Hamiltonicity and Colorings of Arrangement Graphs

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Abstract

We study connectivity, Hamilton path and Hamilton cycle decomposition, 4-edge and 3-vertex coloring for geometric graphs arising from pseudoline (affine or projective) and pseudocircle (spherical) arrangements. While arrangements as geometric objects are well studied in discrete and computational geometry, their graph theoretical properties seem to have received little attention so far. In this paper we show that they provide well structured examples of families of planar and projective-planar graphs with very interesting properties. Most prominently, spherical arrangements admit decompositions into two Hamilton cycles and 4-edge colorings, but other classes have interesting properties as well: 4-connectivity, 3-vertex coloring or Hamilton paths and cycles. We show a number of negative results as well: there are projective arrangements which cannot be 3-vertex colored. A number of conjectures and open questions accompany our results.

Keywords: line and pseudoline arrangement, circle and pseudocircle arrangement, Hamilton path, Hamilton cycle, Hamilton decomposition, coloring, connectivity, planar graph, projective-planar graph.

1 Introduction

We study connectivity, vertex and edge coloring and Hamiltonicity properties for classes of geometric graphs arising from finite collections of pseudolines (resp. pseudo-circles) in the Euclidean and Projective planes or on the sphere **S**. Our objects of study, known as arrangement graphs in the computational or discrete geometry literature, are 4-regular and planar (or projective-planar). They arise in connection with many combinatorial or algorithmic questions involving finite sets of planar lines or (via polar-duality) points (see [4]).

We proceed to a systematic study of these properties and report a number of positive and negative results, as well as a few still open questions which resisted our methods. Our most striking result, described in Section 3, is the existence of two Hamilton path and cycle decompositions for spherical arrangements, obtained via a short and easy to describe construction based on wiring diagrams.

Finding Hamilton paths and cycles in graphs is an NP-hard problem, even for planar graphs, and even for arrangement graphs of Jordan curves (see [11]). It is known that 4-connected planar graphs always have a Hamilton cycle (Tutte [20], see also [19] and [16]). The same property holds for 4-connected projective-planar graphs (Thomas and Yu [18]). It is therefore interesting to see if the Hamilton cycles could be explicitly constructed for particular classes of graphs. We have such a simple construction for spherical arrangements and odd projective arrangements.

Two Hamilton path (2HP) and cycle (2HC) decompositions for 4-regular graphs have been studied in the graph theory community. It is known that under certain conditions the number of such decompositions is even, but as far as we know, there are no explicit families of graphs where a strictly positive number of such decompositions can be guaranteed. Our pseudo-circle and separating-circle arrangement graphs provide such examples.

Coloring vertices of planar graphs with few (3 or 4) colors is known either via the Four Color Theorem, or for particular classes of planar graphs (such as 3-colorability of outerplanar graphs). 4-edge colorability of 4-regular planar graphs arising from arrangements of planar curves is known only for special cases. There are some graph theoretical conjectures (see Jaeger and Shank[12]) about 4-edge

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colorings of certain circle arrangements: a simple proof of them would imply a simple proof for the Four Color Theorem (see also Jensen and Toft [13], page 45). Although our 4-edge coloring result for spherical arrangement graphs does not seem to lead in the direction of Jaeger and Shank's conjecture, some ideas might prove relevant.

The paper is organized as follows. In section 2 we present the definitions, preliminaries and basic results on connectivity, coloring and Hamiltonicity pertaining to our three geometric models: projective, Euclidean and spherical. In section 3 we present the wiring diagram technique for constructing Hamilton path and cycle decompositions for spherical arrangements and partial results in the projective setting. Open problems and conjectures follow the natural flow of the paper.

2 Arrangement Graphs: Preliminaries

The general objects of our study are arrangement graphs arising from finite sets of curves obeying specific intersection rules and which live in the Euclidean or projective plane or on the 2-dimensional sphere. In this section we introduce three classes of arrangements and their corresponding arrangement graphs. We illustrate the definitions by examples and provide proofs of some elementary structural properties concerning connectivity and coloring.

2.1 Projective lines Arrangements of straight lines are among the most basic objects one may study in the real projective plane **P**. Accordingly they have been and still are studied under a vast variety of aspects. See the overviews by Grünbaum [10] and Erdős and Purdy [5] for further pointers to the field. Many combinatorial properties of arrangements of lines do not depend on the fact that the lines are straight, but rather on the nature of their incidence properties. This leads to the natural generalization, first done by Levi [15], to arrangements of pseudolines. See [8] for a comprehensive survey.

An *arrangement of pseudolines* in the projective plane **P** is a family $\{p_1, \dots, p_n\}$ of simple closed curves (called *pseudolines*) such that every two curves have exactly one point in common, where they cross. If no point belongs to more than two of the (pseudo)lines the arrangement is called *simple*, otherwise it is *non-simple*.

Pseudoline arrangements provide generic models for the (purely combinatorial) oriented matroids of rank 3 (see [1]). In this paper we will work only with this model.

A few simplifying assumptions: we will work only with simple arrangements. We also simplify the terminology by dropping the *pseudo* prefix from *pseudoline*: all the results of this paper hold in this more general context, and straightness of lines is no issue.

With an arrangement we associate the cell complex of vertices, edges and 2-dimensional regions into which the lines of the arrangement decompose the underlying space **P**. Arrangements are *isomorphic* provided their cell complexes are isomorphic. A projective arrangement graph is the graph of vertices and edges of an arrangement of pseudolines. See Figure 1 for an example.

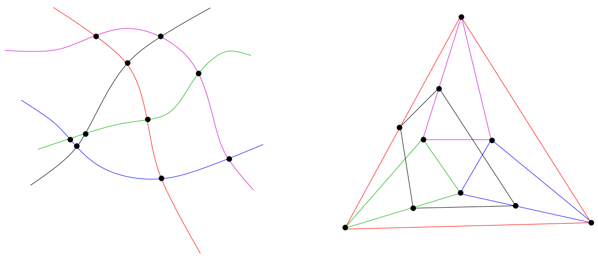

Figure 1: A projective arrangement of pseudolines and its graph

Let G be the graph of a simple projective arrangement of $n \geq 4$ lines. The following list collects some basic facts about G:

- G is 4-regular.
- G has $\binom{n}{2}$ vertices and $n(n-1)$ edges.
- G is planar only for $n = 4$ but always projective-planar.

A less trivial result is given in the next proposition.

Proposition 2.1. *The graph of a simple projective arrangement of $n \geq 4$ lines is 4-connected.*

Proof. Let G be such a graph and u, v any two vertices of G. To show 4-connectedness we will exhibit four internally disjoint paths connecting u and v in G. In the arrangement $\mathcal{A}$ defining G let p_1, p_2 be the lines through u and let q_1, q_2 be the lines through v. If $B = \{p_1, p_2, q_1, q_2\}$ contains only three lines augment B by an arbitrary fourth line. Now consider the graph H of the arrangement of the four lines in B. Note

- The vertices of H are also vertices of G and u and v are vertices of H.
- To an edge e of H connecting vertices w and w' there is a path connecting w and w' in G such that all edges of this path are supported by the line supporting e. Call this the canonical path of e.
- The canonical paths corresponding to the edges of H are pairwise internally disjoint, i.e., they can only meet at endpoints.

From these observations it follows that four disjoint paths between u and v in H can be lifted to disjoint paths in G by replacing edges by their canonical paths. Fortunately there is only one projective arrangement of four lines and hence only one projective arrangement graph H with six vertices. This graph is the skeleton graph of the octahedron. By the high regularity of this graph there are only two cases to consider, see Figure 2. $\square$

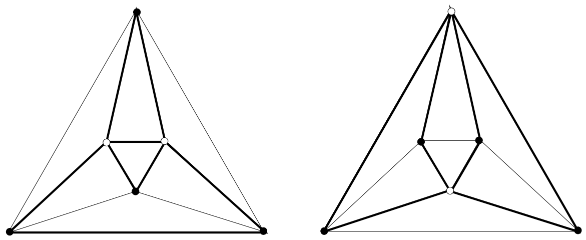

Figure 2: Four path between two vertices of H: adjacent vertices, non-adjacent vertices.

Particularly nice pictures of arrangements of pseudolines and of their graphs are given by the *wiring diagrams* introduced in Goodman [7] (see also [9, 6] and Figure 3). In this representation the n curves are restricted to n wires with different y-coordinates, except for some local switches where adjacent lines cross. These switches are the vertices of the graph. The half-edges extending to the left and right of the picture have to be identified in reverse order, as the numbers indicate in Figure 3. Sometimes a further simplification is made in drawings of wiring diagrams and the switches are only indicated by vertical segments, as in Figure 4.

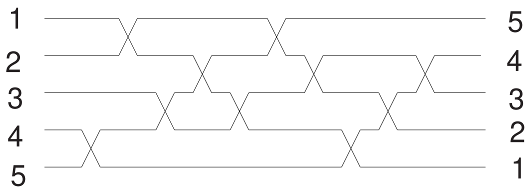

Figure 3: Wiring diagram of an arrangement of 5 pseudolines.

The *cyclic arrangement* of n lines is the arrangement where line i has the crossings with the other lines in the order $1, 2, \ldots, i-1, i+1, \ldots, n$. The *vee-shape wiring diagram* of the cyclic arrangement is the diagram where the crossings form a triangle of bricks (see Figure 4).

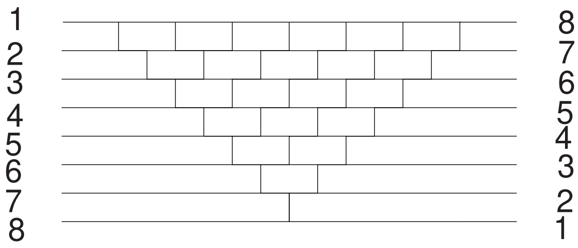

Figure 4: Wiring diagram of the cyclic arrangement of 8 lines.

We close this introductory section on projective arrangement graphs with some remarks on colorings.

By Vizing's theorem the edge chromatic number of a projective arrangement graph is either 4 or 5. If it is 4 every color class has to consist of $n(n-1)/4$ edges. This is only possible if $n \equiv 0, 1 \pmod 4$.

Conjecture 2.1. *The necessary condition* $n \equiv 0, 1 \pmod 4$ *is sufficient for the four edge colorability of projective arrangement graphs.*

With respect to the chromatic number we observe the following:

- $\chi(G) \geq 3$ for every projective arrangement graph G. This is because G always contains a triangle (see e.g. [6]).
- The graph of the cyclic arrangement of 5 lines has $\chi = 4$. We also have found an arrangement of 6 lines with $\chi = 4$.
- The graph of the cyclic arrangement has $\chi = 3$ for every $n > 5$. To see this for $n \equiv 0 \pmod 3$

color all vertices (switches) in each column of the vee-shape wiring diagram with the same color, start with 1 and repeat using 1,2,3 in cyclic order.

For $n \equiv 1 \pmod 3$ do as in the previous case but recolor the right leg of the vee-shape as 32 312 312 . . . 312 1. If $n \equiv 2 \pmod 6$ color columns in order 123 123 . . . 123 121**3**2 123 . . . 123 12. Finally, if $n \equiv 5 \pmod 6$ color columns in order 123 . . . 123 132**3** 123 . . . 123 121321, and recolor the right leg of the vee as 32 123 . . . 123 21. (In the last two cases the digit in boldface corresponds to the apex of the vee.)

An upper bound of 4 for the chromatic number of every arrangement graph is straightforward because of the degree: just use Brooks' theorem (see [2]). Results about Euclidean arrangement graphs will allow us to find a 4-coloring very efficiently.

THEOREM 2.1. *The chromatic number of projective arrangement graphs is at most 4. A 4-coloring can be efficiently found by a simple linear (in the number of vertices) time algorithm.*

2.2 Euclidean lines Given an arrangement $\{p_0, p_1, \ldots, p_n\}$ of $n+1$ lines in the projective plane we may specify a line p_0 as the "line at infinity". This induces the Euclidean arrangement of the n lines $\{p_1, \ldots, p_n\}$ in $\mathbf{E} = \mathbf{P} \setminus \mathbf{p_0}$.

The graph of an Euclidean arrangement is the graph of the bounded edges of the arrangement. A nice thing about Euclidean arrangement graphs is that they come with a natural planar embedding. The parameters of the graph G of a simple Euclidean arrangement of $n \geq 4$ lines are as follows:

- G has minimum degree 2 and maximum degree 4.
- G has $\binom{n}{2}$ vertices.
- G has $n(n-2)$ edges.
- G is 2-connected.

As in the case of projective arrangement graphs the wiring diagram is a useful form of representing Euclidean arrangement graphs. To illustrate the power of this tool we give two examples concerning colorings.

PROPOSITION 2.2. *The edge-chromatic number of an Euclidean arrangement graph is 4.*

Proof. Consider a wiring diagram W of the arrangement defining G. Note that an edge e of G is assigned to a single wire, let $w(e)$ be the number of this wire counted from top to bottom. Color the edges on each odd numbered wire alternating with colors 1 and 2 and the edges on even numbered wires alternating with colors 3 and 4. The coloring thus obtained is obviously a legal edge coloring of G. □

PROPOSITION 2.3. *The chromatic number of an Euclidean arrangement graph G is 3.*

Proof. Consider a wiring diagram W of the arrangement defining G and let the left-to-right orientation of W induce an orientation on the edges of G. Note the following facts about this oriented graph $\overrightarrow{G}$:

- $\overrightarrow{G}$ is acyclic.
- The indegree and the outdegree of vertices of $\overrightarrow{G}$ are at most 2.

A 3-coloring of G is obtained by coloring the vertices in the order given by a topological sorting of $\overrightarrow{G}$. When it comes to color v at most two neighbours (the in-neighbours) of v have been colored. Hence, one of the three available colors can legally be assigned to v. □

The two coloring results are exemplified in Figure 5. The vertex coloring was obtained by coloring from left to right and assigning colors in order of preference 1-2-3.

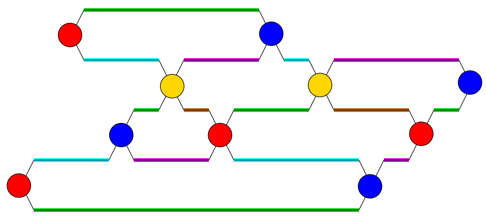

Figure 5: An Euclidean arrangement graph with 3-vertex coloring and 4-edge coloring

Proof [Theorem 2.1]. Let $\{p_0, p_1, \ldots, p_n\}$ be a projective arrangement and G its graph. Declare p_0 the line at infinity and consider the Euclidean arrangement $\{p_1, \ldots, p_n\}$ with graph G'. Note that G' is an induced subgraph of G. The vertices of G which are not in G' form an n-cycle $C = (v_1, v_2, \ldots, v_n)$ (the edges of G supported by p_0) and every vertex of C has exactly two neighbours in G'. Fix a coloring of G' with colors $\{1, 2, 3\}$ (see Proposition 2.3) and for every vertex $v_i \in C$ choose a color $c_i \in \{1, 2, 3\}$ which has not been used for a neighbour of v_i in G'.

If n is even we complete a 4-coloring of G by coloring the vertices of C of even index i with color c_i and those of odd index with a new color 4.

If n is odd and it is possible to choose the c_i such that there is an i with $c_i \neq c_{i+1}$, w.l.o.g. $i = 1$, then we complete the 4-coloring of G by coloring v_1 with c_1 and the other vertices of odd index with color 4 and those of even index i with c_i.

In the remaining case the two neighbours of all vertices of C in G' use the same two colors, say 1 and 2, so that $c_i = 3$ for all i. In this situation we choose a vertex x in G' which has two neighbours on C (this is possible since there exist triangles with a side on p_0, see [15]). W.l.o.g. we may assume that these are the vertices v_1 and v_2. Recolor x with color 4 and change c_1 to the old color of x. This brings us back to the previous case and completes the proof. $\square$

2.3 Circles on the sphere

Arrangements of pseudocircles on the sphere **S** consist of a family $\{c_1, \ldots, c_n\}$ of simple closed curves (called *circles*) such that

- every two circles have exactly two points in common at which they cross
- for three different indices $i, j, k \in \{1, \ldots, n\}$ circle c_k separates the two intersections of c_i and c_j.

The motivating examples for arrangements of circles are arrangements of great circles on the sphere. In this case **S** is a sphere centered at the origin and the circles are the interections of planes containing the origin with **S**. In Figure 6 such an arrangement of four circles on the sphere is shown (thanks to Cinderella [17] for this picture).

If we identify points on the frontside of the sphere with their antipodal counterparts on the backside we obtain a projective arrangement of n lines. If we remove the horizon-circle we obtain two isomorphic Euclidean arrangements.

Let G be the graph of a simple circle arrangement of $n \geq 3$ circles. We summarize some elementary facts about G:

- G is 4-regular.
- G has $n(n-1)$ vertices and $2n(n-1)$ edges.

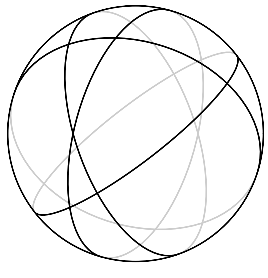

Figure 6: An arrangement of four circles on the sphere

- G is planar.

In Figure 7 we show planar embeddings of the unique simple circle arrangement graphs of tree, four and five circles. In each case one of the circles is bold-dashed, the other circles can be obtained by rotations.

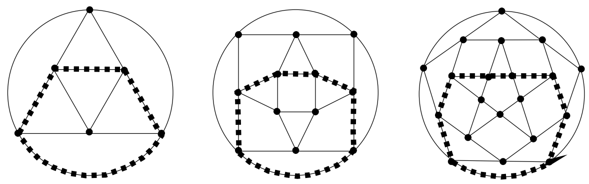

Figure 7: Circle arrangement graphs of tree, four and five circles.

The connectivity of circle arrangement graphs is as high as the degree allows:

PROPOSITION 2.4. *The graph of a simple circle arrangement of $n \geq 3$ circles is 4-connected.*

Proof. Given the graph G of a simple circle arrangement and two vertices u, v of G we exhibit four internally disjoint paths connecting u and v. Let $B = \{c_1, c_2, c_3, c_4\}$ be the circles defining the two vertices. We distinguish three cases depending on the size of B. If $|B| = 2$, i.e., if the two vertices are antipodal the four paths are given by the four arcs connecting u and v along the two cycles. If $|B| = 3$ the three cycles induce the first graph of Figure 7 and the two vertices are adjacent in this graph. Since the graph is isomorphic to the graph of Figure 2 we can refer to that figure which shows the four paths. In the last case $|B| = 4$ the two vertices are the nonadjacent vertices of a qudrilateral face of the induced graph of the four circles (this is the second graph of Figure 7). Its symmetry allows us to assume that the quadrilateral is the central one of the drawing, in which case the four paths can be choosen as shown in Figure 8. □

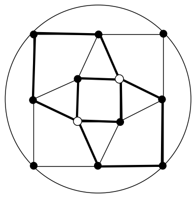

Figure 8: Four connecting paths for the white vertices.

Wiring diagrams are again a useful representation for this class of arrangements. We now give an intuitive idea of how the wiring diagram of an arrangement of n great circles can be obtained. Imagine the sphere to be a globe with the great circles drawn onto it. Now observe the shadow of the frame while the sphere moves on a full rotation around its axis. Label the circles such that in the initial position they occur in the order $1, 2, \ldots, n$ and start drawing them on n wires. When the frame passes a crossing the two circles involved in it change their order and in the wiring diagram a switch has to be drawn. After a half rotation every two circles have interchanged their order. Hence all circles are in reversed order $n, \ldots, 2, 1$. The second half of the rotation is an upside down copy of the first half. After the full rotation the frame reaches its initial position. Figure 9 shows the wiring diagram of a circle arrangement with the two halfs emphasized. To read the graph of a circle arrangement from the wiring diagram the half-edges extending to the left and right have to be identified in the same order as the numbers indicate in Figure 9.

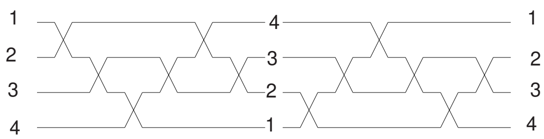

Figure 9: Two copies of the wiring diagram of an Euclidean arrangement glued together, the second copy taken upside down, give a wiring diagram of a circle arrangement.

The process described above for the construction of the wiring diagram is known as *sweeping* an arrangement. With some care in technical details it can be shown that arrangements of pseudocircles on the sphere are sweepable and also admit wiring diagrams which decompose into two halfs, one being the mirror image of the other (see [6] for related results). The diagram shown in Figure 9 has the additional property that from left to right the first crossing of every circle c_i, $i \neq 1$, is the crossing with circle c_1. Every circle arrangement has a wiring diagram with this property, which we call the *one-down property.* To transform an arbitrary diagram into one with the one-down property, move all the switches which block the visibility of circle c_1 from the left to the right side.

Using a diagram with the one-down property we will show in Section 3 that the edge set of a circle arrangement graph can be decomposed into two Hamiltonian cycles. Since each Hamiltonian cycle has $n(n-1)$ edges, an even number, we may alternatingly use colors 1 and 2 for the edges of one of the Hamiltonian cycles and colors 3 and 4 for the eges of the other Hamiltonian cycle. This proves the following proposition as a corollary.

Proposition 2.5. *Circle arrangement graphs are four edge colorable.*

Concerning vertex colorings, we have a conjecture and an efficient procedure for 4-coloring. The existence of such a coloring is implied by Brooks' theorem, but our procedure is much simpler.

Conjecture 2.2. *Circle arrangement graphs are 3-vertex colorable.*

We have verified this conjecture for all cyclic arrangements of circles. These are the arrangements obtained from Fig. 4 by gluing a mirror image of the corresponding wiring diagram.

Proposition 2.6. *Circle arrangement graphs are four vertex colorable.*

Proof. Let $\{c_0, c_1, \ldots, c_n\}$ be a circle arrangement and G its graph. Declare c_0 to be the equator and consider the Euclidean arrangements on the two hemispheres of $\mathbf{S} \setminus \mathbf{c_0}$. Let G' and G'' be the graphs of these arrangements. The vertices of G which are not in G' or G'' form an $2n$-cycle $C = (v_1, v_2, \ldots, v_{2n})$ (the edges of G supported by c_0) and every vertex of C has exactly one neighbour in G' and one in G''. Fix colorings of G' and G'' with colors $\{1,2,3\}$ (see Proposition 2.3) and for every vertex $v_i \in C$ choose a color $\gamma_i \in \{1,2,3\}$ which has not been used for a neighbour of v_i in $G' \cup G''$.

Since n is even we complete a 4-coloring of G by coloring the vertices of even index i on the cycle C with color γ_i and those of odd index with a new color 4. □

There are several generalizations of circle arrangements. We mention two of them.

- *Separating circle arrangements* consist of a family $\{c_1, \ldots, c_n\}$ of simple closed curves in the plane or on the sphere (called *circles*) such that: (1) Every two circles cross exactly twice, and (2) For any two different indices $i, j \in \{2, \ldots, n\}$ circle c_1 separates the two intersections of c_i and c_j.
- *Digon-free circle arrangements* consist of a family $\{c_1, \ldots, c_n\}$ of simple closed curves in the plane or on the sphere (called *circles*) such that: (1) Every two circles cross exactly twice, and (2) The arrangement contains no cell with only two edges and two vertices (digon).

All the results we have for circle arrangement graphs still hold for the class of separating circle arrangement graphs. Digon-free circle arrangements have been studied by Grünbaum [10]. They are much more general and have less favourable properties. E.g., in Figure 10 a digon-free arrangement is shown whose graph is only 3-connected. The completly unrestricted class of 2-intersecting closed curves has the disadvantage that the resulting graphs may have double edges.

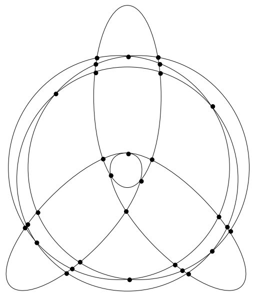

Figure 10: A digon-free arrangement with a cutset of size three.

3 Hamilton Paths and Cycles in Arrangement Graphs

In this section we study Hamiltonicity properties of spherical and projective arrangements. The Euclidean case has been settled in [3] with a negative answer (there are non-Hamiltonian Euclidean line arrangement graphs).

As shown in the previous section, both the pseudo-circle and the projective arrangements are 4-connected. A well-known theorem of Tutte [20] on 4-connected planar graphs guarantees a Hamilton cycle. An even stronger result follows from Thomassen's [19] strengthening of Tutte's theorem: every 4-connected planar graph is Hamilton connected (there exists a Hamilton path connecting any two prescribed vertices).

Theorem 3.1. *Every spherical arrangement graphs has a Hamilton cycle and is Hamilton connected.*

Thomas and Yu [18]'s theorem on 4-connected projective - planar graphs implies a similar result for projective arrangements.

Theorem 3.2. *Every projective arrangement graph is Hamiltonian.*

We now proceed to strengthen these results with explicit constructions. For spherical arrangements, we find not just one, but two such Hamilton paths and cycles, which, moreover, yield a decomposition of the edges of the graph.

3.1 Pseudo-circle and Separating Circle arrangements

Theorem 3.3. *Every pseudo-circle arrangement and and separating circle arrangement can be decomposed into two edge-disjoint Hamilton paths (plus two extra edges), and the decomposition can be found efficiently.*

Proof. The construction is based on the representation of these arrangements as wiring diagrams. As shown in the previous section, we can assume that the wiring diagram representation has the *one-down property*, as in Fig.9. The construction of the two Hamilton paths, red and blue, is described in Fig.11 for 5 wires, but it can be easily generalized to any number of wires by repeating the pattern of colors going up along the switches on line 1. The figure needs some explanations, as it looks incomplete: we did not draw all the switches corresponding to the vertices of the arrangement. We did this to draw the attention to the structure of the construction and avoid cluttering the picture. A continuously colored line along a wire of the wiring diagram denotes a path in the arrangement graph, whose edges are colored in that color and which goes along the edges incident with that wire and touches all vertices connected to them. Remember that the one-down property, and the choice of the wiring diagram drawing, insured that there are no switches left of the one-down switches.

The pictured illustrates a key element of the construction: the *one-down* property. the red, resp. blue Hamilton paths walk along the edges of a level (wire) (visiting all vertices adjacent to it) then go down by two levels at the switches (vertices) corresponding to pseudo-circle 1 (the one going *one-down*).

The crucial observation is that the red (resp. blue) path never touches the same vertex twice, and visits them all, therefore guaranteeing Hamiltonicity.

The correctness of the construction follows from the following easy to establish properties.

- Each switch, except the one involving pseudo-circle 1, is touched by the red path on an odd-numbered wire and by a blue path on an even-numbered wire.
- Each edge (with the two exceptions left uncolored (dashed)) is colored either red (thick) or blue (thin).
- All red edges are connected in a path, and so are the blue edges.
- A path in one color never visits the same vertex twice, and covers all the switches (vertices).

□

Since the spherical and projective graphs are 4-regular graphs, removing a Hamilton cycle (guaranteed by Theorem 3.1) leaves a 2-regular graph. It is a remarkable feature of the pseudo-circle arrangements that we can in fact *partition* the edges of the graph into two Hamilton cycles.

THEOREM 3.4. *Every pseudo-circle arrangement can be decomposed into two edge-disjoint Hamilton cycles, and the decomposition can be found efficiently.*

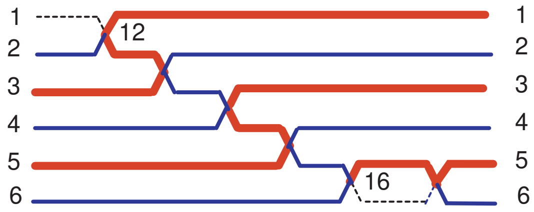

Figure 11: Two Hamilton paths in a pseudo-circle arrangement.

Proof. The proof is based on the construction illustrated in Fig.12 for $n = 6$. It uses not just the switches of line 1 but also of line 2. This is to allow each Hamilton cycle to go up by 4 levels to make room for the other Hamilton cycle to switch levels in between. One important property to make the proof work is that on the top wire there are no switches between the crossings 12 and $2x$ (where x is whatever line happens to cross line 2 right after the crossing with 1), and similarly on the bottom wire, between $1y$ and the second crossing 12. We should remark that Figure 12 gives only one case of the gluing pattern between the two Hamilton cyles, for $n \equiv 2 \pmod 4$. There are three more cases mod 4, all of which can be similarly depicted and which we omit in this abstract. The correctness of this constructive pattern follows from the following properties.

- Each switch is touched by the red path on an odd-numbered wire and by a blue path on an even-numbered wire.
- Each edge is colored either red (thick) or blue (thin).
- All red edges are connected in a cycle, and so are the blue edges.
- The path of red (resp. blue) edges never visits the same vertex twice, and covers all the switches.

□

Since these arguments do not depend on how the switches are arranged on the wires, our argument generalizes to a wider class of 4-regular planar graphs. Each 4-regular planar graph can be decomposed into closed curves crossing properly (not necessarily simple). Some of these graphs can be

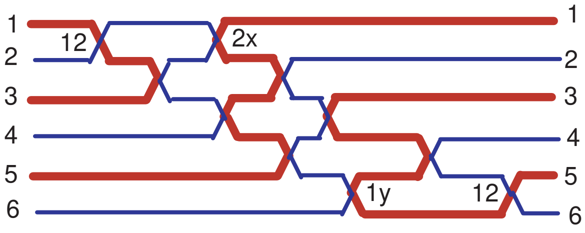

Figure 12: Two Hamilton cycle decomposition of a pseudo-circle arrangement.

drawn as wiring diagrams (*leveled*): this is a necessary condition. To make the previous construction of 2HC decomposition work, they also have to have two *one-down* strands of these curves, as in Fig. 12.

3.2 Projective arrangements

THEOREM 3.5. *Every projective arrangement with an odd number of pseudo-lines can be decomposed into two edge-disjoint Hamilton paths (plus two unused edges), and the decomposition can be found efficiently.*

Proof. The proof is based on a construction for which one example for $n = 9$ is depicted in Fig.13. The construction uses the switches of line 1 to allow each path to go up. The two dashed edges are unused, the others partition the graph into two Hamilton paths. The correctness follows from similar properties as described for pseudo-circles. □

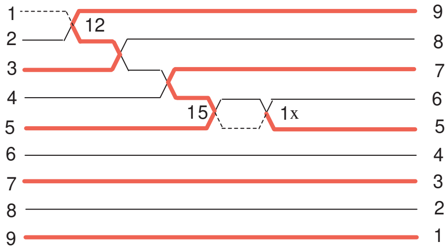

Figure 13: Two Hamilton Path decomposition of an odd projective arrangement.

The projective case is not completely settled, as we have not been able to extend this general type of argument in the case of an even number of lines. Neither do we have a counter-example, as all the examples with small number of projective pseudo-lines that we worked out turned out to be decomposable.

Since the projective graphs are also 4-regular, removing a Hamilton cycle leaves a 2-regular graph. We would expect a similar construction as in the spherical case, but so far the projective case is open: we have neither been able to find counterexamples (for small values of n, as well as for all the cyclic arrangements, we did find 2HC decompositions), neither to prove it is true.

CONJECTURE 3.1. *All projective arrangements admit 2-Hamilton cycle decompositions.*

4 Conclusion

2-Hamilton path and cycle decompositions show a high degree of structure in the geometric arrangement graphs. We have exhibited a general technique for constructing such decompositions based on wiring diagrams. It would be interesting to extend this study to 1-skeletons of arrangements in higher dimensions, where some of the tools we used (wiring diagrams, sweeps) are not available.

Several other directions for further research are open, besides the various conjectures already described in the paper. It would be interesting to count the number of 2HP and 2HC decompositions of spherical arrangements, or to characterize those graphs for which our technique of 2HP and 2HC construction works. It might be possible to generalize these techniques to classes of $2k$-regular graphs, including 1-skeletons of rank $k+1$ pseudo-sphere arrangements. We leave these problems open for further investigations.

Finally, we'd like to add a few comments on algorithmic issues. Arrangement graphs of circles on the sphere can be recognized efficiently. Since the graphs are 4-connected they have unique embeddings, from which we define circles by going straight through each vertex. The verification of the incidence properties is straightforward. It is interesting to note that for projective arrangement graphs this idea would fail: there are 5-connected projective planar graphs with many embeddings, see [14]. Concerning the vertex-coloring of projective arrangements, an interesting problem is to find a polynomial time algorithm for deciding whether χ is equal to 3 or 4.

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