approx r(u)\Theta^2(z)/2$ and $\Delta(x,v,z) \approx r(v)\Theta^2(z)/2$. This requires $r(u) \lesssim r(v)$. Since $u(1) > v(1)$, we know that $r(u) >> r(v)$. The contradiction completes the proof of Case 2.

10. Part 4: Case 3 of 3

In this case, we assume that H is **NC** and Φ is **RAM**. Because this case is dual to Case 2, we outline only the statements necessary to complete the proof. Of course, the key idea here is to focus on the function K.

From Claim 7, we know that Φ is dominated by coordinate 1. So first, we prove the following claim.

Claim 12. *The function K is **RAM**.* □

The reader should note that the proof will hinge on the situation where $h(x, y, z)$ is nearly constant for all 3-element chains $x < y < z$. But this will lead to a contradiction by considering the same four points as in the proof of Claim 9.

Next, the following claims are established.

Claim 13. *For all 4-element chains $x < y < z < w$, $h(x, y, w) \approx h(x, z, w) \approx H(x)$.* □

Claim 14. *If $x(1) < y(1) < z(1)$, $x(2) < z(2)$ and $x(2) < y(2)$, then $h(x, y, z) \approx H(x)$.* □

To complete the argument, we consider the following points: $x = (1, 1, 1)$, $u = (2, 2, 2)$, $w = (3, 0, 5)$ and $z = (4, 4, 4)$. In this case, we conclude that

$$\Delta(x, u, z) \approx h^2(x, u, z)/2r(u) \approx H^2(x)/2r(u),$$

while

$$\Delta(x, w, z) \approx h^2(x, w, z)/2r(w) \approx H^2(x)/2r(w).$$

Thus, we must have $r(w) \lesssim r(u)$. Instead, we know $r(w) >> r(u)$. With this remark, the proof of Case 3 and our principal theorem is complete. □

11. Concluding Remarks

Not surprisingly, our original proof was quite different from the one given here. It was specific to the plane and showed only that there was a finite 3-dimensional poset that was not a circle order. Many details of this approach did not extend to the general problem, and some new techniques were necessary to work around the apparent obstacles. In the end, the proof of the general result is simpler.

It is tempting to conjecture that there is a poset of modest size, say at most 100 points, which is not a sphere order. Certainly, new ideas will be required to prove the existence of such a poset.

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Fachbereich Mathematik und Informatik, Insitut für Informatik, Freie Universität Berlin, Takustr. 9, 14195 Berlin, Germany
E-mail address: felsner@inf.fu-berlin.de

AT&T Labs-Research / C227, 180 Park Avenue, Florham Park, NJ 07932, U.S.A
E-mail address: fish@research.att.com

Department of Mathematics, Arizona State University, Tempe, AZ 85287, U.S.A.
E-mail address: trotter@ASU.edu