# Approximation Algorithms for the Earth Mover's Distance Under Transformations Using Reference Points

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**Abstract**

The Earth Mover's Distance (EMD) on weighted point sets is a distance measure with many applications. Since there are no known exact algorithms to compute the minimum EMD under transformations, it is useful to estimate the minimum EMD under various classes of transformations. For weighted point sets in the plane, we will show a 2-approximation algorithm for translations, a 4-approximation algorithm for rigid motions and an 8-approximation algorithm for similarity operations. The runtime of the translation approximation is $O(T^{EMD}(n,m))$, the runtime of the latter two algorithms is $O(nmT^{EMD}(n,m))$, where $T^{EMD}(n,m)$ is the time to compute the EMD between two weighted point sets with $n$ and $m$ points, respectively. We will also show that these algorithms can be extended to arbitrary dimension, giving higher worse time and approximation bounds, however. All these algorithms are based on a more general structure, namely on reference points, which lead to the elegant generalizations to higher dimensions. We give a comprehensive discussion of reference points for weighted point sets with respect to the EMD. Finally, we will extend our discussion to a variant of the EMD, namely the Proportional Transportation Distance (PTD) and we will show similar results.

## 1 Introduction

The Earth Mover's Distance on weighted point sets is a very useful distance measure for e.g. shape matching, colour-based image retrieval and music score matching, see [5], [6], [7] and [11] for more information. For these applications it is useful to have a quick estimation on the minimum distance between two weighted point sets which can be achieved under a considered class of transformations $\mathcal{T}$. Thus we want to find algorithms to compute an approximation where $EMD^{apx}(A,B) \leq \alpha \cdot \min\{EMD(A,\Phi(B)) : \Phi \in \mathcal{T}\}$. This problem was first regarded by Cohen ([5]). He constructed an iterative Flow-Transformation algorithm, which he proved to converge, but not necessarily to the global minimum. In this paper we will take a different approach and use reference points to get an approximation on the problem. A reference point is a Lipschitz-continuous mapping which is equivariant under the considered class of transformations. These points have already been introduced in [1] to construct approximation algorithms for matching compact subsets of $\mathbb{R}^d$ under translations, rigid motions and similarity operations with respect to the Hausdorff-distance. The authors of [2], [3] and [12] follow a similar approach and use pseudo-reference points to get an approximation on the minimum Hausdorff-distance of simple polygons ([2]) and the minimum area of symmetric difference of convex shapes ([3], [12]). A pseudo-reference point is a mapping which is equivariant under the class of transformations and leads to a constant factor

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approximation. As you will see later, this definition is weaker than the definition using Lipschitz-continuity, which implicates the constant factor approximation. A general discussion of reference point methods for matching according to the Hausdorff-distance has been given in [1]. Here we will extend the definition of reference points to weighted point sets and get fast constant factor approximation algorithms for matching weighted point sets under translations, rigid motions and similarity operations with respect to the Earth Mover's Distance. Quite recently, Cabello et al. ([4]) have been working on similar problems. The advantage of our approach is that the results given can be applied to arbitrary dimension and distance measure on the ground set, even to more than the in this paper mentioned $L_p$-distances. Therefore the results are widely applicable.

In the last section we will show how the results can be applied to matching of weighted point sets with respect to a variant of the Earth Mover's Distance, namely the Proportional Transportation Distance, see [6].

## 2 Basic Definitions

**Definition 2.1 (Weighted Point Set)** *([6]) Let $A = \{a_1, a_2, ..., a_n\}$ be a weighted point set such that $a_i = (p_i, \alpha_i)$ for $i = 1, ..., n$, where $p_i$ is a point in $\mathbb{R}^d$ and $\alpha_i \in \mathbb{R}_0^+$ its corresponding weight. Let $W^A = \sum_{i=1}^n \alpha_i$ be the total weight of $A$. Let $\mathbb{W}^d$ be the set of all weighted point sets in $\mathbb{R}^d$ and $\mathbb{W}^{d,G}$ be the set of all weighted point sets in $\mathbb{R}^d$ with total weight $G \in \mathbb{R}^+$.*

In the following we will use a considered class of transformations on both weighted point sets and discrete subsets of $\mathbb{R}^d$. By a transformation on a weighted point set we mean to transform the coordinates of the weighted points and leave their weights unchanged.

We now introduce the center of mass, a point related to each weighted point set. This point plays an important role in our approximation algorithms. The computation time of this point is linear, so it does not affect the runtime.

**Definition 2.2 (Center of Mass)** *Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\} \in \mathbb{W}^{d,G}$ be a weighted point set for some $G \in \mathbb{R}^+$. The center of mass of $A$ is defined as*

$$C(A) = \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i p_i.$$

As we will see, the center of mass is an instance of a more general class of mappings, namely reference points. Later, we will prove the correctness of abstract algorithms based on this class of mappings. By plugging in the center of mass we will get concrete and implementable algorithms.

**Definition 2.3 (Reference Point)** *([1]) Let $\mathcal{K}$ be a subset of $\mathbb{W}^d$ and $\delta : \mathcal{K} \to \mathbb{R}_0^+$ be a distance measure on $\mathcal{K}$. A mapping $r : \mathcal{K} \to \mathbb{R}^d$ is called a $\delta$-reference point for $\mathcal{K}$ with respect to a set of transformations $\mathcal{T}$ on $\mathcal{K}$, if the following two conditions hold:*

*a) Equivariance with respect to $\mathcal{T}$: For all $A \in \mathcal{K}$ and $\Phi \in \mathcal{T}$ we have*

$$r(\Phi(A)) = \Phi(r(A)).$$

*b) Lipschitz-continuity: There is a constant $c \geq 0$, such that for all $A, B \in \mathcal{K}$,*

$$||r(A) - r(B)|| \leq c \cdot \delta(A, B).$$

*We call $c$ the quality of the $\delta$-reference point $r$.*

In section 4.3 we will construct approximation algorithms for similarities. For this reason we will have to rescale one of the weighted point sets. Unfortunately, rescaling in a way that the diameters of the underlying point sets in $\mathbb{R}^d$ are equal, does not work.

The key to a working algorithm is to rescale the weighted point set in a way that the normalized first moments with respect to their reference points coincide. Here we give the well known definition of the normalized first moment of a weighted point set with respect to an arbitrary point $p \in \mathbb{R}^d$.

**Definition 2.4 (Normalized First Moment)** *Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\} \in \mathbb{W}^{d,G}$ be a weighted point set for some $G \in \mathbb{R}^+$ and let $p \in \mathbb{R}^d$ be an arbitrary point. We call*

$$m_p(A) = \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i ||p_i - p||$$

*the normalized first moment of $A$ with respect to $p$.*

Note that the normalized first moment of a weighted point set with respect to an arbitrary point can be calculated efficiently in linear time.

Next we will introduce the Earth Mover's Distance, a distance measure on weighted point sets.

**Definition 2.5 (Earth Mover's Distance)** *([5]) Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\ldots,m}\} \in \mathbb{W}^d$ be weighted point sets with total weights $W^A$, $W^B > 0$. Let $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ be a distance measure on the ground set $\mathbb{R}^d$. The Earth Mover's Distance between $A$ and $B$ is defined as*

$$EMD(A, B) = \frac{\min_{F \in \mathcal{F}} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} D(p_i, q_j)}{\min\{W^A, W^B\}}$$

*where $F = \{f_{ij}\}$ is a feasible flow, i.e.*

*a)* $f_{ij} \geq 0, i = 1, ..., n, j = 1, ..., m$
*b)* $\sum_{j=1}^{m} f_{ij} \leq \alpha_i, i = 1, ..., n$
*c)* $\sum_{i=1}^{n} f_{ij} \leq \beta_j, j = 1, ..., m$
*d)* $\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} = \min\{W^A, W^B\}$

Throughout this paper we will often deal with weighted point sets with equal total weights. In this case, the definition of the EMD can be simplified:

**Lemma 2.6** *Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\ldots,m}\} \in \mathbb{W}^{d,G}$ be weighted point sets with equal total weight $G \in \mathbb{R}^+$. Let $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ be a distance measure on the ground set $\mathbb{R}^d$. Then, the Earth Mover's Distance between $A$ and $B$ can be calculated as*

$$EMD(A, B) = \frac{1}{G} \min_{F \in \mathcal{F}} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} D(p_i, q_j)$$

*where $F = \{f_{ij}\}$ is a feasible flow, i.e.*

*a)* $f_{ij} \geq 0, i = 1, ..., n, j = 1, ..., m$
*b)* $\sum_{j=1}^{m} f_{ij} = \alpha_i, i = 1, ..., n$
*c)* $\sum_{i=1}^{n} f_{ij} = \beta_j, j = 1, ..., m$

$\square$

As you can easily see, condition *d)* of Definition 2.5 is implied in the case of weighted point sets with equal total weight:

**Lemma 2.7** *Let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\dots,m}\} \in \mathbb{W}^{d,G}$ be weighted point sets with equal total weight $G \in \mathbb{R}^+$. Let $F = \{f_{ij}\}$ be a feasible flow in the sense of Lemma 2.6. Then*

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} = G$$

□

For the rest of the paper the distance measure $D$ used in the definition of the EMD is the same as the one used in the definition of the EMD-reference point. When working with weighted point sets in $\mathbb{R}^d$ we will call $\mathbb{R}^d$ the ground set and $D : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_0^+$ the ground distance. If $D$ is the Euclidean Distance, we will also use EEMD as a notation for the Euclidean Earth Mover's Distance. If $D$ is any $L_p$-distance for $1 \leq p \leq \infty$ we will write $EMD_p$ to denote the Earth Mover's Distance based on this distance measure.

# 3 EMD-Reference Points

In this section we discuss the existence of EMD-reference points. We start with a negative result.

## 3.1 Non-Existence of EMD-Reference Points for Unequal Total Weight

**Theorem 3.1** *There is no EMD-reference point for weighted point sets with unequal total weights with respect to all transformation sets that include the set of translations.*

**Proof.** Assume there is an EMD-reference point $r$ with quality $c \geq 0$. Let $p, q \in \mathbb{R}^d$ be any two distinct points. Define the three weighted point sets $A := \{(p, 1)\}$, $B := \{(q, 1)\}$ and $C = A \cup B$.

Since $EMD(A, C) = 0$ we see by using Lipschitz-continuity that $||r(A) - r(C)|| = 0$, meaning $r(A) = r(C)$. For the same reason holds $r(B) = r(C)$, implicating $r(A) = r(B)$. Conversely, observing that $B$ is $A$ translated by $q - p$, we see that $r(B)$ is $r(A)$ translated by $q - p$ using the equivariance under translation. Since $q - p \neq 0$ and therefore $r(A) \neq r(B)$ we have a contradiction.

□

Since the points $p$ and $q$ can be chosen independently, the result is valid for any diameter of the weighted point set. Additionally, since all weights are chosen to be 1, it is independent of the ratio of the weights.

**Corollary 3.2** *The same result holds, even if you restrict yourself to the consideration of bounded diameter of the ground set or bounded ratio of point weights.*

Unfortunately, Theorem 3.1 has a deep impact on the usability of the reference point approach for shape matching since it makes it impossible to use this approach for partial matching applications. For a more detailed discussion on partial matching using Mass Transportation Distances, see [6].

Now we will extend the last result and show, that there is no mapping on weighted point sets with unequal total weights which is equivariant under translations and leads to a constant factor approximation, namely we show that there is no pseudo-reference point for those sets.

**Theorem 3.3** *There is no EMD-pseudo-reference point for weighted point sets with unequal total weights with respect to all transformation sets that include the set of translations.*

**Proof.** Let $K \in \mathbb{R}, K \geq 1$, $\mathcal{O}$ the origin and $e_1$ the first unit vector in $\mathbb{R}^d$. Let $EMD^r$ denote the Earth Mover's Distance where the pseudo-reference points coincide and $EMD^{opt}(A, B)$ the EMD under an optimal translation. Let $c : \mathbb{W}^d \to \mathbb{R}^d$ be a pseudo-reference point with approximation factor $\alpha$. Consider the following four weighted point sets:

$$\begin{aligned} A &= \{(\mathcal{O}, 1)\} \\ B &= \{(e_1, 1)\} \\ C_1 &= \{(\mathcal{O}, K), (e_1, 1)\} \\ C_2 &= \{(\mathcal{O}, 1), (e_1, K)\} \end{aligned}$$

By equivariance we know that

$$r(B) = r(A) + e_1.$$

Let $i \in \{1, 2\}$. Then

- $$\begin{aligned} EMD^{opt}(C_i, A) &= 0 \\ \Rightarrow EMD^r(C_i, A) &= 0 \\ \Rightarrow EMD(C_i, A + r(C_i) - r(A)) &= 0 \\ \Rightarrow EMD(C_i, \{(r(C_i) - r(A), 1)\}) &= 0 \\ \Rightarrow (r(C_i) - r(A) = \mathcal{O} &\vee \quad r(C_i) - r(A) = e_1) \end{aligned}$$

- Analogously:

$$\begin{aligned} EMD^{opt}(C_i, B) &= 0 \\ \Rightarrow (r(C_i) - r(B) = \mathcal{O} &\vee \quad r(C_i) - r(B) = e_1) \end{aligned}$$

Let us now consider the four possibilities left:

a)

$$\begin{aligned} (r(C_i) - r(A) = \mathcal{O} &\wedge \quad r(C_i) - r(B) = \mathcal{O}) \\ \Rightarrow r(B) &= r(A) \leadsto \text{ Contradiction.} \end{aligned}$$

b)

$$\begin{aligned} (r(C_i) - r(A) = \mathcal{O} &\wedge \quad r(C_i) - r(B) = e_1) \\ \Rightarrow r(A) - r(B) &= e_1 \leadsto \text{ Contradiction.} \end{aligned}$$

c)

$$\begin{aligned} (r(C_i) - r(A) = e_1 &\wedge \quad r(C_i) - r(B) = \mathcal{O}) \\ \Rightarrow r(B) &= r(C_i) \end{aligned}$$

d)

$$\begin{aligned} (r(C_i) - r(A) = e_1 &\wedge \quad r(C_i) - r(B) = e_1) \\ \Rightarrow r(B) + e_1 - r(A) &= e_1 \leadsto \text{ Contradiction.} \end{aligned}$$

Thereby we have show that for $i = 1, 2$ we have $r(B) = r(C_i)$ and therefore $r(C_1) = r(C_2)$. Now, the minimum EMD under translations is smaller or equal the EMD when $C_2$ is translated in a way that the two points of $C_1$ and $C_2$ with weight $K$ coincide. Therefore we have

$$EMD^{opt}(C_1, C_2) \leq \frac{2}{K+1}.$$

On the other hand, since the pseudo-reference points of $C_1$ and $C_2$ coincide, we see

$$EMD^r(C_1, C_2) = \frac{2(K-1)}{K+1}$$

and follow

$$\frac{EMD^r(C_1, C_2)}{EMD^{opt}(C_1, C_2)} \geq \frac{2(K-1)(K+1)}{(K+1) \cdot 2} = K + 1.$$

This, of course, leads to a contradiction because we assumed that the EMD-pseudo reference point $r$ induces a constant factor approximation and $K$ can be chosen arbitrary high. $\square$

## 3.2 The Center of Mass as a Reference Point

In the next section we will present approximation algorithms for the EMD under transformations using EMD-reference points. Since this would be useless if there was no EMD-reference point, we will restrain the consideration to weighted point sets with equal total weight and show that in this case the center of mass is a reference point:

**Theorem 3.4** *The center of mass is an EMD-reference point for weighted point sets with equal total weight with respect to affine transformations. Its quality is* 1. *This result holds for any dimension* $d$ *and any norm on the ground set.*

**Proof.** Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\ldots,m}\} \in \mathbb{W}^{d,G}$ be two arbitrary weighted point sets with equal total weight $G \in \mathbb{R}^+$ of dimension $d \in \mathbb{N}$. We have to prove to the equivariance of the center of mass under affine transformations and its Lipschitz-continuity:

a) Equivariance:
The fact that the center of mass is equivariant under affine transformations is well known.

b) Lipschitz-Continuity:
Note that this proof appeared already in [10] as a proof for a lower bound on the EMD. We have to show that

$$||C(A) - C(B)|| \leq EMD(A, B).$$

Let $F = \{f_{ij}\}_{i=1,\ldots,n, j=1,\ldots,m}$ be a flow determining $EMD(A, B)$. Then

$$\begin{aligned} ||C(A) - C(B)|| &= ||\frac{1}{G}\sum_{i=1}^{n} \alpha_i p_i - \frac{1}{G}\sum_{j=1}^{m} \beta_j q_j|| \\ &= \frac{1}{G}||\sum_{i=1}^{n} \alpha_i p_i - \sum_{j=1}^{m} \beta_j q_j|| \end{aligned}$$

Using the flow conditions of Lemma 2.6, we get

$$||C(A) - C(B)|| = \frac{1}{G}||\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij} p_i - \sum_{j=1}^{m}\sum_{i=1}^{n} f_{ij} q_j||$$

$$
\begin{aligned}
&= \frac{1}{G}||\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}p_i - \sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}q_j|| \\
&= \frac{1}{G}||\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}(p_i - q_j)|| \\
&\leq \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}||p_i - q_j|| \\
&= EMD(A,B)
\end{aligned}
$$

□

### 3.2.1 Reference Points and Different Transformations

We have seen that the center of mass is a reference point for weighted point sets with equal total weights with respect to affine transformations. Now we will discuss the existence of other reference points with respect to different classes of transformations.

Starting with translations we can prove the following result:

**Theorem 3.5** *Let $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ be an arbitrary EMD-reference point for weighted point sets with equal total weight with respect to translations. Then $r' : \mathbb{W}^{d,G} \to \mathbb{R}^d, A \mapsto r(A) + v$, where $v \in \mathbb{R}^d$ is any fixed vector, is an EMD-reference point with respect to translations. The quality $c'$ of $r'$ is equal to the quality $c$ of $r$.*

**Proof.** Again we have to show the equivariance and the Lipschitz-continuity of $r'$:

a) Equivariance:
Let $T$ be any translation and let $T_v$ be the translation by $v$. We have to show that $r'(T(A)) = T(r'(A))$.

$$
\begin{aligned}
r'(T(A)) &= r(T(A)) + v \\
&= T(r(A)) + v \\
&= T_v(T(r(A))) \\
&= T(T_v(r(A))) \\
&= T(r(A) + v) \\
&= T(r'(A))
\end{aligned}
$$

b) Lipschitz-Continuity:
We have to show that $||r'(A) - r'(B)|| \leq EMD(A,B)$:

$$
\begin{aligned}
||r'(A) - r'(B)|| &= ||r(A) + v - (r(B) + v)|| \\
&= ||r(A) - r(B)|| \\
&\leq c \cdot EMD(A,B)
\end{aligned}
$$

□

We have just seen that there are infinitely many EMD-reference points with respect to translations. However, these reference points are not really distinct because in our approximation algorithm they lead to the same position of the sets with respect to each other and therefore lead to the same value of the approximative EMD. An attempt finding a really different reference point can be found in Section 3.4.

We will now show that the center of mass as a reference point with respect to affine transformations is unique for weighted point sets with 3 points where the weights on the points are equal. This is done by mimicking a proof by Knauer ([8]). The proof is done for weighted point sets in $\mathbb{R}^2$. An extension to higher dimensions is straightforward.

**Theorem 3.6** *Let $r : \mathbb{W}^{2,G} \to \mathbb{R}^2$ be an EMD-reference point with respect to affine transformations and let $A = \{(p_i, \alpha)\}_{i=1,\ldots,3} \in \mathbb{W}^{2,G}$. Then $r(A) = C(A)$.*

**Proof.** Let $\Delta = \{(q_i, \alpha)\}_{i=1,\ldots,3} \in \mathbb{W}^{2,G}$ be a weighted point set where the coordinates of the points are the vertices of an equilateral triangle in counterclockwise order. Let $R$ be the rotation by $2\pi/3$ around the center of mass in counterclockwise order. Let $\Delta_R$ be the image of $\Delta$ under this rotation. Of course, the geometry and the weights of $\Delta_R$ and $\Delta$ are equal and we have that $EMD(\Delta, \Delta_R) = 0$. Using Lipschitz-continuity we get that $r(\Delta) = r(\Delta_R)$. Then

$$r(\Delta) = r(\Delta_R) = r(R(\Delta)) = R(r(\Delta)).$$

Therefore, $r(\Delta)$ is a fixpoint under $R$ and, since the center of mass is the only fixpoint of $R$, it follows that $r(\Delta) = C(\Delta)$. To prove the lemma, we can consider the weighted point set $A$ as the image of $\Delta$ under some affine transformation $F$. We now get

$$r(A) = r(F(\Delta)) = F(r(\Delta)) = F(C(\Delta)).$$

Since the center of mass is invariant under affine transformations,

$$r(A) = F(C(\Delta)) = C(F(\Delta)) = C(A).$$

□

We can now use the last lemma to prove the following result:

**Theorem 3.7** *There are no EMD-reference points with respect to every transformation class containing the projective transformations.*

**Proof.** Since the class of projective transformations contains all affine transformations, the only candidate for an EMD-reference point for weighted point sets with 3 weighted points and equal weight in each point is the center of mass according to the last Lemma 3.6. Since the center of mass is not equivariant for these weighted point sets, the theorem follows. □

## 3.3 Lower Bound on the Quality of an EMD-Reference Point

Using the center of mass to construct implementable algorithms raises the question if there is a better reference point. In this section we will show that the center of mass as an EMD-reference point is optimal in the sence that there is no EMD-reference point with quality smaller than 1. This holds for any transformation set including the set of translations and any distance measure on the ground set.

**Theorem 3.8** *Let $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ be an EMD-reference point with respect to any transformation set including the translations for some $G \in \mathbb{R}^+$ and some dimension $d$, and let $c$ be its quality. Then $c \geq 1$. This holds for any distance measure on the ground set.*

**Proof.** Assume there is a reference point $r$ with quality $c < 1$. Let $\mathcal{O}$ be the origin and $e_1$ be the first unit vector. Consider the two weighted point sets

$$\begin{aligned} A &= \{(\mathcal{O}, 1)\} \\ B &= \{(e_1, 1)\} \end{aligned}$$

Since $r$ is Lipschitz-continuous, we see

$$||r(A) - r(B)|| \leq c \cdot EMD(A, B) = c \cdot ||e_1|| < ||e_1||.$$

On the other hand, using the equivariance of $r$ with respect to translations, we see that $r(B) = r(A) + e_1$. Therefore

$$||r(A) - r(B)|| = ||r(A) - (r(A) - e_1)|| = ||e_1||.$$

Contradiction. □

## 3.4 Fermat-Weber Point

A point which comes to mind thinking about another reference point on weighted point sets is the so-called Fermat-Weber Point:

**Definition 3.9 (Fermat-Weber Point)** *Let $A = \{(p_i, \alpha_i)\} \in \mathbb{W}^{d,G}$ be a weighted point set. Then*

$$FW(A) = \arg\min_{p \in \mathbb{R}^d} \sum_{i=1}^{n} \alpha_i ||p_i - p||$$

*is called the Fermat-Weber point of $A$.*

Unfortunately, the Fermat-Weber point does not fulfill the Lipschitz-continuity condition, which immediately proves the following lemma:

**Lemma 3.10** *The Fermat-Weber point is no EMD-reference point for weighted point sets with equal total weight.*

□

**Example 3.11** *Consider the following two sets ($\delta, \mu \in \mathbb{R}^+, 0 < \mu < 1/2$):*

$$\begin{aligned} A &= \{((0,0), \tfrac{1}{2}(1-\delta)), ((1,0), \tfrac{1}{2}(1-\delta)), ((\mu, 0), \delta)\} \\ B &= \{((0,0), \tfrac{1}{2}(1-\delta)), ((1,0), \tfrac{1}{2}(1-\delta)), ((1-\mu, 0), \delta)\} \end{aligned}$$

*Obviously, $FW(A) = (\mu, 0)$, $FW(B) = (1-\mu, 0)$ and therefore*

$$\begin{aligned} ||FW(A) - FW(B)|| &= 1 - 2\mu \\ EMD(A, B) &= \delta(1 - \mu - \mu) = \delta(1 - 2\mu) \end{aligned}$$

*Assuming that $FW(\cdot)$ is Lipschitz-continuous, there is a constant $c > 0$ such that*

$$\begin{aligned} & ||FW(A) - FW(B)|| \leq c \cdot EMD(A, B) \\ \Leftrightarrow \quad & 1 - 2\mu \leq c\delta(1 - 2\mu) \\ \Leftrightarrow \quad & 1 \leq c\delta \end{aligned}$$

*If now $\delta$ tends to zero, $c$ has to be arbitrary high. Contradiction.*

**Remark 3.12** *The example deals with more than 2 points on a line. The same example holds if you move the inner points slightly to the top, thereby getting two weighted point sets with points in general position showing the non-Lipschitz-continuity of the Fermat-Weber point.*

In the last lemma we have shown that the Fermat-Weber point is no reference point. Since being a reference point is only a sufficient condition to induce an approximation algorithm, it may be possible that this point does induce an approximation algorithm anyway, namely that the Fermat-Weber point is a pseudo-reference point. But that this is not the case is proven by the same sets given in the last proof:

**Example 3.13** *Let us consider the set of translations. The position of $A$ and $B$ in the last proof easily shows that*

$$EMD^{opt}(A,B) \leq \delta(1-2\mu).$$

*On the other hand, matching the Fermat-Weber points of the sets leads to*

$$EMD^{FW}(A,B) = (1-2\mu)(1-\delta).$$

*And therefore*

$$\frac{EMD^{FW}(A,B)}{EMD^{opt}(A,B)} \geq \frac{1-\delta}{\delta},$$

*which tends to infinity as $\delta$ tends to zero.*

# 4 Approximation Algorithms Using EMD-Reference Points

The following three sections are organized as follows: In each section we consider a class of transformations, construct an approximation algorithm for matching under these transformations for general EMD-reference points and finally use the center of mass to get a concrete algorithm.

For the whole chapter let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\dots,m}\} \in \mathbb{W}^{d,G}$ be two weighted point sets of dimension $d$ with positive equal total weight $G \in \mathbb{R}^+$. Please be reminded that the following results do not hold for weighted point sets with unequal total weight. Further, let $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ be an EMD-reference point for weighted point sets with respect to the considered class of transformations with quality $c$. Let $T^{ref}(n)$ be the time to compute the EMD-reference point of $A$, $T^{EMD}(n,m)$ and $T^{EEMD}(n,m)$ be the time to compute the $EMD$ and $EEMD$ between $A$ and $B$ and $T^{rot}(n,m)$ be the time needed to find a rotation $R$ around a fixed point minimizing $EMD(A, R(B))$.

An upper bound on $T^{EMD}(n,m)$ and $T^{EEMD}(n,m)$ is $O((nm)^2 \log(n+m))$ using a strongly polynomial minimum cost flow algorithm by Orlin ([9]). In practice, an algorithm using the simplex method to solve the linear program will be faster. Since we are developing approximation algorithms anyway, one can consider using an $(1+\varepsilon)$-approximation algorithm for the Earth Mover's Distance by Cabello et al. ([4]) with runtime $O(\frac{n^2}{\varepsilon^2}\log^2(\frac{n}{\varepsilon}))$.

## 4.1 Translations

Consider the following algorithm to get an approximation on the problem of finding a translation minimizing the EMD under translations:

Algorithm *TranslationApx*:

a) Compute $r(A)$ and $r(B)$ and translate $B$ by $r(A) - r(B)$. Let $B'$ be the image of $B$.

b) Output $B'$ as an approximately optimal solution together with the approximate distance $EMD(A, B')$.

**Theorem 4.1** *Algorithm TranslationApx finds an approximately optimal matching for translations with approximation factor* $c+1$ *in time* $O(T^{ref}(\max\{n,m\})+T^{EMD}(n,m))$*. This holds for arbitrary dimension* $d$ *and distance measure on the ground set.*

**Proof.** Let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\dots,m}\} \in \mathbb{W}^{d,G}$ for some $G \in \mathbb{R}^+$ and dimension $d \in \mathbb{N}$ be two weighted point sets in optimal position. Let $F^* = \{f^*_{ij}\}$ be a flow determining $EMD^{opt}(A,B)$ and $T^r := r(A) - r(B)$ be the translation moving $B$ in a way that the EMD-reference points $r(A)$ and $r(B)$ coincide. When $B$ is translated by $T^r$, denote the EMD of the two sets by $EMD^r(A,B)$ and the distance of two points $p_i \in A$, $q_j \in T^r(B)$ by $D^r_{ij}$. Then

$$\begin{aligned}
EMD^r(A,B) &= \frac{1}{G}\min_{F\in\mathcal{F}}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}D^r_{ij}\\
&\leq \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f^*_{ij}D^r_{ij}\\
&= \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f^*_{ij}||p_i-(q_j+T^r)||\\
&\leq \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f^*_{ij}(||p_i-q_j||+||-T^r||)\\
&= \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f^*_{ij}||p_i-q_j|| + \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f^*_{ij}||T^r||\\
&= EMD^{opt}(A,B) + \frac{||T^r||}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f^*_{ij}\\
&= EMD^{opt}(A,B) + ||T^r||\\
&= EMD^{opt}(A,B) + ||r(A)-r(B)||\\
&\leq EMD^{opt}(A,B) + c\cdot EMD(A,B)\\
&= (1+c)\cdot EMD^{opt}(A,B)
\end{aligned}$$

□

**Corollary 4.2** *Algorithm TranslationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor* $2$*. Its runtime is* $O(T^{EMD}(n,m))$*.*

**Proof.** The quality of the center of mass as an EMD-reference point is 1 and can be computed in $O(\max\{n,m\})$ time. The overall runtime is clearly dominated by the time to compute the Earth Mover's Distance. □

### 4.1.1 Lower Bound for Algorithm TranslationApx

We presented the center of mass as an EMD-reference point with quality 1, thus inducing an approximation algorithm for translations with factor 2. We now show that this bound is tight:

**Theorem 4.3** *There are weighted point sets where the upper bound on the approximation factor for algorithm TranslationApx using the center of mass as an EMD-reference point is assumed.*

**Proof.** Consider the following two weighted point sets, where $K \in \mathbb{R}^+$ is some constant.

a) $A := \{((0,0),1),((1,0),K)\}$

Figure 1: $A$ with center of mass

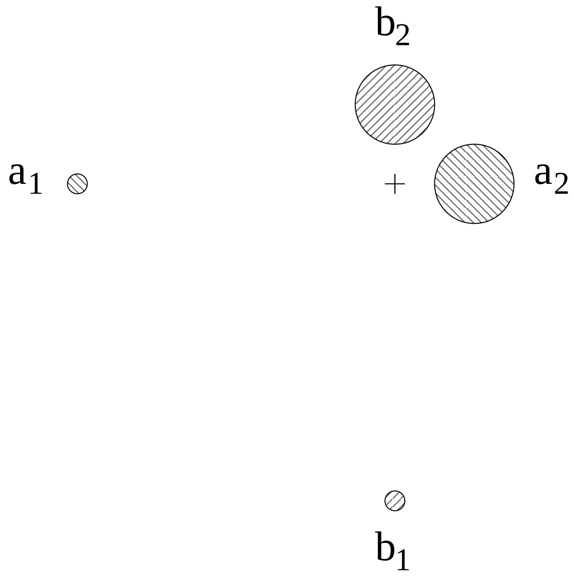


Figure 2: Matching according to center of mass

b) $B := \{((0,0),1),((0,1),K)\}$ ($A$ rotated by 90 degrees)

We now show that $\frac{EMD^C(A,B)}{EMD^{opt}(A,B)} \to 2$ as $K \to \infty$, where $EMD^C$ denotes the Earth Mover's Distance where the centers of mass coincide and $EMD^{opt}(A,B)$ the EMD under an optimal translation.

a) Calculation of $EMD^C(A,B)$. First we start calculating the centers of mass of both sets. By definition

$$\begin{aligned} C(A) &= \frac{1}{W^A}\sum_{i=1}^{n}\alpha_i p_i \\ &= \frac{1}{K+1}(1\cdot(0,0)^T + K\cdot(1,0)^T) \\ &= \frac{K}{K+1}\cdot(1,0)^T \end{aligned}$$

Since $B$ is just a rotation of $A$ it follows directly that

$$C(B) = \frac{K}{K+1}\cdot(0,1)^T.$$

See Figure 2 for an illustration of the matching according to the centers of mass.

For the following calculation, note that the distance of $a_2$ and $b_2$ to the center of mass is $1 - \frac{K}{K+1} = \frac{1}{K+1}$. To avoid case distinctions, we calculate the EMD for both of the two

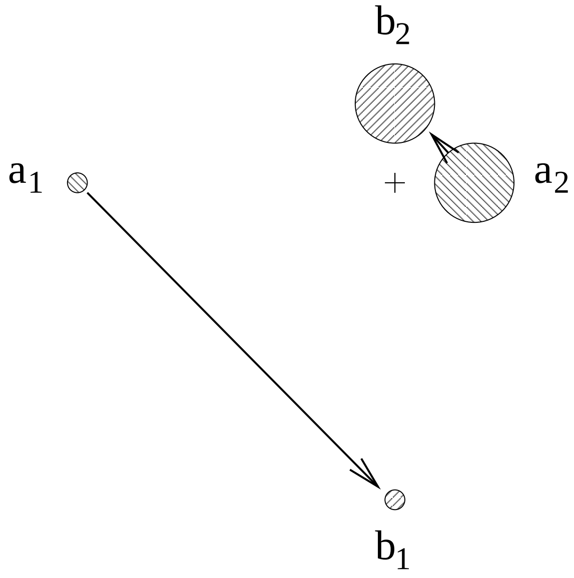


Figure 3: Matching in Version 1

following possibilities of the flow. Note that according to general flow theory these are the only two possibilities to get a minimum cost flow.

(a) $f_{11} = 1, f_{12} = 0, f_{21} = 0$ and $f_{22} = K$, see Figure 3:

$$
\begin{aligned}
EMD^C(A,B) &= \frac{1}{K+1}\left(K\sqrt{2(\frac{1}{K+1})^2} + 1 \cdot \sqrt{2(\frac{K}{K+1})^2}\right) \\
&= \frac{1}{K+1}\left(\frac{K}{K+1} \cdot \sqrt{2} + \frac{K}{K+1} \cdot \sqrt{2}\right) \\
&= \frac{1}{K+1} \cdot \frac{K}{K+1} \cdot 2 \cdot \sqrt{2} \\
&= \frac{K}{(K+1)^2} \cdot 2 \cdot \sqrt{2}
\end{aligned}
$$

(b) $f_{11} = 0, f_{12} = 1, f_{21} = 1$ and $f_{22} = K-1$, see Figure 4:

$$
\begin{aligned}
EMD^C(A,B) &= \frac{1}{K+1}(2\sqrt{(\frac{K}{K+1})^2 + (\frac{1}{K+1})^2} + (K-1)\sqrt{2\frac{1}{K+1}^2}) \\
&= \frac{1}{K+1}(2\sqrt{\frac{K^2+1}{(K+1)^2}} + (K-1)\frac{1}{K+1}\sqrt{2}) \\
&= \frac{1}{K+1}(2\sqrt{\frac{K-1}{K+1}} + \frac{K-1}{K+1}\sqrt{2})
\end{aligned}
$$

b) Calculation of $EMD^{opt}(A,B)$. We do not calculate it exactly but find an upper bound by fixing the translations where $a_2$ and $b_2$ coincide.

In this situation we have

$$EMD^{opt} \leq \frac{1}{K+1}\sqrt{2}.$$

c) Estimation of $\frac{EMD^C}{EMD^{opt}}$

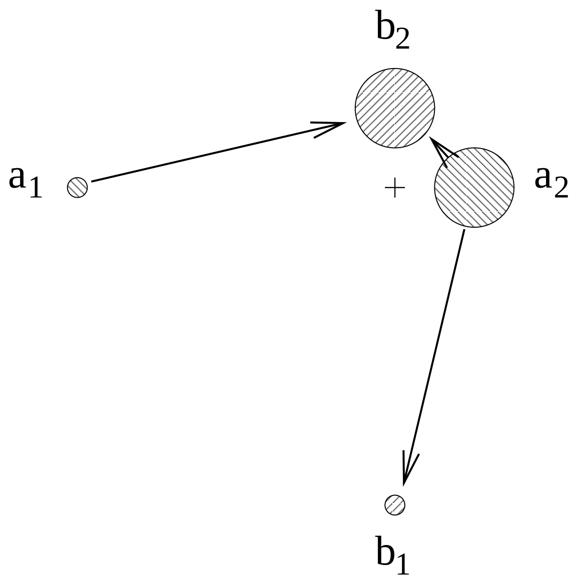


Figure 4: Matching in Version 2

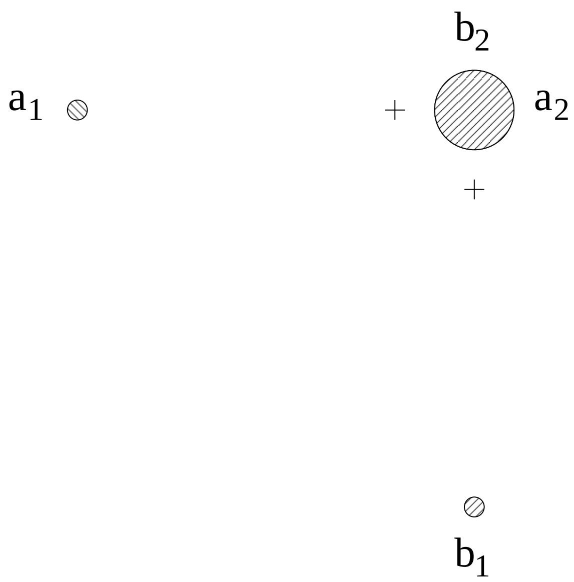


Figure 5: Translation where $a_2$ and $b_2$ coincide

(a) $f_{11} = 1, f_{12} = 0, f_{21} = 0$ and $f_{22} = K$:

$$\frac{EMD^C}{EMD^{opt}} \geq \frac{\frac{K}{(K+1)^2} \cdot 2 \cdot \sqrt{2}}{\frac{1}{K+1}\sqrt{2}} = \frac{2K}{K+1} \to 2 \text{ as } K \to \infty$$

(b) $f_{11} = 0, f_{12} = 1, f_{21} = 1$ and $f_{22} = K - 1$:

$$\frac{EMD^C}{EMD^{opt}} \geq \frac{\frac{1}{K+1}(2\sqrt{\frac{K-1}{K+1}} + \frac{K-1}{K+1}\sqrt{2})}{\frac{1}{K+1}\sqrt{2}} = \sqrt{2}\sqrt{\frac{K-1}{K+1}} + \frac{K-1}{K+1} \to \sqrt{2}+1 \text{ as } K \to \infty$$

Since $\sqrt{2}+1 > 2$ the statement follows.

□

**Remark 4.4** *The proof is independent of the considered diameter, so bounding this will not lead to a better approximation factor. The proof depends on the weights of the points, exploiting an unbounded ratio of weights. It would be nice to see a lower bound not depending on high numbers.*

## 4.2 Rigid Motions

The following algorithm gives a first approach to get an approximation of the EMD under rigid motions, i.e. combinations of translations and rotations:

Algorithm *RigidMotionApx*:

a) Compute $r(A)$ and $r(B)$ and translate $B$ by $r(A) - r(B)$. Let $B'$ be the image of $B$.
b) Find an optimal matching of $A$ and $B'$ under rotations of $B'$ around $r(A) = r(B')$. Let $B''$ be the image of $B'$ under this rotation.
c) Output $B''$ as an approximately optimal solution together with the approximate distance $EMD(A, B'')$.

**Theorem 4.5** *Algorithm RigidMotionApx finds an approximately optimal matching for rigid motions with approximation factor $c+1$ in time $O(T^{ref}(\max\{n,m\}) + T^{EMD}(n,m) + T^{rot}(n,m))$. This holds for arbitrary dimension $d$ and distance measure on the ground set.*

**Proof.** Let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\dots,m}\} \in \mathbb{W}^{d,G}$ for some $G \in \mathbb{R}^+$ and dimension $d \in \mathbb{N}$ be two weighted point sets. Let $M^*(B)$ be an optimal rigid motion of $B$ and $\delta := EMD(A, M^*(B))$ the minimum $EMD$ of $A$ and $B$ under rigid motions. Let $T^r := r(A) - r(M^*(B))$. Then $\tilde{M} := T^r \circ M^*(B)$ is a rigid motion mapping $r(B)$ onto $r(A)$. Let the rigid motion $M'$ minimize $EMD(A, M(B))$ while mapping $r(B)$ onto $r(A)$. Note that $M'$ is the solution we get by algorithm *RigidMotionApx*. Then

$$EMD(A, M'(B)) \leq EMD(A, \tilde{M}(B))$$

Further:

$$EMD(A, M'(B)) \leq EMD(A, \tilde{M}(B))$$

$$
\begin{aligned}
&= EMD(A, T^r \circ M^*(B)) \\
&\leq EMD(A, M^*(B)) + EMD(M^*(B), T^r \circ M^*(B)) \\
&= \delta + EMD(M^*(B), T^r \circ M^*(B)) \\
&\leq \delta + \frac{1}{G}\sum_{i=1}^{m} \beta_i ||M^*(q_i) - (M^*(q_i) + T^r)|| \\
&\leq \delta + \frac{||T^r||}{G}\sum_{i=1}^{m} \beta_i \\
&= \delta + ||T^r|| \\
&= \delta + ||r(A) - r(M^*(B))|| \\
&\leq \delta + c \cdot EMD(A, M^*(B)) \\
&= (1 + c) \cdot \delta
\end{aligned}
$$

The runtime of this algorithm depends on the time to compute the EMD-reference points, translate $B$ such that the EMD-reference points coincide, find the optimal rotation of the translated version of $B$ around $r(A)$ and compute the EMD of $A$ and the optimal rotation of the translated version of $B$. □

In the next corollary we again apply the center of mass as an EMD-reference point to the last result:

**Corollary 4.6** *Algorithm RigidMotionApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor 2 in time* $O(T^{rot}(n,m) + T^{EMD}(n,m))$. *This holds for arbitrary dimension $d$ and distance measure on the ground set.*

**Proof.** The quality $c$ of the center of mass as an EMD-reference point is 1 and this point can be computed in $O(\max\{n,m\})$ time, which is clearly dominated by the time to compute the EMD. □

Since the position of the EMD-reference point as rotation center is fixed, several degrees of freedom have been eliminated and the problem to find the optimal rotation is easier than the one finding the optimal rigid motion itself. Unfortunately, even for this problem no efficient algorithm is known so far. Therefore it would be nice to have at least an approximation algorithm for this problem. In the next lemma we will show an approximation for the Euclidean distance as the ground distance. This result was already published in [4]. After that we will use this lemma to extend the result to all $L_p$-distances for $1 \leq p \leq \infty$. Unfortunately, as you will see, the approximation factor will be worse than 2 for $p \neq 2$.

**Lemma 4.7** *Let $A$ and $B$ be two weighted point sets and $p^*$ be any point. Let $Rot(p^*)$ be the set of all rotations around $p^*$. Then there is a rotation $R' \in Rot(p^*)$ such that*

$$EEMD(A, R'(B)) \leq 2 \cdot \min_{R \in Rot(p^*)} EEMD(A, R(B)),$$

*where $R'$ aligns $p^*$ and any two points of $A$ and $B$.*

**Proof.** Let w.l.o.g. $A$ and $B$ be in optimal position with respect to rotations of $B$ around $p^*$. Let $F^* := \{f^*_{ij}\}$ be a flow defining $EMD(A,B)$. Let $\mathcal{R} := \{R \in Rot(p^*) : R(B)$ alignes $p^*$ and any two points $p_i \in A$ and $q_j \in B\}$. For all $R \in \mathcal{R}$ let $\phi(R) \in (-\pi, \pi]$ be the rotation angle induced by $R$. Take $R' \in \mathcal{R}$, such that for all $R \in \mathcal{R} : |\phi(R')| \leq |\phi(R)|$. Let $\phi'$ be this angle and, w.l.o.g., $R'$ be a counterclockwise rotation. Since we took the minimum angle, $\phi' \in [0, \pi/2]$. The last fact can be seen easily. Assume all angles between pairs of points are bigger than $\pi/2$. Then, rotating $B$ by an angle of $\pi$ around $p^*$ will lead to a position where

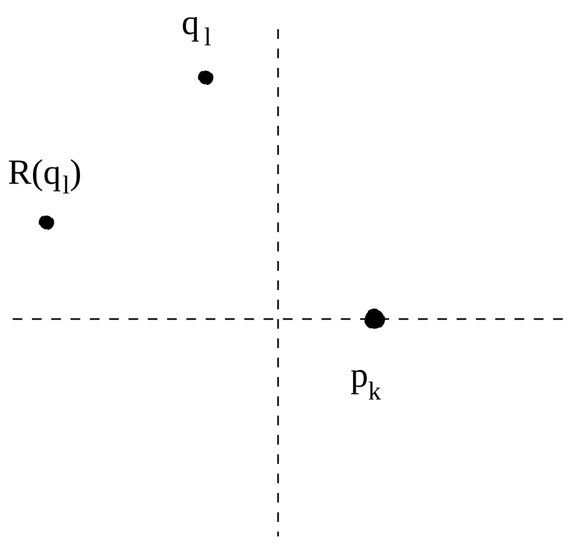


Figure 6: Case 1.

every point $a_i \in A$ is closer to any point $b_j$ in the rotated version of $B$. Therefore the EMD in this constellation is smaller than before rotating, which leads to a contradiction.

Let now $p_k \in A$ and $q_l \in B$ be two matched points. We now show that

$$||R'(q_l) - p_k||_2 \leq 2||q_l - p_k||_2. \tag{1}$$

This will imply that after rotation the distance is at most double and therefore, since the amount of moved mass between each pair of points stays the same, the EMD between the two sets will be at most double. We will assume that $q_l \neq 0$ and $p_k \neq 0$ because in those cases we have $||R'(q_l) - p_k||_2 = ||q_l - p_k||_2$.

Let w.l.o.g. $p^* = (0,0)^T$ and $p_k = (1,0)^T$.

To prove Equation 1 we make a case distinction according to the position of $q_l$ and $R'(q_l)$.

- x-coordinate of $q_l$ and $R'(q_l) \leq 0$.
  Then

  $$\begin{aligned} ||q_l - p_k||_2 &\geq ||q_l||_2 = ||R'(q_l)||_2 \\ ||q_l - p_k||_2 &\geq ||p_k||_2 \end{aligned}$$

  Therefore, we have:

  $$||R'(q_l) - p_k||_2 \leq ||R'(q_l)||_2 + ||p_k||_2 \leq 2||q_l - p_k||_2$$

- x-coordinate of $q_l \leq 0$ and $R'(q_l) \geq 0$.
  Then

  $$||R'(q_l) - p_k||_2 \leq ||q_l - p_k||_2 \leq 2||q_l - p_k||_2$$

- x-coordinate of $q_l \geq 0$.
  Let $R_\alpha$ be the rotation counterclockwise by angle $\alpha \in [0, \pi]$. Then for $\alpha \geq \alpha'$

  $$||R_\alpha(q_l) - p_k||_2 \geq ||R_{\alpha'}(q_l) - p_k||_2.$$

  Therefore, the function $f(\alpha) := \frac{||R_\alpha(q_l) - p_k||_2}{||q_l - p_k||_2}$ is monotone increasing.

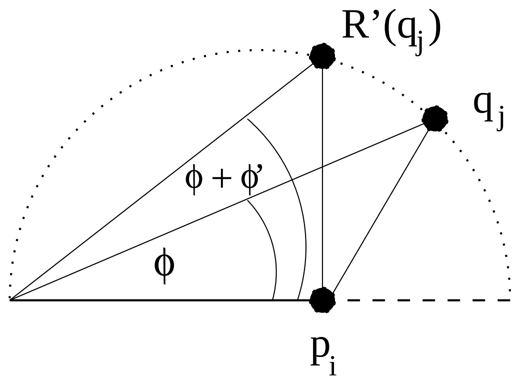


Figure 7: x-coordinate of $q_l \geq 0$

We can now write $q_l$ as a rotation of $p_k$ around $p^*$ by some angle $\phi$ and scaling by some number $\lambda > 0$. Then $q_l = \lambda R_\phi(p_k)$ and $R'(q_l) = \lambda R_{\phi+\phi'}(p_k)$. It remains to show that

$$\tilde{f} : \mathbb{R}^+ \to \mathbb{R}, \lambda \mapsto \frac{||\lambda R_{\phi+\phi'}(p_k) - p_k||_2}{||\lambda R_\phi(p_k) - p_k||_2}$$

is bounded by two. Since $\pi \geq 2\phi \geq \phi + \phi'$ and according to the above remark it suffices to show that

$$f : \mathbb{R}^+ \to \mathbb{R}, \lambda \mapsto \frac{||\lambda R_{2\phi}(p_k) - p_k||_2}{||\lambda R_\phi(p_k) - p_k||_2}$$

is bounded by 2.

Proof for that:

$$\begin{aligned} f(\lambda) &= \frac{||\lambda R_{2\phi}(p_k) - p_k||_2}{||\lambda R_\phi(p_k) - p_k||_2} \\ &= \frac{\sqrt{-4\lambda\cos^2(\phi) + \lambda^2 + 2\lambda + 1}}{\sqrt{-2\lambda\cos(\phi) + \lambda^2 + 1}} \end{aligned}$$

Easy computation shows that this function has a maximum at $\lambda = 1$ independent of the concrete angle $\phi$, therefore the maximum is always attained at a point $q_l$ having the same distance to the rotation center as $p_k$.

Now,

$$g : (0, \pi/2] \to \mathbb{R}, \beta \mapsto \frac{\sqrt{-4\cos^2(\beta) + 4}}{\sqrt{-2\cos(\beta) + 2}}$$

is the function $f$ for points located at the same distance to the rotation center as $p_k$. We are now interested at what angle this function is maximized:

$$\begin{aligned} \frac{\sqrt{-4\cos^2(\beta) + 4}}{\sqrt{-2\cos(\beta) + 2}} &= \sqrt{2}\frac{\sqrt{1 - \cos^2(\beta)}}{\sqrt{1 - \cos(\beta)}} \\ &= \sqrt{2}\frac{\sqrt{(1 - \cos(\beta))1 + \cos(\beta))}}{\sqrt{1 - \cos(\beta)}} \\ &= \sqrt{2}\sqrt{1 + \cos(\beta)} \end{aligned}$$

This function is clearly maximized for $\beta = 0$. This leads to $\tilde{f} \leq 2$ and therefore we have proven that

$$||R'(q_l) - p_k||_2 \leq 2||q_l - p_k||_2$$

The remaining cases can be easily derived be the cases above and altogether we have:

$$||R'(q_l) - p_k||_2 \leq 2||q_l - p_k||_2. \tag{2}$$

We can now use Equation 2 to prove the lemma:

$$\begin{aligned}
EMD(A, R'(B)) &= \frac{1}{G} \min_{F \in \mathcal{F}} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} D(p_i, R'(q_j)) \\
&\leq \frac{1}{G} \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} D(p_i, R'(q_j)) \\
&\leq \frac{1}{G} \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} ||p_i - R'(q_j)||_2 \\
&\leq \frac{1}{G} \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} \cdot 2 \cdot ||p_i - q_j||_2 \\
&\leq 2 \cdot \frac{1}{G} \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} \cdot D(p_i, q_j) \\
&= 2 \cdot EMD(A, B).
\end{aligned}$$

□

As mentioned above, we will now use the last lemma to extend the result to all $L_p$-distances for $1 \leq p \leq \infty$:

**Lemma 4.8** *Let $A, B \in \mathbb{W}^d$ be two weighted point sets and $p^*$ be any point. Let $Rot(p^*)$ be the set of all rotations around $p^*$. Then there is a rotation $R' \in Rot(p^*)$ such that*

$$EMD_p(A, R'(B)) \leq 2\sqrt{d} \cdot \min_{R \in Rot(p^*)} EMD_p(A, R(B)),$$

*where $R'$ aligns $p^*$ and any two points of $A$ and $B$.*

**Proof.** Let $1 \leq p \leq \infty$. Following the strategy of the proof of Lemma 4.7, we have to prove that

$$||R'(q_l) - p_k||_p \leq 2\sqrt{d}||q_l - p_k||_p.$$

a) $1 \leq p < 2$
Then:

$$\begin{aligned}
||R'(q_l) - p_k||_p &\leq \sqrt{d}||R'(q_l) - p_k||_2 \\
&\leq 2\sqrt{d}||R'(q_l) - p_k||_2, \text{ proof of Lemma 4.7} \\
&\leq 2\sqrt{d}||R'(q_l) - p_k||_p
\end{aligned}$$

b) $2 < p \leq \infty$
Then:

$$\begin{aligned}
||R'(q_l) - p_k||_p &\leq ||R'(q_l) - p_k||_2 \\
&\leq 2||R'(q_l) - p_k||_2, \text{ proof of Lemma 4.7} \\
&\leq 2\sqrt{d}||R'(q_l) - p_k||_p
\end{aligned}$$

Like in the proof of Lemma 4.7 the claim of this lemma follows immediately. □

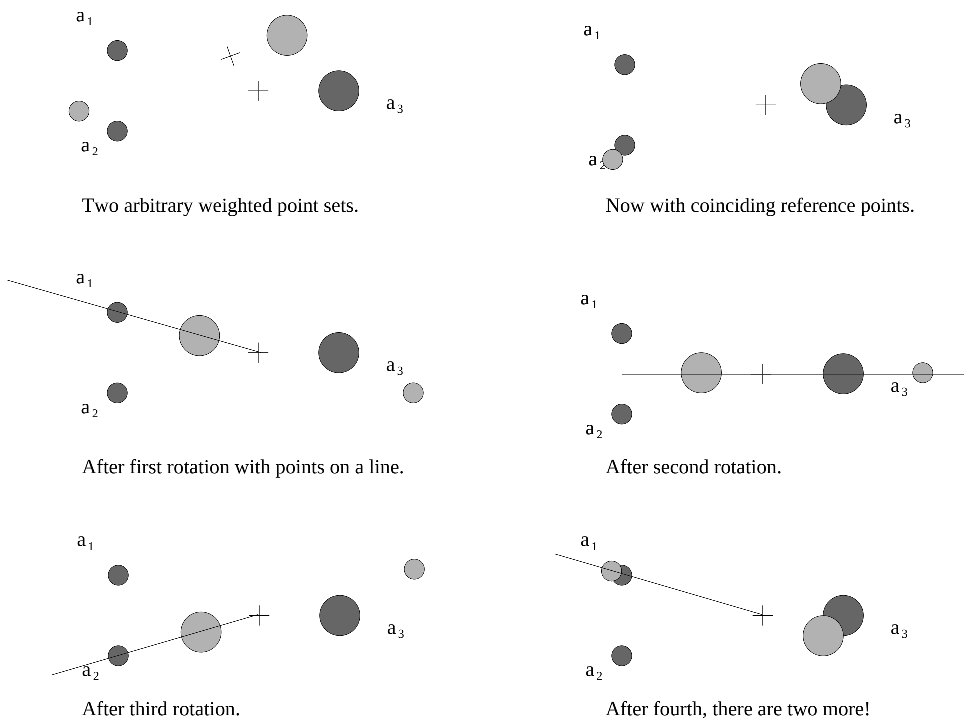


Figure 8: Illustration of algorithm *RigidMotionApxUsingRotationApx*.

### 4.2.1 An Applicable Algorithm in the Plane

Based on the last lemma we are able to construct approximation algorithms for the problem of finding an optimal rotation of a weighted point set around their coinciding reference points.

In this section we will discuss the case of weighted point sets in the plane. The general case will be discussed in the following section.

| Algorithm *RotationApx* |
|---|
| a) Compute the minimum $EEMD$ over all possible alignments of the coinciding reference points and any two points of $A$ and $B$. |

Since there are $O(nm)$ possibilities to align the reference point and any two points of $A$ and $B$, the runtime of this algorithm is $O(nmT^{EEMD}(n,m))$. Using this algorithm combined with EMD-reference points we now get an easy to implement and fast approximation algorithm for rigid motions. Unfortunately, the fact that we now constructed an implementable algorithm must be paid by the increased approximation factor $2(c+1)$. Figure 8 shows an illustration of this algorithm.

| Algorithm *RigidMotionApxUsingRotationApx* |
|---|
| a) Compute $r(A)$ and $r(B)$ and translate $B$ by $r(A) - r(B)$. Let $B'$ be the image of $B$. |
| b) Find a best matching of $A$ and $B'$ under rotations of $B'$ around $r(A) = r(B')$ where $r(A)$ and any two points in $A$ and $B'$ are aligned. Let $B''$ be the image of $B'$ under this rotation. |
| c) Output $B''$ as an approximately optimal solution together with the approximate distance $EEMD(A, B'')$. |

**Theorem 4.9** *Regarding EEMD in the plane, Algorithm RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor* $2(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{EEMD}(n,m))$.

**Proof.** Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\ldots,m}\} \in \mathbb{W}^{2,G}$ for some $G \in \mathbb{R}^+$ be arbitrary weighted point sets. Let $M_{opt}(B)$ be the optimal rigid motion and $\delta := EEMD(A, M_{opt}(B))$. Let $M^*$ be the rigid motion minimizing $EEMD(A, M(B))$ while mapping $r(B)$ onto $r(A)$ and $M^{**}$ be the rigid motion, minimizing $EEMD(A, M(B))$ while mapping $r(B)$ onto $r(A)$ and additionally aligning $r(A)$ and any two points of $A$ and $B$. Note that this is the rigid motion found by the algorithm. We now have that

$$\begin{aligned} EEMD(A, M^{**}(B)) &\leq 2 \cdot EEMD(A, M^*(B)), \text{ Lemma 4.7} \\ &\leq 2(1+c)\delta, \text{ see proof of Theorem 4.5} \end{aligned}$$

The runtime of this algorithm depends on the time to compute the EMD-reference points, translate $B$ such that the EMD-reference points coincide and compute the EEMD at all $O(nm)$ possible alignments of points in $A$, $B$ and $r(A)$. □

**Theorem 4.10** *Algorithm RigidMotionApxUsingRotationApx finds an approximately optimal matching for* $EMD_p$, $1 \leq p \leq \infty$, *under rigid motions in the plane with approximation factor* $2\sqrt{2}(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{EMD_p}(n,m))$.

**Proof.** The proof for this theorem is the same as the proof for Theorem 4.9, using Lemma 4.8 instead of Lemma 4.7. □

In the next corollary we apply the center of mass to the last two theorems and prove results about an implementable algorithm for matching weighted point sets und rigid motions in the plane:

**Corollary 4.11** *Algorithm RigidMotionApxUsingRotationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor* 4 *in case of the Euclidean Distance in the plane and* $4\sqrt{2}$ *for any other* $L_p$ *distance,* $1 \leq p \leq \infty$. *Its runtime is* $O(nmT^{EMD_p}(n,m))$.

**Proof.** The proof of this corollary follows immediately using the last two theorems and the fact that the quality of the center of mass as an EMD-reference point is 1 and can be computed in $O(\max\{n,m\})$ time. □

### 4.2.2 Rigid Motion Approximation in higher dimensions

The algorithm for minimizing the EMD under rigid motions in higher dimensions is similar to the algorithm in the plane. First, we translate $B$ in a way that the two reference points coincide, then compute the EMD whenever at least one point of $A$, one point of $B$ and the reference point are on a line, which is a $(d-1)$-dimensional space in the plane. For arbitrary dimension $d$ we will compute the EMD for every rotation $R$ which rotates $B$ around the reference point in a way that at least $d-1$ points of $A$, $d-1$ points of $B$ and the reference point are on a $(d-1)$-dimensional space.

This gives

$$O\left(\binom{n}{d-1}\binom{m}{d-1}\right) = O(n^{d-1} \cdot m^{d-1})$$

possibilities of rotations.

We will now prove the approximation factor of $O(2^{d-1})$ for matching the weighted point set with respect to the EMD under rotations defined on the Euclidean distance on the ground set. Let $S$

be a fixed $d'$-dimensional space with $0 \leq d' < d-1$. Let w.l.o.g $A$ and $B$ be in optimal position with respect to all rotations leaving $S$ invariant. Then, mimicking the proof of 4.7, we see that there is some rotation $R'$ around the reference point, leaving $S$ invariant and rotating in a way, that $S$, at least one point of $A$ and at least one point of $B$ are in a $d'+1$-dimensional space.

Together with the approximation on the translation with approximation factor $c+1$ we get an approximation algorithm for minimizing the EMD under rigid motions. We will call this algorithm *d-RigidMotionApxUsingRotationApx*. Altogether we have a constructive proof of the following theorem:

**Theorem 4.12** *Algorithm d-RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor* $2^{d-1}(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{EMD}(n,m))$. *This holds for arbitrary dimension $d$.*

Again, using Lemma 4.8 we can extend the result to the EMD defined on $L_p$-distances and get

**Theorem 4.13** *Algorithm d-RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor* $2^{d-1}\sqrt{d}^{d-1}(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{EMD}(n,m))$. *This holds for arbitrary dimension $d$ and $1 \leq p \leq \infty$.*

Using the center of mass, we get to one of our main results in higher dimensions:

**Corollary 4.14** *Algorithm d-RigidMotionApxUsingRotationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor $2^d$ in case of the Euclidean Distance and $2^d\sqrt{d}^{d-1}$ for any other $L_p$ distance, $1 \leq p \leq \infty$. Its runtime is* $O(n^{d-1}m^{d-1}T^{EMD_p}(n,m))$.

**Proof.** The proof of this corollary follows immediately using the last two theorems and the fact that the quality of the center of mass as an EMD-reference point is 1 and can be computed in $O(\max\{n,m\})$ time. □

## 4.3 Similarities

In this section we present approximation algorithms for matching two given weighted point sets under similarity transformations, i.e. combinations of translations, rotations and scalings. More precisely, we want to compute $\min_S EMD(A, S(B))$, where the minimum is taken over all similarity operations $S$. Note that in this case exchanging $A$ and $B$ makes a difference.

Algorithm *SimilarityApx*:

a) Compute $r(A)$ and $r(B)$ and translate $B$ by $r(A) - r(B)$. Let $B'$ be the image of $B$.

b) Determine the normalized first moments $m_{r(A)}(A)$ and $m_{r(B')}(B')$ and scale $B'$ by $\frac{m_{r(A)}(A)}{m_{r(B')}(B')}$ around the center $r(A) = r(B')$. Let $B''$ be the image of $B'$ under this scaling.

c) Find an optimal matching of $A$ and $B''$ under rotations of $B''$ around $r(A) = r(B'')$. Let $B'''$ be the image of $B''$ under this rotation.

d) Output $B'''$ as an approximately optimal solution together with the approximate distance $EMD(A, B''')$.

To show the correctness of this algorithm we use the following two lemmata:

**Lemma 4.15** *Let $A \in \mathbb{W}^d$ be a weighted point set with positive total weight and let $m_p(A)$ be its normalized first moment with respect to some point $p \in \mathbb{R}^d$. Let $\tau_1, \tau_2$ be scalings around the same center $p$ and ratios $\gamma_1$ and $\gamma_2$, respectively. Then*

$$EMD(\tau_1(A), \tau_2(A)) \leq |(\gamma_1 - \gamma_2)| m_p(A).$$

**Proof.**

$$\begin{aligned}
EMD(\tau_1(A), \tau_2(A)) &\leq \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i ||\tau_1(p_i) - \tau_2(p_i)|| \ (1) \\
&= \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i ||p + \gamma_1(p_i - p) - (p + \gamma_2(p_i - p))|| \\
&= \frac{1}{W^A} \sum_{i=1}^{n} \alpha_i ||(\gamma_1 - \gamma_2)(p_i - p))|| \\
&= \frac{|\gamma_1 - \gamma_2|}{W^A} \sum_{i=1}^{n} \alpha_i ||p_i - p|| \\
&= |(\gamma_1 - \gamma_2)| m_p(A)
\end{aligned}$$

In (1) we have chosen the flow between the corresponding points. This, of course, is a feasible flow and therefore the inequality holds. □

The next lemma gives a new lower bound for the EMD of two weighted point sets:

**Lemma 4.16** *Let $A, B \in \mathbb{W}^{d,G}$ for some $G \in \mathbb{R}^+$ and let $r : \mathbb{W}^{d,G} \to \mathbb{R}^d$ be an EMD-reference point with quality $c$. Then*

$$|m_{r(A)}(A) - m_{r(B)}(B)| \leq (1 + c) EMD(A, B).$$

**Proof.** Let $F^* := \{f^*_{ij}\}$ be a flow defining $EMD(A, B)$.

$$\begin{aligned}
|m_{r(A)}(A) - m_{r(B)}(B)| &= \left| \frac{1}{G} \sum_{i=1}^{n} \alpha_i ||p_i - r(A)|| - \frac{1}{G} \sum_{j=1}^{m} \beta_j ||q_j - r(B)|| \right| \\
&= \frac{1}{G} \left| \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} ||p_i - r(A)|| - \sum_{j=1}^{m} \sum_{i=1}^{n} f^*_{ij} ||q_j - r(B)|| \right| \\
&= \frac{1}{G} \left| \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} (||p_i - r(A)|| - ||q_j - r(B)||) \right| \\
&\leq \frac{1}{G} \sum_{i=1}^{n} \sum_{j=1}^{m} f^*_{ij} \, | \, ||p_i - r(A)|| - ||q_j - r(B)|| \, |
\end{aligned}$$

Since

$$\begin{aligned}
||p_i - r(A)|| &= ||p_i - q_j + q_j - r(B) + r(B) - r(A)|| \\
&\leq ||p_i - q_j|| + ||q_j - r(B)|| + ||r(B) - r(A)||
\end{aligned}$$

and analogue

$$||q_j - r(B)|| = ||q_j - p_i + p_i - r(A) + r(A) - r(B)||$$

$$\begin{aligned}
&\leq \; ||q_j - p_i|| + ||p_i - r(A)|| + ||r(A) - r(B)|| \\
&= \; ||p_i - q_j|| + ||p_i - r(A)|| + ||r(B) - r(A)||
\end{aligned}$$

we get

$$\begin{aligned}
|m_{r(A)} - m_{r(B)}| &= \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}^{*} \,|\; ||p_i - r(A)|| - ||q_j - r(B)|| \;| \\
&\leq \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}^{*}(||p_i - q_j|| + ||r(B) - r(A)||) \\
&= \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}^{*}||p_i - q_j|| + \frac{1}{G}\sum_{i=1}^{n}\sum_{j=1}^{m} f_{ij}^{*}||r(B) - r(A)|| \\
&= EMD(A,B) + ||r(B) - r(A)|| \\
&\leq EMD(A,B) + c \cdot EMD(A,B)
\end{aligned}$$

□

Using these lemmata we can prove the following:

**Theorem 4.17** *Algorithm SimilarityApx finds an approximately optimal matching for similarities with approximation factor $2(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + T^{EMD}(n,m) + T^{rot}(n,m))$. This holds for arbitrary dimension $d$ and distance measure on the ground set.*

**Proof.** Consider an optimal similarity transformation $S_{opt}$. It can be written as $S_{opt} = \tau_{opt} \circ M_{opt}$, where $M_{opt}$ is a rigid motion and $\tau_{opt}$ is a homothety with ratio $\alpha_{opt}$ around some point $p$. Let $\delta$ be the optimal earth movers distance $\delta = EMD(A, S_{opt}(B))$. Then

$$||r(A) - r(S_{opt}(B))|| \leq c\delta,$$

because of the Lipschitz-continuity of the EMD-reference point $r$. Let $T^r$ be the translation by $r(A) - r(S_{opt}(B))$; then $\tilde{S} := T^r \circ S_{opt}$ is a similarity transformation mapping $r(B)$ onto $r(A)$ and

$$EMD(A, \tilde{S}(B)) \leq (c+1)\delta.$$

The proof of the last fact is easy:

$$\begin{aligned}
EMD(A, \tilde{S}(B)) &= EMD(A, T^r \circ S_{opt}(B)) \\
&\leq EMD(A, S_{opt}(B)) + EMD(S_{opt}(B), T^r \circ S_{opt}(B)) \\
&\leq \delta + \frac{1}{G}\sum_{j=1}^{m} \beta_j ||S_{opt}(q_j) - (S_{opt}(q_j) + T^r)|| \\
&\leq \delta + \frac{||T^r||}{G}\sum_{j=1}^{m} \beta_j \\
&= \delta + ||T^r|| \\
&\leq \delta + c\delta
\end{aligned}$$

Write now $\tilde{S}$ as $\tilde{S} = \tilde{\tau} \circ \tilde{M}$, where $\tilde{M}$ is a rigid motion mapping $r(B)$ onto $r(A)$ and $\tilde{\tau}$ is a homothety with center $r(A)$ and ratio $\alpha_{opt}$. Let $\alpha := \frac{m_{r(A)}(A)}{m_{r(B)}(B)}$, $\tau$ the homothety with center $r(A)$ and ratio $\alpha$, and $S = \tau \circ \tilde{M}$. Then

$$EMD(A, S(B)) \leq EMD(A, \tilde{S}(B)) + EMD(\tilde{S}(B), S(B)).$$

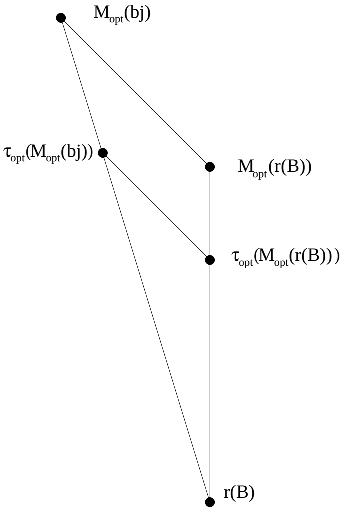


Figure 9: $\frac{||\tau_{opt} \circ M_{opt}(b_j) - \tau_{opt} \circ M_{opt}(r(B))||}{||M_{opt}(b_j) - M_{opt}(r(B))||} = \frac{||\tau_{opt} \circ M_{opt}(r(B)) - p||}{||M_{opt}(r(B)) - p||} = \alpha_{opt}$

Now

$$\begin{aligned}
EMD(\tilde{S}(B), S(B)) &= EMD(\tilde{\tau} \circ \tilde{M}(B), \tau \circ \tilde{M}(B)) \\
&\leq |(\alpha_{opt} - \alpha) m_{r(A)}(\tilde{M}(B))|, \text{ by lemma 4.15} \\
&= |(\alpha_{opt} - \alpha) m_{r(B)}(B)|, \text{ see definition of } \tilde{M} \qquad (3) \\
&= |\alpha_{opt} m_{r(B)}(B) - \alpha m_{r(B)}(B)| \\
&= |\alpha_{opt} m_{r(B)}(B) - m_{r(A)}(A)|, \text{ by definition of } \alpha \\
&= |m_{r(S_{opt}(B))}(S_{opt}(B)) - m_{r(A)}(A)| \qquad (4) \\
&\leq (1+c) EMD(S_{opt}(B), A), \text{ by lemma 4.16}
\end{aligned}$$

It remains to show that equation lines 3 and 4 above hold:

- For 3 it is to show that $m_{r(A)}(\tilde{M}(B)) = m_{r(B)}(B)$. This holds because $\tilde{M}$ is a rigid motion mapping $r(B)$ onto $r(A)$ and a rigid motion does not affect the distances to a point which is translated and rotated in the same way.
- For 4 it is to show that $\alpha_{opt} m_{r(B)}(B) = m_{r(S_{opt}(B))}(S_{opt}(B))$. To proof that:

$$\begin{aligned}
m_{r(S_{opt}(B))}(S_{opt}(B)) &= m_{r(\tau_{opt} \circ M_{opt}(B))}(\tau_{opt} \circ M_{opt}(B)) \\
&= m_{\tau_{opt} \circ M_{opt}(r(B))}(\tau_{opt} \circ M_{opt}(B)) \\
&= \frac{1}{G} \sum_{j=1}^{m} \beta_j ||\tau_{opt} \circ M_{opt}(b_j) - \tau_{opt} \circ M_{opt}(r(B))|| \\
&= \frac{1}{G} \sum_{j=1}^{m} \beta_j (\alpha_{opt} ||M_{opt}(b_j) - M_{opt}(r(B))||), \text{ (see Fig. 9)} \\
&= \alpha_{opt} m_{M_{opt}(r(B))}(M_{opt}(B)) \\
&= \alpha_{opt} m_{r(B)}(B)
\end{aligned}$$

Altogether we have

$$EMD(A, S(B)) \leq 2(c+1)\delta$$

for some similarity transformation $S$ composed of a rigid motion that maps $r(B)$ onto $r(A)$ and a homothety with center $r(A)$ and ratio $\alpha$. Since Algortihm *SimilarityApx* finds the optimum among these similarity transformations the bound holds for it, as well. The runtime of this algorithm depends on the time to compute the EMD-reference points, translate $B$ such that the EMD-reference points coincide, scale the translated version of $B$, find the optimal rotation around $r(A)$ and compute the EMD of $A$ and the optimal rotation of the translated version of $B$. Since computing the normalized first moment and therefore the scaling can be done in linear time, the time bound stays the same as for the algorithm *RigidMotionApx*. □

**Corollary 4.18** *Algorithm SimilarityApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor* 4*. Its runtime is* $O(T^{EMD}(n,m)+T^{rot}(n,m))$*. This holds for any dimension of the ground set and every norm defined on it.*

□

### 4.3.1 An applicable algorithm in the plane

As for *RigidMotionApx*, *SimilarityApx* depends on finding the optimal rotation, which is impractical. Again, we make this algorithm practical and efficient by using *RotationApx* and again we have to pay by a worse approximation factor. Like before, we start in the plane with the EMD defined on the Euclidean distance on the ground set.

Algorithm *SimilarityApxUsingRotationApx*

a) Compute $r(A)$ and $r(B)$ and translate $B$ by $r(A) - r(B)$. Let $B'$ be the image of $B$.
b) Determine the normalized first moments $m_{r(A)}(A)$ and $m_{r(B')}(B')$ and scale $B'$ by $\frac{m_{r(A)}(A)}{m_{r(B')}(B')}$ around the center $r(A) = r(B')$. Let $B''$ be the image of $B'$ under this scaling.
c) Find a best matching of $A$ and $B''$ under rotations of $B''$ around $r(A) = r(B'')$ where $r(A)$ and any two points in $A$ and $B''$ are aligned. Let $B'''$ be the image of $B''$ under this rotation.
d) Output $B'''$ as an approximately optimal solution together with the approximate distance $EEMD(A, B''')$.

**Theorem 4.19** *Regarding EEMD in the plane, Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor* $4(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{EEMD}(n,m))$.

**Proof.** Let $A = \{(p_i, \alpha_i)_{i=1,\ldots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\ldots,m}\} \in \mathbb{W}^{2,G}$ for some $G \in \mathbb{R}^+$ be arbitrary weighted point sets. Let $T^r = r(A) - r(B)$ and $\tau^\alpha$ be the scaling by $\frac{m_{r(A)}(A)}{m_{r(A)}(T^r(B))}$. Let $M^*$ be the rigid motion minimizing $EEMD(A, M \circ \tau_\alpha \circ T^r(B)))$ while mapping $r(B)$ onto $r(A)$ and $M^{**}$ be the rigid motion minimizing $EEMD(A, M \circ \tau_\alpha \circ T^r(B)))$ while mapping $r(B)$ onto $r(A)$ and additionally aligning $r(A)$ and any two points of $A$ and $B$. Let $S^{**} = M^{**} \circ \tau_\alpha \circ T^r(B))$ and $S^* = M^* \circ \tau_\alpha \circ T^r(B))$. Note that $S^{**}$ is the similarity transformation found by algorithm *SimilarityApxUsingRotationApx* and $S^*$ is the similarity found by algorithm *SimilarityApx*. Then

$$\begin{aligned} EEMD(A, S^{**}(B)) &= EEMD(A, M^{**} \circ \tau_\alpha \circ T^r(B)) \\ &\leq 2 \cdot EEMD(A, M^* \circ \tau_\alpha \circ T^r(B)), \text{ Lemma 4.7} \\ &\leq 4(1+c)\delta, \text{ Theorem 4.17} \end{aligned}$$

The runtime of this algorithm depends on the time to compute the EMD-reference points, translate $B$ such that the EMD-reference points coincide and compute the EEMD at all $O(nm)$ possible alignments of points in $A$, $B$ and $r(A)$. □

**Theorem 4.20** *Regarding $EMD_p$ in the plane, $1 \leq p \leq \infty$, Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor $4\sqrt{2}(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + nmT^{EMD_p}(n,m))$.*

**Proof.** This theorem follows immediately using the proof of Theorem 4.19 and Lemma 4.8. □

**Corollary 4.21** *Algorithm SimilarityApxUsingRotationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor 8 in case of the Euclidean Distance in the plane and $8\sqrt{2}$ for any other $L_p$ distance, $1 \leq p \leq \infty$. Its runtime is $O(nmT^{EMD_p}(n,m))$.*

### 4.3.2 Similarities in higher dimensions

We can use Algorithm *d-RigidMotionApxUsingRotationApx* to minimize the EMD under rigid motions in higher dimensions on $A$ and $B$, where as in the last section, $B$ is scaled by the fraction of the normalized first moments. We will call this algorithm *d-SimilarityApxUsingRotationApx* and get the following results:

**Theorem 4.22** *Algorithm d-SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor $2^d(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{EMD}(n,m))$. This holds for arbitrary dimension $d$.*

**Theorem 4.23** *Algorithm d-SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor $2^d\sqrt{d}^{d-1}(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{EMD}(n,m))$. This holds for arbitrary dimension $d$ and $1 \leq p \leq \infty$.*

Applying the center of mass, we can state our main implementable result for minimizung the EMD with respect to similarity operations in any dimension:

**Corollary 4.24** *Algorithm d-SimilarityApxUsingRotationApx using the center of mass as an EMD-reference point induces an approximation algorithm with approximation factor $2^{d+1}$ in case of the Euclidean Distance and $2^{d+1}\sqrt{d}^{d-1}$ for any other $L_p$ distance, $1 \leq p \leq \infty$. Its runtime is $O(n^{d-1}m^{d-1}T^{EMD_p}(n,m))$.*

# 5 Proportional Transportation Distance

In the first sections we introduced EMD-reference points, proved that the center of mass is an EMD-reference point, derived approximation algorithms based on reference points and, finally, by applying the center of mass we got implementable algorithms. Unfortunately we have seen that those results are only true when weighted point sets with equal total weights are considered. In this section we will discuss another distance measure for weighted point sets, the Proportional Transportation Distance (PTD). One of the biggest advantage for our purposes will be the fact, that there are PTD-reference points even for weighted point sets with unequal total weight.

**Definition 5.1 (Proportional Transportation Distance)** *([6]) Let $A = \{(p_i, \alpha_i)_{i=1,\dots,n}\}$, $B = \{(q_j, \beta_j)_{j=1,\dots,m}\}$ be two weighted point sets with $W^A := \sum_{i=1}^n \alpha_i, W^B := \sum_{j=1}^m \beta_j$ the total weights of $A, B$ respectively. Then, the Proprotional Transportation Distance is defined as*

$$PTD(A,B) = \frac{\min_{F \in \mathcal{F}} \sum_{i=1}^n \sum_{j=1}^m f_{ij} d_{ij}}{W^A},$$

*where*

*a)* $f_{ij} \geq 0, i = 1, ..., n, j = 1, ..., m$

*b)* $\sum_{j=1}^{m} f_{ij} = \alpha_i, i = 1, ..., n$

*c)* $\sum_{i=1}^{n} f_{ij} = \frac{\beta_j W^A}{W^B}, j = 1, ..., m$

*d)* $\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} = W^A$

## 5.1 PTD-Reference Points

We start with the main result, showing that the center of mass is a PTD-reference point for weighted point sets with arbitrary total weight with respect to the set of affine transformations.

**Theorem 5.2** *The center of mass of a weighted point set is a PTD-reference point for weighted point sets with respect to affine transformations. Its quality is* 1*. This holds for any dimension* $d$ *of the ground set and every norm defined on it.*

**Proof.**

a) The equivariance of the center of mass under affine transformations for weighted point sets is well known.

b) Lipschitz-Continuity
We have to show that for all weighted point sets $A, B$

$$||C(A) - C(B)|| \leq PTD(A, B)$$

Let $A := \{(p_i, \alpha_i)\}, i = 1, ..., n,\ B = \{(q_j, \beta_j)\}, j = 1, ..., m$ and $W^A := \sum_{i=1}^{n} \alpha_i, W^B := \sum_{j=1}^{m} \beta_j$. Further let $F = (f_{ij})_{i=1,...,j=1,...,m}$ be a flow determining $PTD(A, B)$. Then

$$||C(A) - C(B)|| \quad = \quad ||\frac{1}{W^A} \sum_{i=1}^{n} \alpha_i p_i - \frac{1}{W^B} \sum_{j=1}^{m} \beta_j q_j||$$

By definition we know that

$$\alpha_i = \sum_{j=1}^{m} f_{ij} \text{ and } \beta_j = \frac{W^B}{W^A} \sum_{i=1}^{n} f_{ij}.$$

Therefore we get

$$\begin{aligned} ||C(A) - C(B)|| &= ||\frac{1}{W^A} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} p_i - \frac{1}{W^B} \sum_{j=1}^{m} \frac{W^B}{W^A} \sum_{i=1}^{n} f_{ij} q_j|| \\ &= \frac{1}{W^A} || \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} p_i - \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} q_j|| \\ &= \frac{1}{W^A} || \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} (p_i - q_j)|| \\ &\leq \frac{1}{W^A} \sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij} ||p_i - q_j|| \\ &= PTD(A, B) \end{aligned}$$

□

## 5.2 Approximation Algorithms

Like in the corresponding Section 4 we will now use PTD-reference points to construct approximation algorithms for the PTD under transformations. Since the algorithms are the same and the proofs carry over immediately, we will omit them and just state the results. Note that here all algorithms work for weighted point sets with arbitrary total weight, so $A$ and $B$ in this section are weighted point sets with $W^A, W^B \in \mathbb{R}^+$ and possibly $W^A \neq W^B$. In the description of the algorithms, of course, EMD has to be replaced by PTD.

### 5.2.1 Translations

**Theorem 5.3** *Let $r : \mathbb{W}^d \to \mathbb{R}^d$ be a PTD-reference point for weighted point sets with respect to translations. Let $c$ be its quality. Then algorithm TranslationApx (4.1) finds an approximately optimal matching for translations with approximation factor $c+1$ in time $O(T^{ref}(max\{n,m\}) + T^{PTD}(n,m))$. This holds for arbitrary dimension $d$ and distance measure on the ground set.*

**Corollary 5.4** *Algorithm TranslationApx (4.1) using the center of mass as an PTD-reference point induces an approximation algorithm with approximation factor 2. Its runtime is $O(T^{PTD}(n,m))$. This holds for arbitrary dimension $d$ and distance measure on the ground set.*

### 5.2.2 Rigid Motions

**Theorem 5.5** *Algorithm RigidMotionApx (4.2) finds an approximately optimal matching for rigid motions with approximation factor $c+1$ in time $O(T^{ref}(\max\{n,m\}) + T^{PTD}(n,m) + T^{rot}(n,m))$. This holds for arbitrary dimension $d$ and distance measure on the ground set.*

**Corollary 5.6** *Algorithm RigidMotionApx (4.2) using the center of mass as PTD-reference point induces an approximation algorithm with approximation factor 2 in time $O(T^{rot}(n,m) + T^{PTD}(n,m))$. This holds for arbitrary dimension $d$ and distance measure on the ground set.*

**Theorem 5.7** *Regarding EPTD in the plane, Algorithm RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor $2(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + nmT^{EPTD}(n,m))$.*

**Theorem 5.8** *Regarding $PTD_p$ in the plane, $1 \leq p \leq \infty$, Algorithm RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor $2\sqrt{2}(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + nmT^{PTD_p}(n,m))$.*

**Corollary 5.9** *Algorithm RigidMotionApxUsingRotationApx using the center of mass as PTD-reference point induces an approximation algorithm with approximation factor 4 in case of the Euclidean Distance in the plane and $4\sqrt{2}$ for any other $L_p$ distance, $1 \leq p \leq \infty$. Its runtime is $O(nmT^{PTD_p}(n,m))$.*

**Theorem 5.10** *Algorithm d-RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor $2^{d-1}(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{PTD}(n,m))$. This holds for arbitrary dimension $d$.*

**Theorem 5.11** *Algorithm d-RigidMotionApxUsingRotationApx finds an approximately optimal matching for rigid motions with approximation factor $2^{d-1}\sqrt{d}^{d-1}(c+1)$ in time $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{PTD}(n,m))$. This holds for arbitrary dimension $d$ and $1 \leq p \leq \infty$.*

**Corollary 5.12** *Algorithm d-RigidMotionApxUsingRotationApx using the center of mass as an PTD-reference point induces an approximation algorithm with approximation factor* $2^d$ *in case of the Euclidean Distance and* $2^d\sqrt{d}^{d-1}$ *for any other* $L_p$ *distance,* $1 \leq p \leq \infty$*. Its runtime is* $O(n^{d-1}m^{d-1}T^{PTD_p}(n,m))$*.*

### 5.2.3 Similarities

**Theorem 5.13** *Algorithm SimilarityApx finds an approximately optimal matching for similarities with approximation factor* $2(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + T^{PTD}(n,m) + T^{rot}(n,m))$*. This holds for arbitrary dimension d and distance measure on the ground set.*

**Corollary 5.14** *Algorithm SimilarityApx using the center of mass a PTD-reference point induces an approximation algorithm with approximation factor* $4$*. Its runtime is* $O(T^{PTD}(n,m) + T^{rot}(n,m))$*. This holds for any dimension of the ground set and every norm defined on it.*

**Theorem 5.15** *Regarding EPTD in the plane, Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor* $4(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{EPTD}(n,m))$*.*

**Theorem 5.16** *Regarding* $PTD_p$ *in the plane,* $1 \leq p \leq \infty$*, Algorithm SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor* $4\sqrt{2}(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + nmT^{PTD_p}(n,m))$*.*

**Corollary 5.17** *Algorithm SimilarityApxUsingRotationApx using the center of mass as a PTD-reference point induces an approximation algorithm with approximation factor* $8$ *in case of the Euclidean Distance in the plane and* $8\sqrt{2}$ *for any other* $L_p$ *distance,* $1 \leq p \leq \infty$*. Its runtime is* $O(nmT^{PTD_p}(n,m))$*.*

**Theorem 5.18** *Algorithm d-SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor* $2^d(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{PTD}(n,m))$*. This holds for arbitrary dimension d.*

**Theorem 5.19** *Algorithm d-SimilarityApxUsingRotationApx finds an approximately optimal matching for similarities with approximation factor* $2^d\sqrt{d}^{d-1}(c+1)$ *in time* $O(T^{ref}(\max\{n,m\}) + n^{d-1}m^{d-1}T^{PTD}(n,m))$*. This holds for arbitrary dimension d and* $1 \leq p \leq \infty$*.*

**Corollary 5.20** *Algorithm d-SimilarityApxUsingRotationApx using the center of mass as an PTD-reference point induces an approximation algorithm with approximation factor* $2^{d+1}$ *in case of the Euclidean Distance and* $2^{d+1}\sqrt{d}^{d-1}$ *for any other* $L_p$ *distance,* $1 \leq p \leq \infty$*. Its runtime is* $O(n^{d-1}m^{d-1}T^{PTD_p}(n,m))$*.*

## 6 Conclusion

In this paper we introduced EMD-reference points for weighted point sets and constructed efficient approximation algorithms for matching under various classes of transformations. In contrast to previous work, this approach allows elegant extension to higher dimensions and more general ground distances. Additionally, we presented the center of mass as an EMD-reference point for weighted point sets with equal total weight. This reference point, in fact, turns out to be an optimal reference point in the sence that there is none with a Lipschitz-constant smaller than 1. Unfortunately, the center of mass is no EMD-reference point if you consider the set of all weighted point sets, including those with different total weights. This is not surprising because we have also shown that there is no EMD-reference point for all weighted point sets and, even worse, there is also no pseudo-reference point in this case. This smashes all attempts to find a constant factor approximation just based on an equivariant mapping.

A variation of the EMD is the Proportional Transportation Distance (PTD). We have shown that the center of mass is a PTD-reference point even for weighted point sets with different total weight and all theorems and corollaries mentioned in this paper carry over.

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