LIST OBJECTS AND RECURSIVE ALGORITHMS IN ELEMENTARY TOPOI

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ABSTRACT. The paper generalizes results of [B] by formulating their background in categories with a sufficiently rich internal logic, e. g. elementary topoi, using the well known initial algebra approach. Thus the right setting for program transformations in the sense of [B] is given by embedding them into the generalisation of primitive recursion over the naturals in the sense of [F] to lists. Particularly there is a simple concept of tail recursion, hence an outline on a systematic transformation of naive recursive programs into tail recursive i. e. more efficient iterative forms.

Let $\mathcal{E}$ be an elementary (LAWVERE-TIERNEY-)topos with a natural number object (NNO) $1 \xrightarrow{0} N \xrightarrow{s} N$ (see [F] or [J] for details).

The following standard notations are used:

$A \xrightarrow{A} A$ is the identity and $A \xrightarrow{gf} C$ the composition of $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$. 0 is the initial and 1 the terminal object, $0 \xrightarrow{?} A$ and $A \xrightarrow{!} 1$ the related unique morphisms.

Products are denoted by $\times$ and projections by π (with the appropriate indices), the uniquely determined morphism h for a pair $A \xrightarrow{f} B$ and $A \xrightarrow{g} C$ s. t. $\pi_B h = f$ and $\pi_C h = g$ by $A \xrightarrow{(f,g)} B \times C$. The notations $+$ and σ are reserved for coproducts and injections resp.

A^B is the exponential with $\mathcal{E}(C \times B, A) \cong \mathcal{E}(C, A^B)$, $A^B \times B \xrightarrow{\mathrm{ev}_{AB}} A$ the evaluation morphism. Toggling between morphisms and their exponential transposes by adjointness is denoted by a bar.

$1 \xrightarrow{\mathrm{true}} \Omega$ is the subobject classifier in $\mathcal{E}$.

Not everywhere in this paper the full power of the topos axioms is used — so most results are true in the more general situation of $\mathcal{E}$ just being cartesian closed.

Underlying intuitive ideas are indicated in footnotes, in general by using a sort of functional programming language.

1. Universal Characterisation and Properties of Lists.

We start off with the basic definition and some consequences of it.

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1.1 Definition. *For any object E in $\mathcal{E}$, an object L with a pair of morphisms*[1] $1 \xrightarrow{e} L$ *and* $E \times L \xrightarrow{c} L$ *is called a list object over E, iff this situation is initial in the following sense:*

for all pairs of morphisms[2] $1 \xrightarrow{x} X$ *and* $F \times X \xrightarrow{f} X$ *and all morphisms*[3] $E \xrightarrow{u} F$ *there is a unique* $L \xrightarrow{r} X$ *such that the diagrams*

$$\begin{tikzcd} E \times L \arrow[r, "c"] \arrow[d, "u \times r"] & L \arrow[d, "r"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ F \times X \arrow[r, "f"] & X & 1 \arrow[l, "x"'] \end{tikzcd}$$

commute. (Equivalently, $E \xrightarrow{u} F$ *could have been replaced by* $E \xrightarrow{E} E$*.)*

This situation could of course be expressed in terms of an adjoint situation, although in this paper no particular use will be made of that:

1.1a Alternative description. *Let $\mathcal{E}_E$ be the category of actions of E on objects of $\mathcal{E}$ with objects and morphisms of the type*

$$\begin{tikzcd} E \times X \arrow[r, "E \times f"] \arrow[d, "h"'] & E \times Y \arrow[d, "k"'] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

If the underlying functor $U_E \colon \mathcal{E}_E \to \mathcal{E}$, sending the above morphism to $X \xrightarrow{f} Y$, *has a left adjoint* $\mathcal{E} \xrightarrow{F_E} \mathcal{E}_E$, *then* $E \times L \xrightarrow{c} L$ *is just $F_E(1)$ and* $1 \xrightarrow{e} L$ *is the front adjunction* $1 \xrightarrow{\eta_1} U_E F_E(1)$.

Proof. Simply by describing the adjoint situation $F_E \dashv U_E$ in terms of the corresponding universal morphism problem.

1.2 Corollary. *For every pair* $F \times Y \xrightarrow{h} Y$, $1 \xrightarrow{y} Y$ *and any* $E \times L \xrightarrow{v} F$ *there is a unique* $L \xrightarrow{r} Y$ *s. t. the diagram*

$$\begin{tikzcd} E \times L \arrow[r, "c"] \arrow[d, "(v, r\pi_L)"] & L \arrow[d, "r"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ F \times Y \arrow[r, "h"] & Y & 1 \arrow[l, "y"'] \end{tikzcd}$$

commutes.

Proof. By choosing $X = L \times Y$ and $f = E \times L \times Y \xrightarrow{(c,v) \times Y} L \times F \times Y \xrightarrow{L \times h} L \times Y$ in 1.1 one finds a unique $L \xrightarrow{(l,r)} L \times Y$ s. t.

$$\begin{tikzcd} E \times L \arrow[r, "c"] \arrow[d, "E \times (l,r)"'] & L \arrow[d, "(l,r)"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ E \times L \times Y \arrow[r, "f"] & L \times Y & 1 \arrow[l, "(e,y)"'] \end{tikzcd}$$

[1] lists of elements of type E, the empty list and the construction morphism, `[]` and `:`

[2] "initial states of X" and "actions of F on X"

[3] "changes of base"

commutes. Combining this with

$$\begin{tikzcd} E\times L\times Y \arrow[r, "(c,v)\times Y"] \arrow[d, "\pi_{E\times L}"] & L\times F\times Y \arrow[r, "L\times h"] & L\times Y \arrow[d, "\pi_L"] & 1 \arrow[l, "(e,y)"'] \arrow[d, equal] \\ E\times L \arrow[r, "c"] & L \arrow[r, equal] & L & 1 \arrow[l, "e"'] \end{tikzcd}$$

gives by uniqueness $l = L$, combining it with

$$\begin{tikzcd} E\times L\times Y \arrow[r, "f"] \arrow[d, "v\times Y"] & L\times Y \arrow[d, "\pi_Y"] & 1 \arrow[l, "(e,y)"'] \arrow[d, equal] \\ F\times Y \arrow[r, "h"] & Y & 1 \arrow[l, "e"'] \end{tikzcd}$$

yields the result.

By universality, one then immediately has:

1.3 Lemma. *A list object over E is uniquely determined up to an isomorphism rendering the appropriate diagrams commutative.*

1.4 Example. *The NNO is the list object over* 1.

Proof. Choose $E = 1$, $e = 1 \xrightarrow{0} N$ and the successor map $N \xrightarrow{s} N$ for c, identifying $1 \times N \cong N$.

First, we are going to derive some conclusions of this concept.

1.5 Proposition. *If (E, L, e, c) is a list object, then $E \times L \xrightarrow{c} L \xleftarrow{e} 1$ is a coproduct in $\mathcal{E}$, i. e. there exists an isomorphism[4] $L \xrightarrow{p} 1 + E \times L$ whose inverse is given by $1 + E \times L \xrightarrow{(e,c)} L$*

Proof. Let σ_1 and $\sigma_{E\times L}$ denote the coproduct injections into $1+E\times L$; furthermore let $E \times (1 + E \times L) \xrightarrow{d} 1 + E \times L$ be defined as composition

$$E \times (1 + E \times L) \xrightarrow{E\times(e,c)} E \times L \xrightarrow{\sigma_{E\times L}} 1 + E \times L.$$

By 1.1 there is a unique $L \xrightarrow{p} 1 + E \times L$ such that

$$\begin{tikzcd} E\times L \arrow[r, "c"] \arrow[d, "E\times p"'] & L \arrow[d, "p"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ E\times(1+E\times L) \arrow[r, "d"] & 1+E\times L & 1 \arrow[l, "\sigma_1"'] \end{tikzcd}$$

commutes. We now prove the proposition by showing that the bottom row of this diagram constitutes a list object over E:

[4] a list is empty or consists of head and tail

Let $E \xrightarrow{u} F$ and $F \times X \xrightarrow{f} X \xleftarrow{x} 1$ be given. By taking the unique $L \xrightarrow{r} X$ from 1.1, we consider the diagonal

$$f(u \times r) = rc \colon E \times L \to X.$$

This allows for $g = (x, rc) \colon 1 + E \times L \to X$, such that

$$\begin{array}{ccccc} E \times (1 + E \times L) & \xrightarrow{d} & 1 + E \times L & \xleftarrow{\sigma_1} & 1 \\ {\scriptstyle u \times g}\downarrow & & \downarrow{\scriptstyle g} & & \| \\ F \times X & \xrightarrow{f} & X & \xleftarrow{x} & 1 \end{array}$$

commutes.

Combining the two diagrams we get $gp = L \xrightarrow{r} X$ with regard to the uniqueness property of 1.1. Particularly for $(F, x, f) = (L, e, c)$ we get $(e, c)p = L$; therefore $g\sigma_{E \times L} = g\sigma_{E \times L}(E \times (e, c))(E \times p) = gd(E \times p) = gpc = rc$, which together with $g\sigma_1 = x$ determines g uniquely.

The equation $p(e, c) = 1 + E \times L$ is simply a consequence of the main argument.

Alternatively, one could say that $T(L) = 1 + E \times L \xrightarrow{(e,c)} L$ is a least fixpoint in an appropriate category.

1.6 Lemma. *Let the head and the tail morphism*[5] $L \xrightarrow{\kappa} 1 + E$ *and* $L \xrightarrow{\tau} 1 + L$ *be given by*

$$\kappa = L \xrightarrow{p} 1 + E \times L \xrightarrow{1 + \pi_E} 1 + E \quad \text{and}$$
$$\tau = L \xrightarrow{p} 1 + E \times L \xrightarrow{1 + \pi_L} 1 + L.$$

Then the following diagrams commute:

$$\begin{array}{ccccc} E & \xrightarrow{\sigma_E} & 1 + E & \xleftarrow{\sigma_1} & 1 \\ {\scriptstyle \pi_E}\uparrow & & {\scriptstyle \kappa}\uparrow & & \| \\ E \times L & \xrightarrow{c} & L & \xleftarrow{e} & 1 \\ {\scriptstyle \pi_L}\downarrow & & {\scriptstyle \tau}\downarrow & & \| \\ L & \xrightarrow{\sigma_L} & 1 + L & \xleftarrow{\sigma_1} & 1 \end{array}$$

Proof. Immediate by $pe = \sigma_1$ and $pc = \sigma_{E \times L}$.

1.7 Lemma. *For any list object* $E \times L \xrightarrow{c} L \xleftarrow{e} 1$, *the construction morphism* c *and the projection* $E \times L \xrightarrow{\pi_L} L$ *define a coequalizer diagram* $E \times L \rightrightarrows L \xrightarrow{!} 1$.

Proof. Let $L \xrightarrow{t} X$ be given such that $tc = t\pi_L$. Then the problem 1.1 for $E = F$, $f = F \times X \xrightarrow{\pi_X} X$ and $x = 1 \xrightarrow{te} X$ is solved by t and $te!$, which therefore are equal. Since $!_L$ is a retraction, the factorization of t through 1 is unique.

[5] `hd(x:xs) = x, tl(x:xs) = xs`

1.8 Lemma. *Let the list length morphism*[6] *be the unique* $L \xrightarrow{\nu} N$*, for which*

$$\begin{tikzcd} E\times L \arrow[r, "c"] \arrow[d, "!\times\nu\cong\nu\pi_L"'] & L \arrow[d, "\nu"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ 1\times N\cong N \arrow[r, "s"] & N & 1 \arrow[l, "0"'] \end{tikzcd}$$

commutes. Then both these squares are pullbacks, i. e. $(s\colon \mathbb{N}\to\mathbb{N}, 0\colon 1\to\mathbb{N})$ *is a "list classifier".*

Proof. The proof rests on some typical elementary topos arguments relying on the existence of the adjunction $\nu^* \dashv \Pi_\nu : \mathcal{E}/L \to \mathcal{E}/N$.

The left sqare and the outer diagram in

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E\times L \arrow[r, "\nu\pi_L"] \arrow[d, "c"'] & N \arrow[d, "s"] \\ 1 \arrow[r, "e"] & L \arrow[r, "\nu"] & N \end{tikzcd}$$

are coproduct diagrams, hence pushouts; so the right sqare is a pushout, which is bound to be a pullback, because by 1.5 $E\times L \xrightarrow{c} L$ is monic (in a topos the pushout of a mono gives a pullback).

Let

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "!"'] & L \arrow[d, "\nu"] \\ 1 \arrow[r, "0"] & N \end{tikzcd}$$

be a pullback and $1 \xrightarrow{a} A \xrightarrow{h} L$ the unique factorization of $1 \xrightarrow{e} L$. By pulling back the coproduct $1 \xrightarrow{0} N \xleftarrow{s} N$ along $L \xrightarrow{\nu} N$ we get the coproduct diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E\times L \arrow[d, "c"] \\ A \arrow[r, "h"] & L. \end{tikzcd}$$

We consider the morphism $1+E\times L \xrightarrow{a+E\times L} A+E\times L$; composing it with the resulting iso $A+E\times L \xrightarrow{(h,c)} L$ yields $1+E\times L \xrightarrow{(e,c)} L$, from which by 1.5 follows that it is an isomorphism itself.

Since in a topos all diagrams of the form

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sigma_A"'] & B \arrow[d, "\sigma_B"] \\ A+C \arrow[r, "f+g"] & B+D \end{tikzcd}$$

are pullbacks and pullbacks of epimorphisms are again epic, $1 \xrightarrow{a} A$ turns out to be an isomorphism.

[6] `#[] = 0, #(x:xs) = 1+#xs`

1.9 Definition. *Let the singleton morphism*[7] *be* $E \xrightarrow{\eta} L = E \xrightarrow{(E,e!)} E \times L \xrightarrow{c} L$. *Then the diagrams*

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\sigma_E"'] & E \arrow[r, "e!"] \arrow[d, "\eta"] & L \arrow[d, "\sigma_L"] \\ 1+E & L \arrow[l, "\kappa"'] \arrow[r, "\tau"] & 1+L \end{tikzcd}$$

are commutative and

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[d, "!"'] & L \arrow[d, "\nu"] \\ 1 \arrow[r, "1=s0"] & N \end{tikzcd}$$

is a pullback.[8]

Proof. The first claim follows immediately from the definitions, for the second compose the pullbacks 1.9 with the trivial one $E \xrightarrow{(E,e!)} E \times L \xrightarrow{\pi_L} L = E \xrightarrow{!} 1 \xrightarrow{e} L$.

2. Algorithms on Lists.

We now give some examples, how to carve out several recursively defined algorithms, just by using the approach of the previous section. Throughout this chapter, let $(E \times L \xrightarrow{c} L \xleftarrow{e} 1)$ generally denote a list object over E.

First of all, we have a general form of primitive recursion [F, Proposition 5.22] for list objects:

2.1 Proposition. *Let* $E \times L \times A \xrightarrow{u} F$, $A \xrightarrow{g} B$ *and* $F \times B \xrightarrow{h} B$ *be given. Then there is a unique* $L \times A \xrightarrow{f} B$ *such that the diagrams*

$$\begin{tikzcd} E \times L \times A \arrow[r, "c \times A"] \arrow[d, "(u, f\pi_{LA})"'] & L \times A \arrow[d, "f"] & A \arrow[l, "(e!, A)"'] \arrow[d, equal] \\ F \times B \arrow[r, "h"] & B & A \arrow[l, "g"'] \end{tikzcd}$$

commute.

Proof. The situation is shifted by adjointness to

$$\begin{tikzcd} E \times L \arrow[r, "c"] \arrow[d, "(\overline{u}, \overline{f}\pi_L)"'] & L \arrow[d, "\overline{f}"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ F^A \times B^A \cong (F \times B)^A \arrow[r, "h^A"] & B^A & 1, \arrow[l, "\overline{g}"'] \end{tikzcd}$$

because the transpose of the left bottom row is $h(\mathrm{ev}_{AF}\, \pi_{EA}, \mathrm{ev}_{AB}\, \pi_{LA})$. The unique existence of $\overline{F}$ follows from 1.2, which proves the proposition.

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[7] `[]x = [x]`

[8] elements can be considered as lists of length 1

2.1a Remark. *With respect to 1.1a, $E \times L \times A \xrightarrow{c\times A} L \times A$ is just the value of the left adjoint $F_E \dashv \mathcal{E}_E \xrightarrow{U_E} \mathcal{E}$ on A, $A \xrightarrow{(e!,A)} L \times A$ being the front adjunction.*

2.2 Example. *Let the element morphism*[9] *$L \times E \xrightarrow{\varepsilon} E$ be given by 2.1 for $A = E$, $F = B = \Omega$, $u = \delta_E$ (characterizing the diagonal), $g = \mathrm{false}_E$ and $h = \vee$, i.e. ε is uniquely determined by the commutativity of the diagrams*

$$\begin{tikzcd}
E \times L \times E \arrow[r, "c\times E"] \arrow[d, "(\delta_E\pi_{EE},\varepsilon\pi_{LE})"'] & L \times E \arrow[d, "\varepsilon"] & E \arrow[l, "(e!,E)"'] \arrow[d, "!"] \\
\Omega \times \Omega \arrow[r, "\vee"] & \Omega & 1. \arrow[l, "\mathrm{false}"']
\end{tikzcd}$$

Then one has immediately the equation $E \xrightarrow{(\eta,E)} L \times E \xrightarrow{\varepsilon} \Omega = E \xrightarrow{\mathrm{true}_E} \Omega$.

2.3 Proposition. *Let the concatenation morphism*[10] *be the unique $L \times L \xrightarrow{\gamma} L$ such that*

$$\begin{tikzcd}
E \times L \times L \arrow[r, "c\times L"] \arrow[d, "E\times\gamma"'] & L \times L \arrow[d, "\gamma"] & L \arrow[l, "(e!,L)"'] \arrow[d, equal] \\
E \times L \arrow[r, "c"] & L & L \arrow[l, "L"']
\end{tikzcd}$$

commutes. Then $L \times L \xrightarrow{\gamma} L$ is unitary and associative, i. e. the diagrams

$$\begin{tikzcd}
L \arrow[r, "(e!,L)"] \arrow[d, equal] & L \times L \arrow[d, "\gamma"] & L \arrow[l, "(L,e!)"'] \arrow[d, equal] \\
L \arrow[r, "L"] & L & L \arrow[l, "L"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
L \times L \times L \arrow[r, "L\times\gamma"] \arrow[d, "\gamma\times L"'] & L \times L \arrow[d, "\gamma"] \\
L \times L \arrow[r, "\gamma"] & L
\end{tikzcd}$$

commute. Particularly,[11] *$E \times L \xrightarrow{c} L = E \times L \xrightarrow{\eta\times L} L \times L \xrightarrow{\gamma} L$.*

Proof. The left side of the first diagram commutes by definition of γ, the right side by commutativity of

$$\begin{tikzcd}
E \times L \arrow[r, "c"] \arrow[d, "E\times(L,e!)"'] & L \arrow[d, "(L,e!)"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\
E \times L \times L \arrow[r, "c\times L"] \arrow[d, "E\times\gamma"'] & L \times L \arrow[d, "\gamma"] & 1 \arrow[l, "(e,e)"'] \arrow[d, equal] \\
E \times L \arrow[r, "c"] & L & 1, \arrow[l, "e"']
\end{tikzcd}$$

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[9] `element[]x = False, element(x:xs)x = True, element(x:xs)y = element xs y`

[10] `[]++ys = ys, (x:xs)++ys = x:(xs++ys)`

[11] `x:xs = [x]++xs`

and uniqueness. Futhermore, because the diagram

$$\begin{array}{ccccc}
E\times L\times L\times L & \xrightarrow{c\times L\times L} & L\times L\times L & \xleftarrow{(e!,L\times L)} & L\times L \\
{\scriptstyle E\times L\times\gamma}\downarrow & & \downarrow{\scriptstyle L\times\gamma} & & \Vert \\
E\times L\times L & \xrightarrow{c\times L} & L\times L & \xleftarrow{(e!,\gamma)} & L\times L \\
{\scriptstyle E\times\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} & & \Vert \\
E\times L & \xrightarrow{c} & L & \xleftarrow{\gamma} & L\times L
\end{array}$$

and its counterpiece with $\gamma\times L$ instead of $L\times\gamma$ commute, associativity holds. The last equation is shown by composing $E\times(e!,L)$ with the defining diagram of γ.

2.4 Remark. *L is the free monoid over E with multiplication $L\times L\xrightarrow{\gamma}$ and neutral element $1\xrightarrow{e}L$* [J, Theorem 6.41].

Proof. For any monoid $M\times M\xrightarrow{m}M\xleftarrow{u}1$ and any $E\xrightarrow{h}M$ let $L\xrightarrow{\overline{h}}M$ be the unique morphism such that

$$\begin{array}{ccccc}
E\times L & \xrightarrow{c} & L & \xleftarrow{e} & 1 \\
{\scriptstyle h\times\overline{h}}\downarrow & & {\scriptstyle\overline{h}}\downarrow & & \Vert \\
M\times M & \xrightarrow{m} & M & \xleftarrow{u} & 1.
\end{array}$$

Then $\overline{h}$ is a monoid homomorphism, because

$$\begin{array}{ccccc}
E\times L\times L & \xrightarrow{c\times L} & L\times L & \xleftarrow{(e!,L)} & L \\
{\scriptstyle h\times\xi}\downarrow & & {\scriptstyle\xi}\downarrow & & \Vert \\
M\times M & \xrightarrow{m} & M\times M & \xleftarrow{\overline{h}} & L
\end{array}$$

is made commutative by $\xi=\overline{h}\gamma$ as well as by $\xi=m(\overline{h}\times\overline{h})$, which follows from the definitions of γ and $\overline{h}$ and the monoid structure on M.

Furthermore, there is the unique factorization

$$\begin{array}{ccc}
E & \xrightarrow{h} & M \\
{\scriptstyle\eta}\downarrow & & \Vert \\
L & \xrightarrow{\overline{h}} & M,
\end{array}$$

given by the composition of $E\xrightarrow{(E,e!)}E\times L$ with the defining diagram of $\overline{h}$.

2.5 Corollary. *In the special case of 1.3, γ is just the addition of natural numbers $N\times N\xrightarrow{+}N$ and $L\xrightarrow{\nu}N$ is a monoid homomorphism.*

2.6 Proposition. *Let the append morphism*[12] *be the unique* $L \times E \xrightarrow{\alpha} L$*, for which*

$$\begin{tikzcd} E \times L \times E \arrow[r, "c \times E"] \arrow[d, "E \times \alpha"'] & L \times E \arrow[d, "\alpha"] & E \arrow[l, "(e!, E)"'] \arrow[d, equal] \\ E \times L \arrow[r, "c"] & L & E \arrow[l, "\eta"'] \end{tikzcd}$$

commutes, where $E \xrightarrow{\eta} L$ *is the singleton morphism.*

Then[13] $\alpha = L \times E \xrightarrow{L \times \eta} L \times L \xrightarrow{\gamma} L$ *and the following diagrams*[14,15] *commute:*

$$\begin{tikzcd} L \times L \times E \arrow[r, "\gamma \times E"] \arrow[d, "L \times \alpha"'] & L \times E \arrow[d, "\alpha"] \\ L \times L \arrow[r, "\gamma"] & L \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \times E \times L \arrow[r, "L \times c"] \arrow[d, "\alpha \times L"'] & L \times L \arrow[d, "\gamma"] \\ L \times L \arrow[r, "\gamma"] & L. \end{tikzcd}$$

Proof. The definition of γ and η implies the first equation.

The commutativity of the diagrams is a consequence of that, using the defining properties of γ.

2.7 Lemma. *Let the last element morphism*[16] *be the unique* $L \xrightarrow{\lambda} 1+E$*, for which*

$$\begin{tikzcd} E \times L \arrow[rrr, "c"] \arrow[d, "E \times \lambda"'] & & & L \arrow[d, "\lambda"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ E \times (1+E) \arrow[r, "\cong"] & E + E \times E \arrow[r, "(E, \pi_2)"] & E \arrow[r, "\sigma_E"] & 1+E & 1 \arrow[l, "\sigma_1"'] \end{tikzcd}$$

commutes. Then the diagrams

$$\begin{tikzcd} L \times E \arrow[r, "\alpha"] \arrow[d, "\pi_E"'] & L \arrow[d, "\lambda"] & E \arrow[l, "\eta"'] \arrow[d, equal] \\ E \arrow[r, "\sigma_E"] & 1+E & E \arrow[l, "\sigma_E"'] \end{tikzcd}$$

commute.

Proof. Because the diagram

$$\begin{tikzcd} E \times L \times E \arrow[rrr, "c"] \arrow[d, "E \times \xi"'] & & & L \times E \arrow[d, "\xi"] & E \arrow[l, "(e!, E)"'] \arrow[d, equal] \\ E \times (1+E) \arrow[r, "\cong"] & E + E \times E \arrow[r, "(E, \pi_2)"] & E \arrow[r, "\sigma_E"] & 1+E & E \arrow[l, "\sigma_E"'] \end{tikzcd}$$

commutes for $\xi = L \times E \xrightarrow{\alpha} L \xrightarrow{\lambda} 1+E$ as well as for $\xi = L \times E \xrightarrow{\sigma_E} E \xrightarrow{\pi_E} 1+E$, the first diagram commutes by uniqueness. Composing $(E, e!)$ with the defining diagram of λ gives immediately the commutativity of the second diagram.

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[12] `append[]y = [y], append(x:xs)y = x:append xs y`

[13] `append xs y = xs++[y]`

[14] `xs++(append ys y) = append(xs++ys)y`

[15] `(append xs y)++ys = xs++(y:ys)`

[16] `last[x] = x, last(x:xs) = last xs`

2.8 Lemma. *Let the (simple) list reverse morphism*[17] *be the unique* $L \xrightarrow{\varrho} L$*, for which*

$$\begin{tikzcd} E\times L \arrow[rr, "c"] \arrow[d, "E\times\varrho"'] & & L \arrow[d, "\varrho"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\ E\times L \arrow[r, "\cong"] & L\times E \arrow[r, "\alpha"] & L & 1 \arrow[l, "e"'] \end{tikzcd}$$

commutes. Then the diagrams[18]

$$\begin{tikzcd} L\times L \arrow[r, "\cong"] \arrow[d, "\varrho\times\varrho"'] & L\times L \arrow[r, "\gamma"] & L \arrow[d, "\varrho"] \\ L\times L \arrow[rr, "\gamma"] & & L \end{tikzcd}$$

and particularly[19]

$$\begin{tikzcd} E\times L \arrow[r, "\cong"] \arrow[d, "E\times\varrho"'] & L\times E \arrow[r, "\alpha"] & L \arrow[d, "\varrho"] & E \arrow[l, "\eta"'] \arrow[d, equal] \\ E\times L \arrow[rr, "c"] & & L & E \arrow[l, "\eta"'] \end{tikzcd}$$

are commutative, where $\cong$ *denote the swap morphisms* (π_2, π_1).

Proof. Composing the defining diagrams of γ with the defining diagram of ϱ shows, that the diagram

$$\begin{tikzcd} E\times L\times L \arrow[rr, "c"] \arrow[d, "E\times\xi"'] & & L\times L \arrow[d, "\xi"'] & L \arrow[l, "(e!{,}L)"'] \arrow[d, equal] \\ E\times L \arrow[r, "\cong"] & L\times E \arrow[r, "\alpha"] & L & L \arrow[l, "\varrho"'] \end{tikzcd}$$

is commutative for $\xi = L\times L \xrightarrow{\gamma} L \xrightarrow{\varrho} L$.

Now, the same diagram commutes with $\xi = L\times L \xrightarrow{\cong} L\times L \xrightarrow{\varrho\times\varrho} L\times L \xrightarrow{\gamma} L$, which can be seen by composing the defining diagram of ϱ with 2.6 and some canonical factor commuting isomorphisms.

Hence, by uniqueness of ξ, the first result is proven.

The second is a special case of that by using $\alpha = L\times E \xrightarrow{L\times\eta} L\times L \xrightarrow{\gamma} L$; the third is immediate from the definition of ϱ and η.

2.9 Corollary. $L \xrightarrow{\varrho} L$ *is idempotent, i. e.* $\varrho^2 = L$.

Proof. By Composition of the defining diagram of ϱ with the second result of 2.8.

[17] `rev[] = [], rev(x:xs) = append(rev xs)x`

[18] `rev(xs++ys) = (rev ys)++(rev xs)`

[19] `rev(append xs x) = x:rev xs`

2.10 Corollary. *The following equations*[20] *hold:*

$$L \xrightarrow{\varrho} L \xrightarrow{\lambda} 1+E = L \xrightarrow{\kappa} 1+E \quad \text{and}$$
$$L \xrightarrow{\varrho} L \xrightarrow{\kappa} 1+E = L \xrightarrow{\lambda} 1+E.$$

Proof. By definition of ϱ and 2.7 one has

$$\begin{array}{ccccccc}
E\times L & & & \xrightarrow{c} & L & \xleftarrow{e} & 1 \\
{\scriptstyle E\times\varrho}\downarrow & & & & \downarrow{\scriptstyle \varrho} & & \| \\
E\times L & \xrightarrow{\cong} & L\times E & \xrightarrow{\alpha} & L & \xleftarrow{e} & 1 \\
 & & {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \lambda} & & \| \\
 & & E & \xrightarrow{\sigma_E} & 1+E & \xleftarrow{\sigma_1} & 1,
\end{array}$$

hence $\lambda\varrho(e,c) = 1+\pi_E$. By 1.5 and 1.6 we therefore get $\lambda\varrho = (1+\pi_E)p = \kappa$.

The second equation is a consequence of that, using the preceding corollary.

3. Tail recursion.

3.1 Lemma. *Let $E\times L\times A \xrightarrow{u} A$ and $A \xrightarrow{g} B$ be given. Then there is a unique morphism $L\times A \xrightarrow{f} B$, such that the diagram*

$$\begin{array}{ccccc}
E\times L\times A & \xrightarrow{c\times A} & L\times A & \xleftarrow{(e!,A)} & A \\
{\scriptstyle (\pi_L,u)}\downarrow & & \downarrow{\scriptstyle f} & & \| \\
L\times A & \xrightarrow{f} & B & \xleftarrow{g} & A
\end{array}$$

commutes.

Proof. Consider the internal composition morphism $A^A\times B^A \xrightarrow{d} B^A$, given as exponential transpose by

$$\begin{array}{ccc}
A^A\times B^A\times A & \xrightarrow{d\times A} & B^A\times A \\
{\scriptstyle (\pi_2,\mathrm{ev}_{AA}\,\pi_{13})}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{AB}} \\
B^A\times A & \xrightarrow{\mathrm{ev}_{AB}} & B.
\end{array}$$

Then by 1.1, there is a unique $L \xrightarrow{\overline{f}} B^A$ such that

$$\begin{array}{ccccc}
E\times L & \xrightarrow{c} & L & \xleftarrow{e} & 1 \\
{\scriptstyle (\overline{u}\pi_{E\times L},\overline{f}\pi_L)}\downarrow & & \downarrow{\scriptstyle \overline{f}} & & \| \\
A^A\times B^A & \xrightarrow{d} & B^A & \xleftarrow{\overline{g}} & 1
\end{array}$$

commutes. Crossing this diagram with A and combining it with the above yields the result, since $(\pi_2,\mathrm{ev}_{AA}\,\pi_{13})(\overline{u}\pi_{E\times L},\overline{f}\pi_L,\pi_A) = (\overline{f}\pi_L,u)\times A$.

[20] `last(rev(x:xs)) = x, hd(rev xs) = last xs`

3.2 Corollary. *Let $L \times L \xrightarrow{\varrho'} L$ be the unique morphism, given by the preceding lemma for $A = B = L$, $u = c\pi_{13}$ and $g = L$, s. t. the following diagram commutes:*

$$\begin{tikzcd}
E \times L \times L \arrow[r, "c \times L"] \arrow[d, "(\pi_2, c\pi_{13})"'] & L \times L \arrow[d, "\varrho'"] & L \arrow[l, "(e!, L)"'] \arrow[d, equal] \\
L \times L \arrow[r, "\varrho'"] & L & L. \arrow[l, "L"']
\end{tikzcd}$$

Then

$$\begin{tikzcd}
L \times L \arrow[r, "\varrho \times L"] \arrow[d, equal] & L \times L \arrow[d, "\gamma"'] \\
L \times L \arrow[r, "\varrho'"] & L
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
L \arrow[r, "(L, e!)"] \arrow[d, equal] & L \times L \arrow[d, "\varrho'"'] \\
L \arrow[r, "\varrho"] & L
\end{tikzcd}$$

commute, i. e. in particular, the (simple) reverse morphism ϱ and the fast reverse morphism[21] *$L \xrightarrow{(L,e!)} L \times L \xrightarrow{\varrho'} L$ coincide.*

Proof. $\gamma(\varrho \times L)$ defines ϱ', because by definition of ϱ and 2.6

$$\begin{tikzcd}
E \times L \times L \arrow[r, equal] \arrow[dd, "\cong \times L"'] & E \times L \times L \arrow[rr, "c \times L"] \arrow[d, "E \times \varrho \times L"'] & & L \times L \arrow[d, "\varrho \times L"'] & L \arrow[l, "(e!, L)"'] \arrow[d, equal] \\
& E \times L \times L \arrow[r, "\cong \times L"] & L \times E \times L \arrow[r, "\alpha \times L"] \arrow[d, "L \times c"'] & L \times L \arrow[d, "\gamma"'] & L \arrow[l, "(0, L)"'] \arrow[d, equal] \\
L \times E \times L \arrow[r, "L \times c"] & L \times L \arrow[r, "\varrho \times L"] & L \times L \arrow[r, "\gamma"] & L & L, \arrow[l, "L"']
\end{tikzcd}$$

commutes. Composing the first diagram with $L \xrightarrow{(L,e!)} L \times L$ renders the second one commutative, by neutrality of γ.

3.3 Corollary. *Let $L \times N \xrightarrow{\nu'} N$ be the unique morphism, s. t.*

$$\begin{tikzcd}
E \times L \times N \arrow[r, "c \times N"] \arrow[d, "\pi_L \times s"'] & L \times N \arrow[d, "\nu'"] & N \arrow[l, "(e!, N)"'] \arrow[d, equal] \\
L \times N \arrow[r, "\nu'"] & N & N \arrow[l, "N"']
\end{tikzcd}$$

commutes. Then the following diagrams are commutative:

$$\begin{tikzcd}
L \times N \arrow[r, "\nu \times N"] \arrow[d, equal] & N \times N \arrow[d, "+"] \\
L \times N \arrow[r, "\nu'"] & N
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
L \arrow[r, "(L, 0!)"] \arrow[d, equal] & L \times N \arrow[d, "\nu'"] \\
L \arrow[r, "\nu"] & L.
\end{tikzcd}$$

[21] if `rev'xs = r'xs[] where r'[]ys = ys, r'(x:xs)ys = r'xs(x:ys)`, then `rev' = rev`

Proof. The top left diagram commutes by definition of ν, together with the commutative rest

$$\begin{tikzcd}
E\times L\times N \arrow[rr, "c\times N"] \arrow[d, "\pi_L\times N"'] & & L\times N \arrow[d, "\nu\times N"] & N \arrow[l, "(e,N)"'] \arrow[d, equal] \\
L\times N \arrow[r, "\nu\times N"] \arrow[d, "L\times s"'] & N\times N \arrow[r, "s\times N"] \arrow[d, "N\times s"'] & N\times N \arrow[d, "+"] & N \arrow[l, "(0,N)"'] \arrow[d, equal] \\
L\times N \arrow[r, "\nu\times N"] & N\times N \arrow[r, "+"] & N & N \arrow[l, "N"']
\end{tikzcd}$$

which shows by uniqueness that $\nu' = L\times N \xrightarrow{\nu\times N} N\times N \xrightarrow{+} N$. From that and the equation $+(N,0!) = N$ it follows, that the list length morphism ν is identical with the fast list length morphism,[22] given as composition $L \xrightarrow{(L,0!)} L\times N \xrightarrow{\nu'} L$.

3.4 Lemma. *Let the fold left morphism*[23] *be the unique* $L\times A^{E\times A}\times A \xrightarrow{\varphi} A$, *such that the following diagram is commutative:*

$$\begin{tikzcd}
E\times L\times A^{A\times E}\times A \arrow[r, "c\times A^{A\times E}\times A"] \arrow[d, "(\pi_{23}{,}\mathrm{ev}_{A\times E,A}\,\pi_{341})"] & L\times A^{E\times A}\times A \arrow[d, "\varphi"] & A^{A\times E}\times A \arrow[l, "(e!{,}A^{E\times A}\times A)"'] \arrow[d, equal] \\
L\times A^{A\times E}\times A \arrow[r, "\varphi"] & A & A^{A\times E}\times A. \arrow[l, "\pi_A"']
\end{tikzcd}$$

Then $\nu' = L\times N \xrightarrow{(\pi_L,\overline{s\pi_N},\pi_N)} L\times N^{N\times E}\times N \xrightarrow{\varphi} N$.

Furthermore, if L is the list object over N and $L \xrightarrow{\sigma} N$ is the sum morphism,[24] *defined by*

$$\begin{tikzcd}
N\times L \arrow[r, "c\times L"] \arrow[d, "N\times\sigma"'] & L \arrow[d, "\sigma"'] & 1 \arrow[l, "e"'] \arrow[d, equal] \\
N\times N \arrow[r, "+"] & N & 1, \arrow[l, "0"']
\end{tikzcd}$$

then $\sigma = L \xrightarrow{(L,\overline{+}!,0)} L\times N^{N\times N}\times N \xrightarrow{\varphi} N$.

Proof. Straightforward diagram chasing.

3.5 Lemma on tail recursion over natural numbers. *Let* $(N \xrightarrow{s} N, 1 \xrightarrow{0} N)$ *be the NNO. Let* $A \xrightarrow{u} A$ *and* $A \xrightarrow{g} B$ *be given. Then there is a unique morphism* $N\times A \xrightarrow{f} B$, *such that the diagram*

$$\begin{tikzcd}
N\times A \arrow[r, "s\times A"] \arrow[d, "(\pi_N{,}u\pi_A)"'] & N\times A \arrow[d, "f"] & A \arrow[l, "(0!{,}A)"'] \arrow[d, equal] \\
N\times A \arrow[r, "f"] & B & A \arrow[l, "g"']
\end{tikzcd}$$

commutes.

Proof. The situation is a special case of 3.1, using 1.4.

[22] if `length xs = l xs 0 where l(x:xs)n = l xs sn`, then `length = #`

[23] `foldl u a [] = a, foldl u a(x:xs) = foldl u(u a x)xs`

[24] `sum[] = 0, sum(n:ns) = n+sum ns`

3.6 Example. *Let the replicate morphism be the unique $N \times E \times L \xrightarrow{\vartheta'} L$ such that*

$$\begin{tikzcd}
N \times E \times L \arrow[r, "s \times E \times L"] \arrow[d, "(\pi_N, c\pi_{EL}, \pi_L)"'] & N \times E \times L \arrow[d, "\vartheta'"] & E \times L \arrow[l, "(0!, E \times L)"'] \arrow[d, equal] \\
N \times E \times L \arrow[r, "\vartheta'"] & L & E \times L \arrow[l, "\pi_L"']
\end{tikzcd}$$

and $N \times E \xrightarrow{\vartheta} L = N \times E \xrightarrow{N \times E, e!)} N \times E \times L \xrightarrow{\vartheta'} L$, where $N \times E \xrightarrow{\vartheta} L$ is the unique morphism[25] *with $\vartheta(s \times E) = c(\pi_E, \vartheta)$ and $\vartheta(0!, E) = e!$ by 1.4.*[26]

Proof. Straightforward.

4. Appendix: Standard abstract data types.

4.1 Example: Stacks. *A stack S on E is given by the operations $1 \xrightarrow{new} S$, $S \times E \xrightarrow{push} S$, $S \xrightarrow{top} 1 + E$ and $S \xrightarrow{pop} 1 + S$ with the standard equations describing stacks.*

Then, if $\mathcal{E}$ has list objects, there are enough stacks.

Proof. For a stack over E choose the list object L over E, and set $new = 1 \xrightarrow{e} L$, $push = E \times L \xrightarrow{c} L$, $top = L \xrightarrow{\kappa} 1 + E$ and $pop = L \xrightarrow{\tau} 1 + L$. The stack equations are given by 1.6.

4.2 Example: Queues. *A queue Q on E is given by the operations $1 \xrightarrow{new} Q$, $Q \times E \xrightarrow{add} Q$, $Q \xrightarrow{first} 1 + E$, $Q \xrightarrow{last} 1 + E$, $Q \xrightarrow{dequeue} 1 + Q$ and $Q \times Q \xrightarrow{merge} Q$ with some wellknown equations.*

If $\mathcal{E}$ has list objects, then there are enough queues.

Proof. Let Q be a list object L over E and $new = 1 \xrightarrow{e} L$, $add = L \times E \xrightarrow{\alpha} L$, $first = L \xrightarrow{\kappa} 1 + E$, $last = L \xrightarrow{\lambda} 1 + L$, $dequeue = L \xrightarrow{\tau} 1 + L$, $merge = L \times L \xrightarrow{\gamma} L$. The queue equations are given in 1.6, 2.6, 2.7 by means of the commutative diagram

$$\begin{tikzcd}
L \times L \arrow[r, "\gamma"] \arrow[d, "\tau \times L"'] & L \arrow[d, "\tau"'] \\
(1 + L) \times L \cong L + L \times L \arrow[r, "(\tau, \sigma_L \gamma)"] & 1 + L.
\end{tikzcd}$$

4.3 Definition. *For any object E in $\mathcal{E}$, an object T with a pair of morphisms*[27] *$1 \xrightarrow{e} T$ and $T \times E \times T \xrightarrow{c} T$ is called a tree object over E, iff this situation is initial in the following sense: for all pairs of morphisms $1 \xrightarrow{x} X$ and $X \times F \times X \xrightarrow{f} X$ and all morphisms $E \xrightarrow{u} F$ there is a unique $T \xrightarrow{r} X$ such that the diagrams*

$$\begin{tikzcd}
T \times E \times T \arrow[r, "c"] \arrow[d, "r \times u \times r"] & T \arrow[d, "r"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\
X \times F \times X \arrow[r, "f"] & X & 1 \arrow[l, "x"']
\end{tikzcd}$$

commute.

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[25] `rep 0 x = [], rep(n+1)x = x:rep n x`

[26] if `r n x = r'n x [] where r'0 x xs = xs, r'(n+1)x xs = r'n x(x:xs)`, then `r = rep`

[27] the constructors s. t. `Tree* = Empty|Node(Tree*)*(Tree*)`

4.4 Lemma. *$T \xrightarrow{(e,c)} 1 + T \times E \times T$ is an isomorphism.*

Proof. By the same technique as in the proof of 1.5.

4.5 Lemma. *Let the tree size morphism[28] be the unique $L \xrightarrow{\zeta} N$, such that*

$$\begin{tikzcd}
T \times E \times T \arrow[rrr, "c"] \arrow[d, "\zeta \times E \times \zeta"'] & & & T \arrow[d, "\zeta"] & 1 \arrow[l, "e"'] \arrow[d, equal] \\
N \times 1 \times N \arrow[r, "\pi_{NN}"] & N \times N \arrow[r, "+"] & N \arrow[r, "s"] & N & 1 \arrow[l, "0"']
\end{tikzcd}$$

commutes. Then these diagrams are pullbacks and $\zeta\varphi = \nu$, where ν is the list length and $\varphi\colon T \to L$ is the flatten morphism,[29] given by

$$\begin{tikzcd}
T \times E \times T \arrow[r, "c"] \arrow[d, "\varphi \times E \times \varphi"'] & T \arrow[d, "\varphi"'] & 1 \arrow[l, "e"'] \arrow[d, equal] \\
L \times E \times L \arrow[r, "\gamma(L \times c)"] & L & 1. \arrow[l, "e"']
\end{tikzcd}$$

Proof. The first result is shown analogously to 1.8, the second is given by the combination of the definitions of ζ, φ and ν with 2.3 and 2.5.

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[28] `size Empty = 0, size(Node l x r) = size l+size r+1`

[29] `flatten Empty = [], flatten (Node l x r) = (flatten l)++(x:flatten r)`